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## Generators for quasi-free completely positive semi-groups

by

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**ABSTRACT.** — We construct quasi-free completely positive (CP) semi-groups on the CCR-C\*-algebra, show that they can be extended, in certain representations, to a dynamical semi-group on the associated von Neumann algebra and determine the infinitesimal generator.

**RÉSUMÉ.** — Nous construisons les semi-groupes complètement positifs et quasi-libres sur la C\*-algèbre de relations de commutation.

Ces sémi-groupes pouvant être étendus à l'algèbre de von Neumann associée à certaines représentations, on détermine le générateur infinitésimal.

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### 1. INTRODUCTION

In the algebraic approach to non-equilibrium statistical mechanics, it is generally assumed that the dynamics of an open system, idealized as a C\*- or a von Neumann algebra, is given by means of a one parameter semi-group of completely positive maps on the algebra [1] [10]. In case this semi-group extends to a group of \*-automorphisms, the system is called conservative, if not the system is called dissipative.

In this paper we study a particular class of dynamical systems, namely quasi-free boson systems. Our algebra will be the CCR-C\* algebra  $\overline{\Delta(H, \sigma)}$

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build over a symplectic space  $(H, \sigma)$ . [2] [3], while the CP maps will be of « quasi-free » type.

These CP maps were introduced in [4] [5]. The full class of these maps was characterized in [6]. Some further results, concerning extremality, dilation, implementation and relaxation, were obtained in [7].

It is clear that semi-groups of quasi-free CP maps cannot be strongly continuous, as they map Weyl-operators into Weyl-operators. We show however (theorem 4.4 below) that in certain representations, determined by quasi-free states [8], the semi-group may be extended to a so-called dynamical semi-group [9] [10] on the von Neumann algebra generated by the representation.

As an ultraweakly continuous semi-group of normal maps on a von Neumann algebra, there exists a densely defined and closed generator. We obtain this generator explicitly in theorems 4.7, 9 below. Formally it is of the Lindblad type [10].

The characterization of unbounded generators of dynamical semi-groups being far from complete, the results obtained here should contain some information on the structure of these generators.

The paper is organized as follows :

In § 2 we gather some results on symplectic spaces, operators and semi-groups on it. We briefly recall the definition of the CCR-C\* algebra  $\overline{\Delta}(H, \sigma)$ . In § 3 we construct the class of quasi-free CP semi-groups. § 4 shows, the extension of the semi-group to certain associated von Neumann algebras, is possible. Finally the explicit form of the generator is obtained. If moreover we ask for the existence of a separating vector in the representation space, the domain of the generator is fully determined.

For the general theory of semi-groups and their generators we refer to [13] [14] [15]. For a treatment of quasi-free semi-groups on the CAR algebra, see [21].

## 2. SYMPLECTIC SPACES AND THE CCR ALGEBRA

The one particle space  $(H, \sigma)$  is a real symplectic space, i. e.

- i)  $H$  is a real linear, possibly infinite dimensional, space.
- ii)  $\sigma$  is a real, bilinear antisymmetric and non degenerated form, defined on  $H$ .

On  $H$ , we define the topology, induced by the family of seminorms  $\{p_\phi\}$ ,

$$p_\phi(\psi) = |\sigma(\phi, \psi)|.$$

The resulting locally convex space is Hausdorff. We call this topology the  $s$ -topology.

Given any continuous operator  $T: H \rightarrow H$ , a unique operator  $T^+$  is

defined through the formula  $\sigma(Tx, y) = \sigma(x, T^+y)$ . A complex structure is an operator  $J: H \rightarrow H$ , such that  $J^+ = -J$ , and  $J^2 = -1$ .

A symplectic base of  $H$ , is a set of vectors  $\{f_i, g_i\}_{i=1, \dots}$  such that

- i)  $\{f_i, g_i\}$  generate  $H$  (we suppose  $H$  separable).
- ii)  $\sigma(f_i, g_j) = \delta_{ij} \quad \forall i, j$
- iii)  $\sigma(f_i, f_j) = \sigma(g_i, g_j) = 0 \quad \forall i, j$

We suppose  $\{f_i, g_i\}$  is ordered as follows:  $\{f_1, g_1, f_2, g_2, f_3, \dots\}$  and denoted as  $\{e_1, e_2, e_3, \dots\}$ . Then defining

$$J: H \rightarrow H$$

by

$$\begin{aligned} J e_{2k+1} &= e_{2k+2} & k &= 0, 1, \dots \\ J e_{2k+2} &= -e_{2k+1} & k &= 0, 1, \dots \end{aligned}$$

it is easily checked that  $J$  extends to a complex structure.

The following formulas hold

$$\forall \phi \in H: \quad \phi = \sum_k \sigma(\phi, J e_k) e_k \tag{1}$$

$$\forall \psi, \phi \in H: \quad \sigma(\phi, \psi) = \sum_k \sigma(\phi, J e_k) \sigma(e_k, \psi) \tag{2}$$

DEFINITION 2.1. [8]. —  $\mathcal{Q}$  is the set of all operators  $Q: H \rightarrow H$  s. t.

- i)  $s_Q(\psi, \phi) \equiv -\sigma(Q\psi, \phi)$  defines a real scalar product on  $H$ .
- ii)  $Q^*Q \geq 1$  where  $*$  and  $\geq$  are taken with respect to  $s_Q$ .

It follows then that  $Q^* = -Q = Q^+$ , hence  $Q$  is bounded for the  $s_Q$ -norm topology on  $H$ ; moreover  $Q$  is invertible.

Let  $\bar{H}^Q$  denote the completion of  $H$  for the  $s_Q$  norm topology. The following properties are well known. Suppose  $H$  is sequentially  $s$ -complete, then  $\forall Q \in \mathcal{Q}$ ,  $H$  is  $s_Q$ -norm complete. [8] Conversely if  $H$  is not sequentially  $s$ -complete, then  $\bar{H}^Q$  is sequentially  $s$ -complete [12], II cor. 29. Henceforth we will suppose  $H$  is sequentially  $s$ -complete.

DEFINITION 2.2. — A continuous semi-group on  $H$  is a 1-parameter family of  $s$ -continuous, everywhere defined, operators  $A_t, t \in \mathbb{R}^+$  s. t.

- i)  $A_0 = 1$
- ii)  $A_t A_s = A_{t+s}$
- iii) the map  $t \rightarrow A_t$  is  $s$ -continuous.

As  $H$  is sequentially  $s$ -complete, by the closed graph theorem,  $\{A_t\}$  is a strongly continuous semi-group on  $H$ , equipped with the  $s_Q$ -norm topology, whenever  $Q \in \mathcal{Q}$ .

The infinitesimal generator  $Z$  of  $A_t$  is defined by

$$Z\psi = \lim_{t \rightarrow 0} \frac{1}{t} (A_t - 1)\psi \quad \text{for any } \psi \in H$$

such that the limit exists (in any of the topologies). When  $H$  is sequentially  $s$ -complete,  $Z$  is  $s$  &  $s_Q$ -norm densely defined. Moreover  $Z$  is  $s_Q$ -norm closed.

In the sequel we make use of the following properties  $((+), (+ +))$ .

i) Let  $A_t$  be a continuous semi-group in  $H$ ,  $Z$  its generator, then for all  $\psi \in \mathcal{D}(Z)$  the map  $t \mapsto ZA_t\psi$  is  $s$ -continuous. Hence  $t \mapsto \sigma(ZA_t\psi, A_t\psi)$  is continuous for all  $\psi \in \mathcal{D}(Z)$  [13] [14].

ii) Let  $A_t$  be continuous, then for fixed  $\psi$ , and finite  $s$ , the set  $\{A_t\psi \mid t \in [0, s]\}$  is contained in a finite dimensional subspace of  $H$ .

The CCR  $C^*$ -algebra  $\Delta(H, \sigma)$  ([2] [3]) is the  $C^*$ -algebra obtained by completing the  $*$ -algebra  $\overline{\Delta(H, \sigma)}$  generated by the Weyl elements  $\delta_\psi$ ,  $\psi \in H$ , satisfying

$$\begin{aligned} \delta_\psi \delta_\phi &= e^{-i\sigma(\psi, \phi)} \delta_{\psi + \phi} \\ (\delta_\psi)^* &= \delta_{-\psi} \end{aligned}$$

We refer to [3] for the exact definition of the norm with respect to which the completion is to be taken.

We recall that ([8]) any  $Q \in \mathcal{Q}$  determines a quasi-free state on  $\overline{\Delta(H, \sigma)}$  through the formula

$$\omega_Q(\delta_\psi) = e^{1/2\sigma(Q\psi, \psi)}.$$

### 3. QUASI-FREE CP SEMIGROUPS

Let  $A$  be any operator on  $H$ .

Denoting  $\sigma_A(\psi, \phi) \equiv \sigma(\psi, \phi) - \sigma(A\psi, A\phi)$ . It was shown in [6], that the map

$$\begin{aligned} \tau : \Delta(H, \sigma) &\rightarrow \Delta(H, \sigma) \\ \tau(\delta_\psi) &= \delta_{A\psi} f(\psi) \end{aligned} \tag{3}$$

$f$  being a functional on  $H$ , such that  $f(0) = 1$ , extends to a CP map on  $\overline{\Delta(H, \sigma)}$  iff  $\omega$ , defined by

$$\omega(\delta_\psi) = f(\psi)$$

extends to a state on the  $C$  algebra  $\overline{\Delta(H, \sigma_A)}$ .

Imposing some regularity conditions, the general form of semi-groups, consisting of CP maps of type (3), was exhibited in [7]. In the following we suppose  $f(\psi)$  to be the generating functional of a quasi-free state on  $\overline{\Delta(H, \sigma_A)}$ .

**THEOREM 3.1.** — Let  $\tau_t : \overline{\Delta(H, \sigma)} \rightarrow \overline{\Delta(H, \sigma)}$  be a one parameter semi-group of quasi-free CP maps, i. e.

$$\tau_t(\delta_\psi) = \delta_{A_t\psi} f(\psi) \tag{4}$$

such that  $A_t$  is continuous and for all  $\psi$  the map  $t \mapsto f_t(\psi)$  is differentiable.

Suppose  $f_t(\psi)$  is the *generating functional* of a quasi-free state on  $\overline{\Delta(H, \sigma_{A_t})}$ , then it is of the form

$$f_t(\psi) = \exp [\sigma(B_t\psi, \psi)] \tag{5}$$

where

$$B_t = \int_0^t A_x^+ Y A_x dx \tag{6}$$

Here  $Y$  satisfies :

- i)  $Y$  is uniquely and everywhere defined
- ii)  $Y^+ = -Y$  is bounded for all  $s_Q$ -norm topologies.
- iii)  $\sigma(Y\psi, \psi) \leq 0 \quad \forall \psi$
- iv)  $\forall \psi, \phi \in \mathcal{D}(Z)$

$$|\sigma(Z\psi, \phi) + \sigma(\psi, Z\phi)| \leq -\frac{1}{2}[\sigma(Y\psi, \psi) + \sigma(Y\phi, \phi)] \tag{7}$$

Conversely, any continuous semi-group  $\{A_t\}$  and operator  $Y$  with the above properties, define a CP semi-group through the formulas (4) (5) (6).

*Proof.* — As  $t \rightarrow f_t(\psi)$  is differentiable for all  $\psi$  and as  $B_t^+ = -B_t$ , we obtain that the map  $t \rightarrow \sigma(B_t\psi, \phi)$  is differentiable for all  $\psi, \phi$  in  $H$ . On the other hand, as  $Q$  is invertible when  $Q \in \mathcal{Q}$

$$\forall t \exists B_t^Q \text{ s. t. } B_t = Q B_t^Q$$

then  $B_t^Q = B_t^{Q*}$  where the adjoint is taken w. r. t.  $s_Q$ . As  $B_t^Q$  is everywhere defined,  $B_t^Q$  is bounded for the  $s_Q$ -norm topology. We have that

$$\lim_{t \rightarrow 0} \frac{1}{t} [\sigma(Q B_t^Q \psi, \phi) - \sigma(Q \psi, \phi)] = -\lim_{t \rightarrow 0} \frac{1}{t} [s_Q(B_t^Q \psi, \phi) - s_Q(\psi, \phi)]$$

exists for all  $\psi$  and  $\phi$  in  $H$ . Thus there is an  $s_Q$ -bounded operator  $Y^Q$  such that

$$\left. \frac{d}{dt} s_Q(B_t^Q \psi, \phi) \right|_{t=0} = s_Q(Y^Q \psi, \phi)$$

Defining  $Y = Q Y^Q$  we obtain i) and ii). Using Prop. 4.2 in [7] we obtain

$$\begin{aligned} f_t(\psi) &= \exp \left[ \int_0^t dx \left( \frac{d}{dt}, f_t, (A_x \psi) \Big|_{t'=0} \right) \right] \\ &= \exp \left[ \int_0^t dx \sigma(Y A_x \psi, A_x \psi) \right]. \end{aligned}$$

To show iii) we note that  $f_t(\psi)$  is the generating functional of a state on a  $C^*$ -algebra and that as such  $|f_t(\psi)| = |\omega_t(\delta_\psi)| \leq 1$  hence, for all  $t$ ,  $\sigma(B_t\psi, \psi) \leq 0$ . On the other hand,  $\tau_0 = 1$ , and we have  $\sigma(B_0\psi, \psi) = 0$  thus

$$\left. \frac{d}{dt} \sigma(B_t\psi, \psi) \right|_{t=0} \leq 0,$$

which is by definition  $\sigma(Y\psi, \psi) \leq 0$ .

Finally we express that

$$f_t(\psi) = \exp \left[ \int_0^t dx \sigma(YA_x\psi, A_x\psi) \right]$$

defines a state on  $\overline{\Delta(H, \sigma_{A_t})}$ . That  $f_t(\psi)$  generates a state, implies ([8])

$$|\sigma_{A_t}(\psi, \phi)| \leq -\frac{1}{2}[\sigma(B_t\psi, \psi) + \sigma(B_t\phi, \phi)];$$

Noting once more that the equality is reached for  $t = 0$ , we derive for  $\psi, \phi \in \mathcal{D}(Z)$

$$|\sigma(Z\psi, \phi) + \sigma(\psi, Z\phi)| \leq -\frac{1}{2}[\sigma(Y\psi, \psi) + \sigma(Y\phi, \phi)]$$

Conversely, suppose  $\{A_t\}$  is a continuous semi-group on  $H$ , and  $Y$  is an operator on  $H$  enjoying properties *i*), *ii*), *iii*) and *iv*).

Taking  $\psi, \phi$  in  $\mathcal{D}(Z)$ , by *iv*) we obtain for all  $x \geq 0$

$$\begin{aligned} -\sigma(ZA_x\psi, A_x\phi) - \sigma(A_x\psi, ZA_x\phi) \\ \leq -\frac{1}{2}[\sigma(YA_x\psi, A_x\psi) + \sigma(YA_x\phi, A_x\phi)] \quad (8) \end{aligned}$$

The left and right hand sides of the inequality are integrable on any bounded interval by property (+); integrating (8) yields

$$\begin{aligned} -\int_0^s [\sigma(ZA_x\psi, A_x\phi)dx + \sigma(A_x\psi, ZA_x\phi)dx] \\ \leq -\frac{1}{2} \int_0^s [\sigma(YA_x\psi, A_x\psi) + \sigma(YA_x\phi, A_x\phi)]dx \end{aligned}$$

Hence

$$\sigma(\psi, \phi) - \sigma(A_t\psi, A_t\phi) \leq -\frac{1}{2}[\sigma(B_t\psi, \psi) + \sigma(B_t\phi, \phi)]$$

using the other inequality we arrive at (7). This together with *i*) *ii*) and *iii*) imply that  $f_t(\psi)$  defines a state on  $\Delta(H, \sigma_{A_t})$ . That  $\tau_t$  defines a semi-group on  $\overline{\Delta(H, \sigma)}$  follows now from [7] prop. 4.2. ■

EXAMPLE 3.2. — [4] [16]. — Let  $H = \mathbb{R}^2$

$$\sigma((x, y), (x', y')) = \frac{1}{2}(xy' - yx')$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A_t z = e^{-\lambda t} z \quad \lambda > 0 \quad t \geq 0 \quad z \in \mathbb{R}^2$$

$$f_t(z) = \exp \left[ -\frac{\theta}{4}(1 - e^{-2\lambda t}) \|z\|^2 \right], \quad \theta \geq 1$$

then  $f_t(z)$  is of the form (6) with

$$Y = -\frac{\lambda\theta}{2} J$$

At the risk of being confusing we now introduce

DEFINITION 3.3. — Any semi-group  $\{\tau_t\}$  as in theorem 3.1 will be called a quasi-free CP semi-group.

#### 4. GENERATORS OF QUASI-FREE CP SEMIGROUPS

DEFINITION 4.1 [10]. — A one parameter semi-group  $\{\tau_t\}$  on a von Neumann algebra  $\mathcal{M}$  is called a dynamical semi-group whenever

- i)  $\forall t \tau_t$  is a CP map  $\mathcal{M} \rightarrow \mathcal{M}$
- ii)  $\tau_t(1) = 1$  for any  $t$
- iii)  $\tau_0 = 1$
- iv)  $\forall t \tau_t$  is an ultraweakly continuous map
- v)  $t \rightarrow \tau_t(x)$  is ultraweakly continuous.

It follows that  $\{\tau_t\}$  is a contraction semi-group.

If  $\{\tau_t\}$  is a dynamical semi-group, there exists an ultraweakly dense set in  $\mathcal{M}$ , called  $\mathcal{D}(\mathcal{L})$  such that for all  $x \in \mathcal{D}(\mathcal{L})$  the

$$\lim_{t \rightarrow 0} \frac{\tau_t(x) - x}{t} \quad \text{exists in ultraweak (u. w.)-sense.}$$

The limit is called  $\mathcal{L}(x)$ ; it is again an element of  $\mathcal{M}$ .  $\mathcal{L}$  is called the generator of  $\{\tau_t\}$ . It can be shown that  $\mathcal{L}$  is uw-closed [13] [15].

Using the methods of [11] [13] [15] the following can be shown.

THEOREM 4.2. — Let  $\tau_t$  be a dynamical semi-group on  $\mathcal{M}$ . Then

1.  $\forall x \in \mathcal{M}, \forall s$ , the element  $x_s = \int_0^s \tau_t(x) dt$  is well defined and  $x_s \in \mathcal{D}(\mathcal{L})$

Moreover  $\tau_s(x) - x = \mathcal{L}(x_s)$

2. Let  $\mathcal{C}$  be an u. w. dense set,  $\mathcal{C} \subseteq \mathcal{D}(\mathcal{L})$ , s. t.

$$\tau_t(\mathcal{C}) \subseteq \mathcal{C}, \quad \text{then } \mathcal{C} \text{ is a core for } \mathcal{L}. \quad \blacksquare$$

Let  $\omega_Q$  be any quasi-free state on  $\overline{\Delta(H, \sigma)}$ , and  $\tau_t$  a quasi-free CP semi-group, then  $\forall t, \tau_t^*(\omega_Q)$  is again quasi-free and

$$\tau_t^* \cdot \omega_Q = \omega_{A_t^* Q A_t + B_t}$$



We now introduce our main hypothesis in order to ensure condition *iv*) of definition 4.1 is satisfied when  $\tau_t$  is considered in a quasi-free representation of the CCR.

DEFINITION 4.3. — A quasi-free state  $\omega_Q$  is said to be approximately  $\tau_t$ -invariant, iff  $\forall t \geq 0$

$$Q - A_t^+ Q A_t - B_t \tag{9}$$

is an  $s_Q$ -trace class operator.

THEOREM 4.4. — Let  $\omega_Q$  a quasi-free state,  $\tau_t$  a quasi-free semi-group on  $\overline{\Delta(H, \sigma)}$ . If  $H$  is sequentially  $s$ -complete, and  $\omega_Q$  is approximately  $\tau_t$  invariant then  $\tau_t$  extends uniquely to a dynamical semi-group on

$$\Pi_{\omega_Q}(\overline{\Delta(H, \sigma)})'' \equiv \mathcal{M}.$$

*Proof* (For convenience we denote  $\Pi_{\omega_Q}(x)$  as  $x$ . —  $\omega_Q$  and  $\tau_t^*(\omega_Q)$  are quasi equivalent. Indeed, as

$$H = \bar{H}^Q = \bar{H}^{A_t^+ Q A_t + B_t}$$

both  $\omega_Q$  and  $\tau_t^*(\omega_Q)$  are factor states [8] and the condition ensures that they are quasi-equivalent [17].

We now show that for any  $\omega \in \mathcal{M}_*^+$ , the positive part in the predual of  $\mathcal{M}$ ,  $\tau_t^*(\omega)$  is again in  $\mathcal{M}_*^+$ .

Denote by  $\tau_\chi$  the gauge automorphism  $\overline{\Delta(H, \sigma)} \rightarrow \overline{\Delta(H, \sigma)}$  defined by  $\tau_\chi(x) = \delta_\chi x \delta_{-\chi}$ ,  $\chi \in H$  and, given  $\omega \in \mathcal{M}_*^+$ , define the state  $\omega^\chi$

$$\omega^\chi = \omega_0 \tau_\chi$$

then  $\tau_t^*(\omega^\chi) = (\tau_t^*(\omega_0))^{\Lambda_t^+ \chi}$ .

Hence,  $\forall \chi \in H$ , the state  $\tau_t^*(\omega^\chi)$  is quasi-equivalent to  $\omega_0$ .

For a general state  $\omega \in \mathcal{M}_*^+$ , there is a sequence  $\rho_n$  where  $\rho_n$  is a finite linear combination of states of the form  $\omega^\chi$  such that  $\rho_n \rightarrow \omega$  in norm [18].

On the other hand,  $\forall t$ , the map  $\tau_t^* : \overline{\Delta(H, \sigma)}^* \rightarrow \overline{\Delta(H, \sigma)}^*$  as the dual of a normalized positive map on a  $C^*$ -algebra, is norm continuous (in fact  $\|\tau_t^*\| = 1$ )

Thus we obtain  $\tau_t^*(\rho_n) \rightarrow \tau_t^*(\omega)$  in norm and as  $\mathcal{M}_*$  is norm closed  $\tau_t^*(\omega) \in \mathcal{M}_*^+$ . This is nothing but saying that for any  $t$ , the map  $\tau_t : \Pi_\omega(\overline{\Delta(H, \sigma)}) \rightarrow \Pi_{\omega_0}(\overline{\Delta(H, \sigma)})$  is ultraweakly continuous.

This implies that  $\forall t$ ,  $\tau_t$  can be extended to an uw-continuous map  $\bar{\tau}_t : \mathcal{M} \rightarrow \mathcal{M}$ . Using a Kaplansky-type approximation, we show that  $\bar{\tau}_t$  is positive.

In the same way we have  $\bar{\tau}_t \otimes 1_n : \mathcal{M} \otimes \mathcal{M}_n \rightarrow \mathcal{M} \otimes \mathcal{M}_n$  are positive, such that  $\bar{\tau}_t$  is CP.

Remains to show *v*) of definition 4.1, i. e. for all  $\omega$  in  $\mathcal{M}_*$  and for all  $x$

in  $\mathcal{M}$  the map  $t \mapsto \omega(\tau_t(x))$  is continuous. This is clearly true for  $\omega = \omega_Q \cdot \tau_x$  and

$$x = \sum_{i=1}^n \lambda_i \delta \psi_i$$

Using once more the norm density of the linear combinations of the states  $\omega_Q \cdot \tau_x$  in  $\mathcal{M}_*^+$ , we obtain the continuity of

$$t \mapsto \omega(\tau_t(x))$$

for all  $\omega \in \mathcal{M}_*$ , and  $x = \sum_{i=1}^n \lambda_i \delta \psi_i$

Finally, for general  $x \in \overline{\Pi(\Delta(H, \sigma))}$ , there is a sequence  $x_n, x_n \in \overline{\Pi(\Delta(H, \sigma))}$

$$x_n \rightarrow x \text{ in norm}$$

hence

$$|\omega(\tau_t(x)) - \omega(\tau_t(x_n))| = |(\omega_Q \tau_t)(x - x_n)| \leq \|x - x_n\| < \varepsilon \quad \text{for } n \text{ large enough.}$$

By the uniform convergence  $t \mapsto \omega(\tau_t(x))$  is continuous.

Then, using the method of [19], we obtain the continuity for all  $x$  in  $\mathcal{M}$ . ■

Let  $\omega_Q$  be any quasi-free state on  $\overline{\Delta(H, \sigma)}$ ; for all  $\psi$  and  $\phi$  in  $H$ , the map

$$\lambda \in \mathbb{R} \mapsto \omega_Q(\delta_{\lambda\psi + \phi})$$

is infinitely differentiable. Hence,  $\forall \psi$ , there exists a selfadjoint operator  $B_Q(\psi)$  on  $\mathcal{H}_{\omega_Q}$ , the GNS space for  $\omega_Q$  such that

$$\Pi_{\omega_Q}(\delta_{\lambda\psi}) = e^{i\lambda B_Q(\psi)},$$

(Remark that  $\mathcal{H}_{\omega_Q}$  is separable).

Moreover  $\forall \psi, \phi, \eta \in H$  we have

$$i) \quad \Pi_{\omega_Q}(\delta_\psi) \Omega_{\omega_Q} \in \mathcal{D}(B_Q(\phi))$$

here  $\Omega_{\omega_Q}$  denotes the cyclic vector in the GNS space.

$$ii) \quad B_Q(\psi) \Pi_{\omega_Q}(\delta_\phi) \Omega_{\omega_Q} \in (B_Q(\eta)).$$

If  $\{e_k\}$  is a symplectic base for  $H$ , we will denote  $B_Q(e_k)$  by  $B_k$ . From now on we drop all indices referring to  $\omega_Q$ .

The following equalities are easily verified:

$$\begin{aligned} &\langle \Pi(\delta_\phi) \Omega, B_k [B_l, \Pi(\delta_\psi)] \Pi(\delta_\eta) \Omega \rangle \\ &= 2[\sigma(e_k, \psi + \phi + \eta) + i\sigma(Qe_k, -\phi + \psi + \eta)] \\ &\quad \cdot \sigma(e_l, \psi) \langle \pi(\delta_\phi) \Omega, \pi(\delta_\psi) \pi(\delta_\eta) \Omega \rangle \end{aligned} \tag{10}$$

$$\begin{aligned} &\langle \Pi(\delta_\phi) \Omega, [B_k, \Pi(\delta_\psi)] B_l \Pi(\delta_\eta) \Omega \rangle \\ &= 2[-\sigma(e_l, \psi + \phi + \eta) + i\sigma(Qe_l, -\phi + \psi + \eta)] \\ &\quad \cdot \sigma(e_k, \psi) \langle \pi(\delta_\phi) \Omega, \pi(\delta_\psi) \pi(\delta_\eta) \Omega \rangle \end{aligned} \tag{11}$$

LEMMA 4.5. — Let  $\tau_t$  and  $\omega_Q$  as in theorem 4.4  $\Pi$  as above. Then  $\forall k, l \in \mathbb{N}$  and  $\forall \psi, \phi \in H$ , and for all finite  $s$  the elements

$$\int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \quad (12)$$

$$\int_0^s \Pi(\tau_t(\delta_\psi))B_l\Pi(\delta_\phi)\Omega dt \quad (13)$$

are well defined and belong to respectively  $\mathcal{D}(B_k B_l)$  and  $\mathcal{D}(B_k)$ .

*Proof* (for (12)). — It is easily checked that the maps

$$t \rightarrow \|\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega\|$$

and

$$t \rightarrow \|\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega\|$$

are continuous; thus the elements

$$\int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt$$

and

$$\int_0^s B_l\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt$$

exist in Bochner sense.

By the continuity of  $t \mapsto \tau_t$ ,  $\int_0^s \Pi(\tau_t(\delta_\psi))dt$  exists as an operator (in ultraweak sense).

Let  $\xi \in \mathcal{D}(B_l)$ , then

$$\begin{aligned} \langle B_l \xi, \int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \rangle &= \int_0^s \langle B_l \xi, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega \rangle dt \\ &= \int_0^s \langle \xi, B_l \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega \rangle dt \\ &= \langle \xi, \int_0^s B_l \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \rangle \end{aligned}$$

Thus, since  $B_l = B_l^*$ ,  $\int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \in \mathcal{D}(B_l)$ . In the same way we prove that

$$\int_0^s B_l \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \text{ is in } \mathcal{D}(B_k). \quad \blacksquare$$

PROPOSITION 4.6. — Let  $H$  be finite dimensional,  $\tau$ , a CP quasi-free semi-group;  $\omega_Q$  a quasi-free state on  $\overline{\Delta(H, \sigma)}$ .

Then there exist real sequences

$$\begin{aligned} \{ a_{lk} \} & \quad k, l = 1, \dots, 2N \\ \{ b_{lk} \} & \quad k, l = 1, \dots, 2N \end{aligned}$$

such that  $\forall x \in \mathcal{D}(\mathcal{L})$  and all  $\xi, \eta$  in a dense set  $\mathcal{D}$ , one has

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle = & -\frac{i}{4} \sum_{k,l} a_{lk} [ \langle \mathbf{B}_l \mathbf{B}_k \xi, x\eta \rangle - \langle \mathbf{B}_k \xi, x\mathbf{B}_l \eta \rangle ] \\ & + a_{kl} [ \langle \mathbf{B}_k \xi, x\mathbf{B}_l \eta \rangle - \langle \xi, x\mathbf{B}_k \mathbf{B}_l \eta \rangle ] \\ & - \frac{1}{4} \sum_{k,l} b_{lk} [ \langle \mathbf{B}_l \mathbf{B}_k \xi, x\eta \rangle - \langle \mathbf{B}_k \xi, x\mathbf{B}_l \eta \rangle ] \\ & - b_{kl} [ \langle \mathbf{B}_k \xi, x\mathbf{B}_l \eta \rangle - \langle \xi, x\mathbf{B}_k \mathbf{B}_l \eta \rangle ] \end{aligned} \tag{12}$$

Moreover there is a core  $\mathcal{C}$  for  $\mathcal{L}$ , such that  $\forall x \in \mathcal{C}$  and  $\forall n \in \mathcal{D}$

$$\begin{aligned} \mathcal{L}(x)\eta = & -\frac{i}{4} \sum_{k,l} (a_{kl} \mathbf{B}_k [\mathbf{B}_l, x]\eta + a_{kl} [\mathbf{B}_k, x] \mathbf{B}_l \eta) \\ & - \frac{1}{4} \sum_{k,l} (b_{kl} \mathbf{B}_k [\mathbf{B}_l, x]\eta - b_{kl} [\mathbf{B}_k, x] \mathbf{B}_l \eta). \end{aligned} \tag{13}$$

*Proof.* — Let  $\mathcal{D}$  be the linear span of  $\{ \Pi(\delta_\psi)\Omega \mid \psi \in \mathbf{H} \}$  in  $\mathcal{H}$ . Denote by  $\mathcal{C}$  the linear span of  $\left\{ \int_0^s \Pi(\tau_t(\delta_\psi)) dt \mid s < \infty, \psi \in \mathbf{H} \right\}$  in  $\mathcal{M}$ .

Then, by theorem 4.2,  $\mathcal{C}$  is a core for  $\mathcal{L}$ .  
Then define ( $e_k$  being a symplectic base)

$$\begin{aligned} a_{kl} &= \sigma(\mathbf{J}e_k, \mathbf{Z}^+ \mathbf{J}e_l) \\ b_{kl} &= \sigma(\mathbf{J}e_k, \mathbf{Y} \mathbf{J}e_l) \end{aligned}$$

By theorem 4.2 we know

$$\mathcal{L} \left( \int_0^s \Pi(\tau_t(\delta_\psi)) dt \right) = \Pi(\tau_s(\delta_\psi)) - \Pi(\delta_\psi)$$

on the other hand, the function

$$t \mapsto \frac{d}{dt} \langle \Pi(\delta_\phi)\Omega, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\eta)\Omega \rangle$$

being continuous,

$$\langle \Pi(\delta_\phi)\Omega, \Pi(\tau_s(\delta_\psi) - \delta_\psi)\Pi(\delta_\eta)\Omega \rangle = \int_0^s \frac{d}{dt} [ \langle \Pi(\delta_\phi)\Omega, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\eta)\Omega \rangle ] dt.$$

For notational convenience, we put  $\phi = \eta = 0$ .

$$\left\langle \Omega, \mathcal{L} \left[ \int_0^s \Pi(\tau_i(\delta_\psi)) dt \right] \Omega \right\rangle = \int_0^s dt \omega_Q(\tau_i(\delta_\psi)) [-\sigma(ZA_i\psi, QA_i\psi) + \sigma(YA_i\psi, A_i\psi)]$$

which by (1) equals

$$= \int_0^s dt \omega_Q(\tau_i(\delta_\psi)) \sum_{k,l} [\sigma(A_i\psi, J e_l) \sigma(Z e_l, J e_k) \sigma(Q e_k, A_i\psi) - \sigma(A_i\psi, J e_l) \sigma(e_l, Y J e_k) \sigma(e_k, A_i\psi)]$$

substituting  $(J e_l)$  for  $(e_l)$  and  $(-e_l)$  for  $(J e_l)$

$$\begin{aligned} &= - \int_0^s dt \omega_Q(\tau_i(\delta_\psi)) \sum_{k,l} [\sigma(A_i\psi, e_l) \sigma(Z e_l, J e_k) \sigma(Q e_k, A_i\psi) - \sigma(A_i\psi, e_l) \sigma(J e_l, Y J e_k) \sigma(e_k, A_i\psi)] \\ &= - \frac{i}{2} \int_0^s dt \omega_Q(\tau_i(\delta_\psi)) \sum_{k,l} a_{lk} [\sigma(e_l, A_i\psi) \sigma(e_k, A_i\psi) + i \sigma(Q e_k, A_i\psi) \sigma(e_l, A_i\psi) + a_{kl} [-\sigma(e_k, A_i\psi) \sigma(e_l, A_i\psi) + i \sigma(e_k, A_i\psi) \sigma(Q e_l, A_i\psi)]] \\ &\quad - \frac{1}{2} \int_0^s dt \omega_Q(\tau_i(\delta_\psi)) \sum_{k,l} b_{kl} [\sigma(e_l, A_i\psi) \sigma(e_k, A_i\psi) + i \sigma(Q e_k, A_i\psi) \sigma(e_l, A_i\psi) + b_{lk} [\sigma(e_k, A_i\psi) \sigma(e_l, A_i\psi) - i \sigma(e_k, A_i\psi) \sigma(Q e_l, A_i\psi)]] \end{aligned}$$

by (10) (11) this equals

$$\begin{aligned} &= - \int_0^s dt \left[ \frac{i}{4} \sum_{k,l} [a_{lk} \langle \Omega, \mathbf{B}_k [\mathbf{B}_l, \tau_i(\delta_\psi)] \Omega \rangle + a_{kl} \langle \Omega, [\mathbf{B}_k, \tau_i(\delta_\psi)] \mathbf{B}_l \Omega \rangle] \right. \\ &\quad \left. - \frac{1}{4} \sum_{kl} [b_{lk} \langle \Omega, \mathbf{B}_k [\mathbf{B}_l, \tau_i(\delta_\psi)] \Omega \rangle - b_{kl} \langle \Omega, [\mathbf{B}_k, \tau_i(\delta_\psi)] \mathbf{B}_l \Omega \rangle] \right] \quad (14) \end{aligned}$$

as all terms in the sum are integrable, we obtain by making use of lemma 4.5.

$$\begin{aligned} &= - \frac{i}{4} \sum_{kl} \left[ a_{lk} \left\langle \Omega, \mathbf{B}_k \left[ \mathbf{B}_l, \int_0^s \tau_i(\delta_\psi) dt \right] \Omega \right\rangle + a_{kl} \left\langle \Omega, \left[ \mathbf{B}_k, \int_0^s \tau_i(\delta_\psi) dt \right] \mathbf{B}_l \Omega \right\rangle \right] \\ &\quad - \frac{1}{4} \sum_{kl} \left[ b_{lk} \left\langle \Omega, \mathbf{B}_k \left[ \mathbf{B}_l, \int_0^s \tau_i(\delta_\psi) dt \right] \Omega \right\rangle - b_{kl} \left\langle \Omega, \left[ \mathbf{B}_k, \int_0^s \tau_i(\delta_\psi) dt \right] \mathbf{B}_l \Omega \right\rangle \right] \quad (15) \end{aligned}$$

which is (13).

For general  $x$  in  $\mathcal{D}(\mathcal{L})$ , we proceed as follows  $\exists x_\alpha \in \mathcal{C}$  such that  $x_\alpha \rightarrow x$  u. w. and  $\mathcal{L}(x_\alpha) \rightarrow \mathcal{L}(x)$  u. w. thus, for  $(\xi, \eta) \in \mathcal{D}$ :

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \lim_\alpha [\langle \xi; \mathcal{L}(x_\alpha)\eta \rangle] \\ &= \lim_\alpha \left[ \frac{-i}{4} \sum_{kl} a_{lk} [\langle \mathbf{B}_l \mathbf{B}_k \xi, x_\alpha \eta \rangle - \langle \mathbf{B}_k \xi, x_\alpha \mathbf{B}_l \eta \rangle + \dots] \right] \end{aligned}$$

since there isn't but a finite number of terms, this equals

$$= -\frac{i}{4} \sum_{kl} a_{lk} \langle \mathbf{B}_l \mathbf{B}_k \xi, x \eta \rangle - \langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle + \dots$$

which is (12). ■

NOTATION 4.7. — For fixed  $x$  in  $\mathcal{D}(\mathcal{L})$ , the right hand side in (12) defines a bilinear form on  $\mathcal{D}$ . We will denote it as

$$\phi_x(\xi, \eta)$$

Let  $(H_n)_{n \in \mathbb{N}}$  be an increasing and absorbing net of finite dimensional regular symplectic subspaces of  $H$ . Then we define  $\mathcal{M}_F \subseteq \mathcal{M}$  as

$$\mathcal{M}_F \equiv \bigcup_{n \in \mathbb{N}} \Pi(\overline{\Delta(H_n, \sigma)})''$$

PROPOSITION 4.8. — Let  $H$  be infinite dimensional and sequentially  $s$ -complete,  $\omega_Q$  and  $\tau_t$  as in theorem 4.4.

Then there exist real infinite sequences  $\{a_{lk}\}, \{b_{ek}\}$  such that  $\forall x \in \mathcal{D}(\mathcal{L}) \cap \mathcal{M}_F$  and all  $\xi$ , in a dense set  $\mathcal{D}$  one has

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \frac{i}{4} \sum_{k,l} [a_{lk} [\langle \mathbf{B}_l \mathbf{B}_k \xi, x \eta \rangle - \langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle] \\ &\quad + a_{kl} [\langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle - \langle \xi, x \mathbf{B}_k \mathbf{B}_l \eta \rangle]] \\ &\quad - \frac{1}{4} \sum_{k,l} [b_{lk} [\langle \mathbf{B}_l \mathbf{B}_k \xi, x \eta \rangle - \langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle] \\ &\quad - b_{lk} [\langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle - \langle \xi, x \mathbf{B}_k \mathbf{B}_l \eta \rangle]] \quad (16) \\ &\equiv \phi_x(\xi, \eta) \end{aligned}$$

Moreover there is a core  $\mathcal{C}$  for  $\mathcal{L}$ , such that  $\forall x \in \mathcal{C}$ , a formula similar to (13) holds.

*Proof.* — Define  $\mathcal{D}$  as in proposition 4.7, and  $\mathcal{C}$  as the linear span of  $\left\{ \int_0^s \Pi(\tau_t(\delta_\psi)) dt \mid s < \infty, \psi \in \mathcal{D}(z) \right\}$ . Again by theorem 4.2  $\mathcal{C}$  is a core for  $\mathcal{L}$ .

Choose a symplectic base in  $\{e_k\}$  in  $\mathcal{D}(Z^+)$  and define

$$a_{kl} = \sigma(Je_k, Z^+Je_l)$$

$$b_{kl} = \sigma(Je_k, YJe_l)$$

If  $\psi \in \mathcal{D}(Z)$ , then by property (+) stated in § 2, the map

$$t \mapsto \frac{d}{dt} \omega_Q(\tau_t(\delta_\psi)) = \omega_Q(\tau_t(\delta_\psi))[-\sigma(ZA_t\psi, QA_t\psi) + \sigma(YA_t\psi, A_t\psi)]$$

is continuous.

The proof is then a mere extension of the method in 4.7. Indeed, by the continuity, we arrive at a formula similar to (14). The elements  $\int_0^s \Pi(\tau_t(\delta_\psi))dt$  belong to  $\mathcal{M}_F$ , by property (+) stated in § 2.

Hence we obtain (15), and thus (13).

For general  $x \in \mathcal{D}(\mathcal{L})$ , there is a net  $\{x_\alpha\} \in \mathcal{C}$  such that  $x_\alpha \rightarrow x$  and  $\mathcal{L}(x_\alpha) \rightarrow \mathcal{L}(x)$  u. w.

Thus for  $\xi, \eta \in \mathcal{D}$

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \lim_\alpha \langle \xi, \mathcal{L}(x_\alpha)\eta \rangle \\ &= \lim_\alpha \phi_{x_\alpha}(\xi, \eta) \end{aligned} \tag{17}$$

If moreover  $x \in \mathcal{M}_F$ , then for  $l$  sufficiently large we obtain e. g.,

$$\langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle = 0.$$

Since any term in (17) is convergent to a term which eventually vanishes, we obtain (16). ■

Formula (16) is of the form

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \lim_{N \rightarrow \infty} \sum_{k,l=1}^N -\frac{i}{4} [a_{lk} \langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle] \\ &\quad + a_{kl} \dots + \dots \\ &\equiv \lim_{N \rightarrow \infty} \phi_x^N(\xi, \eta) \end{aligned}$$

We remark however that in general the bilinear forms  $\phi_x^N$  are not associated to a generator of a quasi-free CP semi-group on  $\Pi(\overline{\Delta(H_N, \sigma)})''$ , the reason being that when  $Z$  generates a semi-group in  $H$ , and  $P_N$  is the projection onto  $H_N$ ,  $P_N Z P_N$  need not generate a semi-group on  $H_N$ , except when  $H_N$  is a reducing subspace.

Remark also that if  $\sum_{k,l} (a_{lk} + a_{kl}) B_k B_l$  can be given a sense as a self-

adjoint operator, then by re-summing the series, we recover the Lindblad form of the generator [10] [22].

If we impose the condition that  $\mathcal{M}$  has a separating vector, then we

can determine  $\mathcal{D}(\mathcal{L})$ , in the same way as was done in [11] for spatial derivations.

**THEOREM 4.9.** — Let  $H, \omega_Q, \tau, \mathcal{D}$  as above. Suppose moreover that  $\omega_Q$  extends faithfully to  $\mathcal{M}$ .

An element  $x$  in  $\mathcal{M}_F$  belongs to  $\mathcal{D}(\mathcal{L})$  iff the bilinear form

$$\phi_x(\cdot, \cdot) : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{C}$$

has a bounded extension to  $\mathcal{H} \times \mathcal{H}$ .

*Proof.* — If  $x \in \mathcal{D}(\mathcal{L}) \cap \mathcal{M}_F$ , then by proposition 4.7 and 4.9, there is an element  $\mathcal{L}(x)$  in  $\mathcal{M}$  such that  $\phi_x(\xi, \eta) = \langle \xi, \mathcal{L}(x)\eta \rangle$ . Hence  $\phi_x$  has a bounded extension.

Let  $\phi_x(\cdot, \cdot)$  have a bounded extension, then there is a bounded operator  $B_x$  such that for all  $\xi, \eta \in \mathcal{D}$

$$\langle \xi, B_x \eta \rangle = \phi_x(\xi, \eta)$$

As  $\mathcal{D}(\mathcal{L}) \cap \mathcal{M}_F$  is u. w. dense in  $\mathcal{M}$ , there is a net  $x_\alpha$  in this set such that

$$x_\alpha \rightarrow x \text{ u. w.}$$

then  $\phi_{x_\alpha}(\xi, \eta) \rightarrow \phi_x(\xi, \eta)$  for all  $\xi, \eta \in \mathcal{D}$  or  $\langle \xi, \mathcal{L}(x_\alpha)\eta \rangle \rightarrow \langle \xi, B_x \eta \rangle$ .

As both  $\mathcal{L}(x_\alpha)$  and  $B_x$  are bounded we have that  $\mathcal{L}(x_\alpha) \rightarrow B_x$  weakly, hence  $B_x \in \mathcal{M}$ .

In case  $\mathcal{M}$  has a separating vector, then the weak and ultraweak topologies coincide [20]. Thus we constructed a net  $x_\alpha$  in  $\mathcal{D}(\mathcal{L})$

$$x_\alpha \rightarrow x \text{ u. w.}$$

and

$$\mathcal{L}(x_\alpha) \rightarrow B_x \text{ u. w.}$$

as  $\mathcal{L}$  is u. w. closed, we obtain  $x \in \mathcal{D}(\mathcal{L})$  and  $\mathcal{L}(x) = B_x$ . ■

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