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G. G. ASGEKAR

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**On the space-time permeated by a viscous,
compressible, thermally conducting,
self-gravitating fluid
with infinite electrical conductivity
and constant magnetic permeability**

by

G. G. ASGEKAR

Department of Mathematics, Shivaji University, Kolhapur-416 004 India

ABSTRACT. — The properties of the space-time filled with a viscous, compressible, thermally conductivity, relativistic fluid with infinite electrical conductivity and constant magnetic permeability (magnetofluid) are investigated. The causes producing the disturbance in the homogeneity and isotropy of the universe filled with the magnetofluid are found by constructing « Maxwell-like » equations. Further the compatibility of a C-space with the gravitational field of the magnetofluid is examined. The propagation equations for the kinematical parameters are obtained and it is proved that for an essentially expanding flow of the magnetofluid, the tidal force is due to magnetic field only. Moreover, if the stream lines are shear-free and irrotational then it is shown that the magnetofluid is of Petrov type I, D or O.

1. INTRODUCTION

The techniques of the classical magnetohydrodynamics are applied to the astronomical systems like magnetic variable stars (Cowling [5]), sun spots (Wilson [19]) and intergalactic space (Parker [16]) by ignoring gravitational fields. However, the astronomical systems possess intense gravitational fields. For such fields the methods of the relativistic magnetohydrodynamics are indispensable.

Relativistic Magnetohydrodynamics (RMHD) is basically developed through Minkowski's electrodynamics of moving bodies. Lichnerowicz ([13]) has given an elegant account of the field equations of RMHD and established the existence and uniqueness of their solutions. His formalism is applied by Yodzis [20] in the study of galactic cosmology, gravitational collapse and pulsar theory. Bray [4] has found Gödel type of universes by solving Lichnerowicz's field equations. These field equations are utilized by Date [6, 7] to study the local behaviour congruences in RMHD and has obtained a class of non-uniform cosmological models. In this context the work of Mason [14] regarding the velocity of Alfvén wave is remarkable. Greenberg [11] has derived the post-Newtonian approximations of RMHD equations under certain assumptions. The action principle is successfully used by Maugin [15] to investigate the electromagnetic field in matter.

In view of the occurrence of the magnetic fields in astronomical systems it is appropriate to consider the matter distributions as cosmic magnetofluids (Yodzis [20]). The stress energy tensor for thermodynamical perfect fluid with infinite electrical conductivity and constant magnetic permeability (perfect magnetofluid) is due to Lichnerowicz [13]. The assumption of infinite conductivity is convenient in analytical work and befits the study of cosmic fluids which possess very high electrical conductivity. Nevertheless, this fluid is too ideal to describe the natural systems in which the matter consists of viscous, compressible thermally conducting self-gravitating fluids with strong magnetic fields.

It is, therefore, desirable to study the properties of a viscous, compressible, thermally conducting self-gravitating fluid with infinite electrical conductivity and constant magnetic permeability (in this connection please refer [1-3]). Throughout the development of this paper we designate such fluid by *magnetofluid*. The local behaviour of the time like congruences, the weak conservation laws and the interaction of the gravitational field with the magnetofluid are the main objectives of this work.

We consider the 4-dimensional space-time with the metric form $ds^2 = g_{ab}dx^a dx^b$. The signature of the metric is $(-, -, -, +)$. Latin indices a, b, \dots run from 1 to 4. Semicolon and the comma represent covariant derivative and partial derivative respectively. Symmetrization and anti-symmetrization are denoted by round and square brackets around the suffixes respectively. Units are so chosen that the gravitational constant and the velocity of light are 1.

2. FIELD EQUATIONS AND CONSEQUENCES

Following the hint given by Greenberg [11] we construct a symmetric stress-energy tensor for viscous, compressible, thermally conducting and

self-gravitating fluid with infinite electrical conductivity by linear superposition of fluid stress-energy tensor and magnetic field stress-energy tensor. The resultant stress-energy tensor for the magnetofluid is ([1])

$$T_{ab} = (\rho + p + \mu h^2)u_a u_b - (p + 1/2\mu h^2)g_{ab} + v\sigma_{ab} + 2u_{(a}q_{b)} - \mu h_a h_b, \quad (2.1)$$

where ρ is the matter energy density, p is the isotropic pressure, μ is the constant magnetic permeability, v is the coefficient of shear viscosity, q^a is the heat-flux vector and h^a is the magnetic field vector. The gradient of the time-like 4-velocity vector u^a has the expression

$$u_{a;b} = \sigma_{ab} + w_{ab} + 1/3\theta p_{ab} + \dot{u}_a u_b, \quad (2.2)$$

where σ_{ab} is the symmetric shear tensor, w_{ab} is skew symmetric rotation tensor, θ is the expansion scalar, $p_{ab} = g_{ab} - u_a u_b$ is the 3-space projection operator and $\dot{u}_a = u_{a;b}u^b$ is the 4-acceleration. The space-like vectors h_a and q_a satisfy the relations

$$h^a u_a = q^a u_a = 0, \quad h^a h_a = -h^2, \quad q^a q_a = -q^2. \quad (2.3)$$

According to Greenberg [11] the matter energy density ρ is connected with the proper matter density ρ_0 and the internal energy density ε as

$$\rho = \rho_0(1 + \varepsilon), \quad (2.4)$$

and the equations connecting the thermodynamical variables are

$$TdS = d\varepsilon + pd(1/\rho_0) \quad (2.5)$$

$$S^a = \rho_0 S u^a + q^a/T \quad (2.6)$$

$$q^a = -K(T_{,b} - T\dot{u}_b)p^{ab}. \quad (2.7)$$

Here T is the rest temperature, S is the specific entropy, S^a is entropy flux vector and K is the coefficient of heat conduction. For the magnetofluid characterized by (2.1) the field equations are the Einstein equations.

$$R_{ab} - 1/2Rg_{ab} = -T_{ab}, \quad (2.8)$$

and the Maxwell equations

$$(u^a h^b - u^b h^a)_{;b} = 0. \quad (2.9)$$

We have from (2.9)

$$\dot{u}_a h^a = -h^a_{;a}, \quad (2.10)$$

$$\sigma_{ab} h^a h^b + 2/3\theta h^2 + 1/2h^2_{,a} u^a = 0. \quad (2.11)$$

The local conservation laws $T^{ab}_{;b} = 0$ provide the equations of stream-lines for fluid particle in the form

$$A_{,b} u^a u^b - B_{,b} g^{ab} + A(\dot{u}^a + \theta u^a) + \dot{q}^a + \theta q^a + u^a_{;b} q^b + q^b_{;b} u^a + v\sigma^{ab}_{;b} - \mu(h^a h^b)_{;b} = 0. \quad (2.12)$$

where $A = \rho + p + \mu h^2$, $B = p + 1/2\mu h^2$.

This equation with (2.9) yields the equation of continuity for the magnetofluid as

$$\dot{\rho} + (\rho + p)\theta + \dot{q}^a u_a + q^b{}_{;b} + v\sigma^{ab}{}_{;b} u_a = 0. \quad (2.13)$$

On making use of the thermodynamical relations (2.4)-(2.7) we obtain

$$S^a{}_{;a} = (\rho_0 u^a)_{;a} + \rho_0 S_{,a} u^a + (q^a/T)_{;a}. \quad (2.14)$$

This manifests the entropy generation in the magnetofluid. The space-components of (2.12) are given by

$$A \dot{u}^c = B_{,b} p^{bc} - v\sigma^{ab}{}_{;b} p_a^c - \dot{q}_a p^{ac} - \theta q^c - u^c{}_{;b} q^b + \mu(h^a h^b)_{;b} p_a^c. \quad (2.15)$$

The equations (2.15) give the factors affecting the deviation of the fluid elements from the geodesic path.

3. MAXWELL-LIKE EQUATIONS FOR MAGNETOFLUID

We have the expression for the Weyl tensor in terms of the Riemann curvature tensor R_{abcd} and the Ricci tensor $R_{ab} = R^c{}_{abc}$ as (Ellis [9]).

$$C_{abcd} = R_{abcd} - g_{a[d} R_{c]b} - g_{b[c} R_{d]a} + R/3 g_{a[d} g_{c]b} \quad (3.1)$$

with the properties

$$C_{abcd} = C_{[ab][cd]}, \quad (3.2)$$

$$C_{a[bc]d} = 0, \quad (3.3)$$

$$C^a{}_{bca} = 0. \quad (3.4)$$

The electric type component E_{ab} and the magnetic type component H_{ab} are defined as

$$E_{ab} = C_{abcd} u^b u^d, \quad (3.5)$$

$$H_{ab} = 1/2 C_a{}^{pqr} \eta_{qrs} u_p u^s \quad (3.6)$$

Consequently (3.1) takes the form

$$C_{abcd} = (g_{abpq} g_{cdrs} - \eta_{abpq} \eta_{cdrs}) u^p u^r E^{qs} - (\eta_{abpq} g_{cdrs} + g_{abpq} \eta_{cdrs}) u^p u^r H^{qs}, \quad (3.7)$$

where

$$g_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc}.$$

These two tensors E_{ab} and H_{ab} are symmetric, trace-free and orthogonal to the flow vector u^a . The divergence of the Weyl tensor is independent of itself and is designated as the matter current J_{abc}^* (Szekerés [18]) viz.

$$C_{abc;d}^{\dots d} = J_{abc}^* \quad (3.8)$$

Accordingly, the equation

$$J^{*abc}{}_{;c} = 0, \quad (3.9)$$

is the conservation equation for the source of the gravitational field. The well-known Bianchi identities $R_{ab[c;d;e]} = 0$ imply (Kundt and Trumper [12])

$$J^{*abc} = R^{c[a;b]} - \frac{1}{6} g^{c[a} R^{b]}. \quad (3.10)$$

On using the Einstein field equations (2.8) the expression (3.10) becomes

$$J^{*abc} = T_{c[a;b]} + \frac{1}{3} g_{c[a} T_{b]}. \quad (3.11)$$

For the magnetofluid (3.11) gives

$$\begin{aligned} J^{*abc} = & u_c(\rho + p + \mu h^2)_{,a} u_b + (\rho + p + \mu h^2) u_{c;[a} u_{b]} \\ & + (\rho + p + \mu h^2) u_c u_{[b;a]} + u_{c;[a} q_{b]} + q_c u_{[b;a]} \\ & + u_c q_{[b;a]} + q_{c;[a} u_{b]} + v \sigma_{c[b;a]} - \mu h_{c;[a} h_{b]} \\ & - \mu h_c h_{[b;a]} + \left(\frac{1}{3} \rho + 1/2 \mu h^2 \right)_{, [b} g_{a]c}. \end{aligned} \quad (3.12)$$

The Bianchi identities (3.10) for the decomposition (3.7) give rise to the « Maxwell-like » equations. By virtue of (3.12) we get following Maxwell-like equations for the magnetofluid

$$\begin{aligned} 3H_{ab} w^b + E_{b;c} P_a{}^b P^{cd} - \eta_{abcd} u^b \sigma^c{}_{,e} H^{de} \\ = -\frac{1}{3} \rho_{,b} P^b{}_a + \frac{3}{2} w_{ba} q^b + 1/2 \sigma_{ba} q^b - \frac{1}{3} \theta q_a \\ - v \sigma^2 u_a - v/2 \sigma^b{}_{a;b} - v/2 \sigma^b{}_a \dot{u}_b + 1/2 \mu h_{a;b} h^b - 1/2 \mu h_{b;c} h^c u^b u_a, \end{aligned} \quad (3.13)$$

$$\begin{aligned} H_{bc;d} P^{cd} P^{ab} - 3E^a{}_b w^b - \eta^{abcd} u_b \sigma_{ce} E^e{}_d \\ = -(\rho + p + \mu h^2) w^a - 1/2 \eta^{abcd} u_b q_{[c;d]} \\ + v/2 \sigma_{ce} (w_d{}^e + \sigma^{de}) \eta^{abcd} u_b + 1/2 \mu \eta^{abcd} u_b u^e h_c h_{e;d}, \end{aligned} \quad (3.14)$$

$$\begin{aligned} H_{ac} P^{ma} P^{tc} - P_a{}^{(m} \eta^{t)rsd} u_r E^a{}_{s;d} + 2E_a{}^{(t} \eta^{m)bpq} u_b u_p \\ + P^{mt} \sigma^{ab} H_{ab} + \frac{1}{3} \theta H^{mt} - 3H_s{}^{(m} \sigma^{t)s} - H_s{}^{(m} \omega^{t)s} \\ = 1/2 \sigma_c{}^{(t} \eta^{m)ebc} u_e q_b + 1/2 \omega_c{}^{(t} \eta^{m)ebc} u_e q_b + 1/2 \theta P_c{}^{(t} \eta^{m)ebc} u_e q_b \\ + q^{(t} \omega^{m)} + v/2 \sigma_b{}^f{}_{;c} P_f{}^{(t} \eta^{m)ebc} u_e - \mu/2 h_{a;c} P^{a(t} \eta^{m)ebc} u_e h_b \\ - \mu/2 h_{b;c} h^{(t} \eta^{m)ebc} u_e. \end{aligned} \quad (3.15)$$

$$\begin{aligned}
\dot{E}^{ac}P_a^tP_b^m + H^a_{s;d}u_rP_a^{(m}\eta^{t)rsd} - 2H_a^{(t}\eta^{m)bpq}u_b\dot{u}_p + \sigma^{ab}E_{ab}P^{mt} \\
+ \frac{1}{3}\theta E^{mt} - 3E_s^{(m}\sigma^{t)s} - E_s^{(m}\omega^{t)s} = -1/2(\rho + p + \mu h^2)\sigma^{tm} \\
+ \dot{u}^{(t}q^{m)} - 1/2P^{at}p^{mb}q_{(a;b)} + v/2P^{at}P^{mb}\dot{\sigma}_{ab} + v/2\sigma^{c(t}\omega^{m)}.c \\
+ v/2\sigma^{c(t}\sigma^{m)}.c + v/6\theta\sigma^{tm} - \frac{1}{6}\{q_a\dot{u}^a - q^a_{;a} \\
- 2v\sigma^2\}P^{tm} - 1/4\mu h^2_{.c}u^c\dot{P}^{tm} - 1/6\mu h^2\theta P^{tm} \\
- \mu/2h_{a;b}u^bh^mp^{at} - \mu h_{[b;c]}u^ch^tP^{bm}. \quad (3.16)
\end{aligned}$$

In the absence of electromagnetic field, the equations (3.13)-(3.16) reduce to the equations given by Ellis [9] for a thermally conducting, viscous, compressible fluid. The extra terms due to the viscosity, the electromagnetic field and the heat flux vector on right hand side of (3.13)-(3.16) produce disturbance in the gravitational radiation.

If we assume that the undisturbed state is conformally flat ($C_{abcd} = 0$), then (3.13)-(3.16) reduce to

$$\begin{aligned}
\frac{1}{3}\rho_{,b}P^b_a - 3/2\omega_{ba}q^b - 1/2\sigma_{ba}q^b + 1/3\theta q_a + v\sigma^2u_a \\
+ v/2\sigma^b_{a;b} + v/2\sigma^b_a\dot{u}_b - 1/2\mu h_{a;b}h^b + 1/2\mu h_{b;c}h^c u^b u_a = 0 \quad (3.17)
\end{aligned}$$

$$\begin{aligned}
(\rho + p + \mu h^2)\omega^a + 1/2\eta^{abcd}u_bq_{[c;d]} - v/2\sigma_{ce}(w_d^e + \sigma_d^e)\eta^{abcd}u_b \\
- 1/2\mu\eta^{abcd}u_bu^e h_c h_{e;d} = 0, \quad (3.18)
\end{aligned}$$

$$\begin{aligned}
1/2\sigma_c^{(t}\eta^{m)ebc}u_e q_b + 1/2\omega_c^{(t}\eta^{m)ebc}u_e q_b + 1/6\theta P_c^{(t}\eta^{m)ebc}u_e q_b \\
+ q^{(t}\omega^{m)} + v/2\sigma_{b;c}^f P_f^{(t}\eta^{m)ebc}u_e - \frac{\mu}{2}h_{a;c}P^{a(t}\eta^{m)ebc}u_e h_b \\
- \frac{\mu}{2}h_{b;c}h^{(t}\eta^{m)ebc}u_e = 0. \quad (3.19)
\end{aligned}$$

$$\begin{aligned}
1/2(\rho + P + \mu h^2)\sigma^{tm} - \dot{u}^{(t}q^{m)} + 1/2p^{at}P^{mb}q_{(a;b)} - v/2P^{at}P^{mb}\dot{\sigma}_{ab} \\
- v/2\sigma^{c(t}\omega^{m)}.c - v/2\sigma^{c(t}\sigma^{m)}.c - v/2\theta\sigma^{tm} \\
+ 1/6(q_a\dot{u}^a + q^a_{;a} - 2v\sigma^2)P^{tm} + 1/4\mu h^2_{.c}u^c P^{tm} \\
+ 1/6\mu h^2\theta P^{tm} + \frac{\mu}{2}h_{a;b}u^bh^mp^{at} + \mu h_{[b;c]}u^ch^tP^{bm} = 0. \quad (3.20)
\end{aligned}$$

In case of the perfect fluid equations (3.17)-(3.20) yield

$$\omega_{ab} = \sigma_{ab} = 0, \quad (3.21)$$

$$\rho_{,a}P^{ab} = \theta_{,a}P^{ab} = 0. \quad (3.22)$$

If the equation of state is

$$p = p(\rho), \quad (3.23)$$

then the stream lines equation with (3.21) and (3.23) implies

$$p_{,a}P^{ab} = \dot{u}^b = 0. \tag{3.24}$$

Thus, the space-time of the perfect fluid turns to be homogeneous and isotropic. From (3.17)-(3.20), it is clear that this homogeneity and isotropy of the universe is disturbed due to the viscosity, the heat-flux vector and the electromagnetic field.

4. C-SPACE

The concept of C-space has been introduced by Szekeres [18] as the space in which the divergence of Weyl tensor vanishes. The compatibility of a C-space with the gravitational field of the magnetofluid is examined.

THEOREM 1. — *In C-space for the magnetofluid with the equation of state $\rho = 3/2 p$ if the stream lines are essentially rotating, then $\omega_{bc}q^b + 1/4(1/2\mu h^2_{,c} - \dot{q}_c) = 0$.*

Proof. — For essentially rotating stream lines we have

$$\theta = \sigma_{ab} = \dot{u}_a = 0, \quad \omega_{ab} \neq 0. \tag{4.1}$$

Accordingly (3.13), (2.15) and Maxwell equations (2.9) give rise to

$$1/3\rho_{,b}P^b_c - 3/2\omega_{bc}q^b - 1/2\mu h_{c;b}h^b = 0 \tag{4.2}$$

$$-p_{,c}P^c_b + \dot{q}_b + \omega_{bc}q^c - 1/2\mu h^2_{,b} + 1/2\mu h^2_{,c}u^c u_b - \mu h_{b;c}h^c - \mu h_{b;c}h^c - \mu h^c_{;c}h_b = 0 \tag{4.3}$$

and

$$h^2_{,a}u^a = h^b_{;b} = 0. \tag{4.4}$$

The equations (4.2)-(4.4) on simplification give

$$1/6(3p - 2\rho)_{,b}P^b_c + 2\omega_{bc}q^b + 1/2(1/2\mu h^2_{,c} - \dot{q}_c) = 0. \tag{4.5}$$

From (4.5) with the equation of state $\rho = 3/2p$, we procure

$$\omega_{bc}q^b + 1/4(1/2\mu h^2_{,c} - \dot{q}_c) = 0. \tag{4.6}$$

Here the proof is complete.

Remark 1. — From (4.6) we get

$$\omega_{bc}q^b = 0 \Leftrightarrow \dot{q}_c = 1/2\mu h^2_{,c}.$$

Hence, the heat-flux vector is orthogonal to the plane of rotation of the magnetofluid if and only if $\dot{q}_c = 1/2\mu h^2_{,c}$.

Remark 2. — By virtue of (4.6), we have

$$\mu \dot{h}^2{}_{,c} q^c = q^2{}_{,c} u^c = 0$$

This implies that for C-space with the essentially rotating flow of the magnetofluid, the magnitude of the magnetic field conserves along the heat-flux vector if and only if the magnitude of the heat-flux vector conserves along the world line.

THEOREM 2. — *In C-space for the magnetofluid the stream lines are expansion-free if and only if the magnitude of the magnetic field remains invariant along these lines.*

Proof. — For the C-space, the equation $J_{abc} u^c P^a{}_{(t} P^b{}_{m)} g^{tm} = 0$ with (3.20) yields

$$-2\nu\sigma^2 - 1/2\mu(\dot{h}^2\theta + 1/2\dot{h}^2{}_{,a}u^a + u_{a;b}h^ah^b) = 0. \quad (4.8)$$

Accordingly, the Maxwell equations imply

$$1/2\mu\dot{h}^2{}_{,a}u^a + \frac{2}{3}\dot{h}^2\theta = 0. \quad (4.9)$$

This gives

$$\theta = 0 \Leftrightarrow \dot{h}^2{}_{,a}u^a = 0, \quad \text{as} \quad \dot{h}^2 \neq 0. \quad (4.10)$$

Here the proof is complete.

COROLLARY. — *For the C-space permeated by the magnetofluid with Killing heat-flux vector, the 4-acceleration is orthogonal to the magnetic field vector.*

Proof. — Transvecting (3.20) with h^ah^b and using the condition (4.8) in the resulting equation we have

$$\dot{u}_a h^a (q_a h^b) - q_{a;b} h^a h^b + 1/6 \dot{h}^2 (q_a \dot{u}^a - q^a{}_{;a}) = 0, \text{ by (4.9)} \quad (4.11)$$

If the heat-flux vector is Killing then

$$q_{(a;b)} = 0.$$

Consequently (4.11) reduces to

$$\dot{u}_a h^a (q_b h^b) = 0.$$

$$\text{i. e.} \quad \dot{u}_a h^a = 0, \quad \text{as} \quad q_a h^b \neq 0. \quad (4.13)$$

Thus, the 4-acceleration is orthogonal to the magnetic field vector.

Remark. — The Maxwell equations with (4.13) produce $h^a{}_{;a} = 0$. Thus, in C-space for the magnetofluid with Killing heat-flux vector, the magnetic lines are divergence-free.

5. PROPAGATION EQUATIONS OF THE KINEMATICAL PARAMETERS

In the general theory of relativity Ricci identities are

$$R_{abcd}\xi^a = 2\xi_{b;[cd]}, \tag{5.1}$$

where ξ is any arbitrary vector field.

A) Propagation equation for the expansion parameter

For the flow vector u^a (5.1) becomes

$$2u_{b;[cd]} = R_{abcd}u^a \tag{5.2}$$

i. e.
$$2u_{b;[cd]}u^d = R_{abcd}u^a u^d. \tag{5.3}$$

On making use of the identity

$$(\dot{u}_b)_{;c} = u_{b;dc}u^d + u_{b;d}u^d_{;c} \tag{5.4}$$

in (5.3), we obtain

$$(u_{b;c})' + u_{b;d}u^d_{;c} - \dot{u}_{b;c} = R_{abcd}u^a u^d. \tag{5.5}$$

The decomposition (2.2) of the gradient of the 4-velocity u^a is used in (5.5) to get

$$\begin{aligned} (u_{b;c})' - \dot{u}_{b;c} + \sigma_{ba}\sigma^d_c + \sigma_{ba}\omega^d_c + \omega_{ba}\omega^d_c \\ + \omega_{ba}\sigma^d_c + 2/3\theta\omega_{bc} + 2/3\theta\sigma_{bc} + 1/9\theta^2P_{bc} - R_{abcd}u^a u^d = 0. \end{aligned} \tag{5.6}$$

The contraction of (5.6) with g^{bc} yields

$$R_{ab}u^a u^b = \dot{\theta} + 1/3\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a_{;a} \tag{5.7}$$

For the magnetofluid we have

$$R_{ab}u^a u^b = -1/2(\rho + 3p + \mu h^2). \tag{5.8}$$

Accordingly (5.7) becomes

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a_{;a} + 1/2(\rho + 3p + \mu h^2) = 0. \tag{5.9}$$

This is the equation of propagation for the expansion parameter. Note that (5.9) is the Raychaudhuri's [17] equation for the magnetofluid. In the light of the time-like, rigid and normal congruence (Ehlers and Kundt [8]) we write

$$u_{a;b} = \dot{u}_a u_b. \tag{5.10}$$

So that (5.9) reduces to

$$\dot{u}^a{}_{;a} = 1/2(\rho + 3p + \mu h^2). \quad (5.11)$$

The isotropic part of the rate of change of the distance Δ of neighbouring particles of the magnetofluid with respect to the time is determined by the expansion parameter θ

$$\text{as} \quad \frac{\dot{\Delta}}{\Delta} = \frac{1}{3} \theta \quad (5.12)$$

$$\text{i. e.} \quad \frac{3\ddot{\Delta}}{\Delta} = \dot{\theta} + \frac{1}{3} \theta. \quad (5.13)$$

Consequently (5.9) produces

$$\frac{3\ddot{\Delta}}{\Delta} = 2(\omega^2 - \sigma^2) + \dot{u}^a{}_{;a} - 1/2(\rho + 3p + \mu h^2). \quad (5.14)$$

Remark 1. — For static space-time permeated by the magnetofluid, active gravitational mass density $-1/2(\rho + 3p + \mu h^2)$, balancing the divergence of the 4-acceleration of the magnetofluid, explains the phenomenon of the gravitational collapse (*vide* 5.9).

Remark 2. — The equation (5.14) explains clearly the effects of the magnetic field, rotation, distortion, divergence of the acceleration and the active gravitational mass density on the second derivative of the curve $\Delta(t)$.

B) Propagation equation for the shear tensor

We differentiate the expression for the shear tensor and use (5.6) in the resulting equation to derive

$$\begin{aligned} \dot{\sigma}_{ab} = & \dot{u}_{(a;b)} - u_{(a}\ddot{u}_{b)} - \dot{u}_a\dot{u}_b - \frac{2}{3}\theta\sigma_{ab} - \omega_{ad}\omega_{;b}^d - \sigma_{ad}\sigma_{;b}^d \\ & - u_{(b}u_{a);d}\dot{u}^d + \frac{2}{3}\theta u_{(a}\dot{u}_{b)} - \frac{1}{3}P_{ab}\left(\dot{\theta} + \frac{1}{3}\theta^2\right) + R_{abcd}u^c u^d, \end{aligned}$$

i. e.

$$\begin{aligned} \dot{\sigma}_{ab} = & \dot{u}_{(a;b)} - \ddot{u}_{(a}u_{b)} - \dot{u}_a\dot{u}_b - \frac{2}{3}\theta\sigma_{ab} - \omega_{ad}\omega_{;b}^d \\ & - \sigma_{ad}\sigma_{;b}^d - u_{(b}u_{a);d}\dot{u}^d + \frac{2}{3}\theta u_{(a}\dot{u}_{b)} - \frac{1}{3}P_{ab}[2(\omega^2 - \sigma^2) + \dot{u}^c{}_{;c}] \\ & + R_{abcd}u^c u^d - \frac{1}{3}R_{cd}u^c u^d P_{ab}, \text{ by (5.7).} \end{aligned} \quad (5.15)$$

The spatial component of $\dot{\sigma}_{ab}$ is given by

$$\begin{aligned} \perp \dot{\sigma}_{ab} = \perp \dot{u}_{(a;b)} - \dot{u}_a \dot{u}_b - \omega_{ad} \omega_{.b}^d - \sigma_{ad} \sigma_{.b}^d \\ - \frac{2}{3} \theta \sigma_{ab} - \frac{1}{3} \{ 2(\omega^2 - \sigma^2) + \dot{u}_{;c}^c \} P_{ab} \\ + R_{mabn} u^m u^n - \frac{1}{3} R_{mn} u^m u^n P_{ab}, \end{aligned} \quad (5.16)$$

where $\perp \dot{\sigma}_{ab} = P^c_a P^d_b \dot{\sigma}_{cd}$.

THEOREM 3. — *For the essentially expanding flow of the magnetofluid, the tidal force is due to magnetic field only.*

Proof. — The essentially expanding flow is characterized by

$$\sigma_{ab} = \omega_{ab} = 0, \quad \dot{u}_a = 0, \quad \theta \neq 0. \quad (5.17)$$

It follows from (5.16) and (5.17).

$$R_{mabn} u^m u^n - \frac{1}{3} R_{mn} u^m u^n P_{ab} = 0, \quad (5.18)$$

i. e.

$$\begin{aligned} C_{mabn} u^m u^n - (g_{m[u} R_{b]a} + g_{a[b} R_{a]m}) u^m u^n \\ - \frac{1}{3} R g_{m[b} g_{n]a} u^m u^n - \frac{1}{3} R_{mn} u^m u^n P_{ab} = 0, \text{ by (3.1)}. \end{aligned} \quad (5.19)$$

For the magnetofluid, (3.5) and (5.19) yield

$$E_{ab} + 1/2 \mu h_a h_b + 1/6 \mu h^2 P_{ab} = 0. \quad (5.20)$$

From (5.20) it is clear that

$$E_{ab} = 0 \Leftrightarrow h^a = 0. \quad (5.21)$$

Hence, the tidal force E_{ab} is due to the magnetic field only.

Remark. — In the absence of electromagnetic field, the propagation equation (5.16) is identical with the equations due to Ellis [9] and Glass [10] for relativistic hydrodynamics.

C) Propagation equation for the rotation tensor

To obtain the propagation equation of rotation tensor, we differentiate he expression for the rotation tensor and use (5.6) in it. Hence, we get

$$\dot{w}_{ba} = \frac{2}{3} \theta \omega_{ab} + \omega_{ad} \sigma_{.b}^d - \omega_{bd} \sigma_{.a}^d + \dot{u}_{[b;a]} + \dot{u}^d u_{[a} u_{b];d} + \ddot{u}_{[b} u_{a]}. \quad (5.22)$$

Thus (5.22) illustrates that in the propagation equation for the rotation tensor, there is no explicit occurrence of dynamical entities.

The propagation equations associated with the unitary space-like congruence $Q^a = q^a / |q|$ are given in Appendix.

D) Classification of fields

The several constraints imposed by the kinematical properties of the magnetofluid on the algebraic structure of the gravitational field be realized by the following theorem:

THEOREM 4. — *The magnetofluid with shear-free and irrotational stream lines is of Petrov type I, D or O.*

Proof. — To prove this we start with (3.1) which gives

$$\mathbf{C}^a{}_{bcd}u_a = R^a{}_{bcd}u_a - T_{b[d}u_{c]} - g_{b[d}T_{c]}{}^a u_a + \frac{2}{3}T_{u_{[c}g_{d]b}}. \quad (5.23)$$

Following the Ricci identities (5.2), we write

$$2u_{b;[cd]} = R^a{}_{bcd}u_a,$$

i. e.

$$\begin{aligned} R^a{}_{bcd}u_a &= 2\omega_{b[c;d]} + 2\sigma_{b[c;d]} + \frac{2}{3}P_{b[c\theta;d]} \\ &\quad + \frac{2}{3}\theta(\omega_{b[c}u_{d]} + \sigma_{b[c}u_{d]} + \frac{1}{3}\theta g_{b[c}u_{d]}) \\ &\quad + u_b\omega_{dc} + u_b\dot{u}_{[d}u_{c]} + 2\dot{u}_{b;[d}u_{c]} + 2\dot{u}_b\omega_{cd} \\ &\quad - 2\dot{u}_b\dot{u}_{[c}u_{d]}, \text{ by (2.2)}. \end{aligned} \quad (5.24)$$

For the magnetofluid we have

$$\begin{aligned} \frac{2}{3}T_{u_{[c}g_{d]b}} - T_{b[d}u_{c]} - u_a g_{b[d}T_{c]}{}^a &= \left(p + \frac{1}{3}\rho\right)g_{b[c}u_{d]} \\ &\quad + \mu h_b h_{[d}u_{c]} + \nu\sigma_{b[c}u_{d]} + q_{[c}u_{d]}u_b + g_{[d}g_{c]}b. \end{aligned} \quad (5.25)$$

Finally, with $\sigma_{ab} = \omega_{ab} = 0$, we derive from (5.23)-(5.25)

$$\begin{aligned} u_a \mathbf{C}^a{}_{bcd} &= \frac{2}{3}P_{b[c\theta;d]} + \frac{2}{3}\theta\left(\dot{u}_{[d}u_{c]}u_b + \frac{1}{3}\theta g_{b[c}u_{d]}\right) \\ &\quad + 2\dot{u}_{b;[d}u_{c]} + 2u_b\dot{u}_{[c}u_{d]} + \left(p + \frac{1}{3}\rho\right)g_{b[c}u_{d]} \\ &\quad + \mu h_b h_{[d}u_{c]} + \nu\sigma_{b[c}u_{d]} + u_{bq[c}u_{d]} + q_{[d}g_{c]}b \end{aligned} \quad (5.26)$$

Consequently we get

$$u_a \mathbf{C}^a{}_{b[cd]u_e} = 0. \quad (5.27)$$

This is possible only if the Weyl tensor is of Petrov type I, D or O.
Hence, the proof is complete.

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APPENDIX

The propagation equations for parameters $\dot{\theta}$, $\dot{\omega}_{ab}$ and $\dot{\sigma}_{ab}$ associated with the unitary space-like congruence Q are as follows:

$$\dot{\theta}_{,c} Q^c + \dot{\theta}^2 + (\dot{\sigma}^2 - \dot{\omega}^2) - 1/2(Q^c_{;d} Q^d)_{;c} + 1/2 \dot{Q}_{a;c} Q^c u^a + (\dot{Q}_a u^a)^2 - 1/2(\dot{Q}_a \dot{Q}^a) - 1/4(\rho - p - \mu h^2) - \nu/2 \sigma_{ab} Q^a Q^b = 0, \quad (1)$$

$$\begin{aligned} \dot{\omega}_{ab;c} Q^c - \dot{P}_b^d Q_{[c;d]} \{ Q_{a;e} Q^e Q^c + Q^c_{;e} Q^a Q^e - u_{a;e} Q^e u^c - u^e_{;c} Q^c u_a \} \\ - \dot{P}_a^c Q_{[c;d]} \{ Q_{b;e} Q^d Q^e + Q^d_{;e} Q^e Q_b - u_{b;e} u^d Q^e - u^d_{;e} Q^e u_b \} \\ - 1/2 \dot{P}_a^c \dot{P}_b^d (Q_{c;de} - Q_{d;ec}) Q^e = 0. \end{aligned} \quad (2)$$

$$\begin{aligned} \dot{\sigma}_{ab;c} Q^c - \dot{P}_b^d Q_{(c;d)} \{ Q_{a;e} Q^e Q^c + Q^c_{;e} Q^e Q_a - u_{a;e} Q^e u^c - u^c_{;e} u_a Q^e \} \\ - \dot{P}_a^c Q_{(c;d)} \{ Q_{b;e} Q^e Q^d + Q^d_{;e} Q^e Q_b - u_{b;e} Q^e u^d - u^d_{;e} u_b Q^e \} \\ - 1/2 \dot{P}_a^c \dot{P}_b^d \{ Q_{c;ed} Q^e + Q_{d;ec} Q^e + C_{ecd} Q^e Q^f + C_{edc} Q^e Q^f \} \\ + 1/2 \dot{\theta} (u_{b;c} u_a Q^c + u_b u_{a;c} Q^c) + (Q_a Q_{b;c} Q^c + Q_b Q_{a;c} Q^c) \\ + P_{ab} \dot{\theta}_{,c} Q^c = -\frac{1}{3} \left(2\rho - 3p + \frac{3}{2} \mu h^2 \right) \dot{P}_{ab} \\ + 1/2 \mu (h_c Q^c)^2 P_{ab} - \nu/2 \sigma_{cd} Q^c Q^d \dot{P}_{ab} + \nu/2 (\sigma_{ab} + \sigma_{ad} Q^d Q_b \\ + \sigma_{cb} Q^c Q_a + \sigma_{cd} Q^c Q^d Q_a Q_b) - \frac{\mu}{2} h_a h_b - \frac{\mu}{2} (h_c Q^c) (h_a Q_b + h_b Q_a + Q_a Q_b). \end{aligned} \quad (3)$$

REFERENCES

- [1] G. G. ASGEKAR and T. H. DATE, *J. Math. Phys.*, t. **18**, 1977, p. 738.
- [2] G. G. ASGEKAR and T. H. DATE, *J. Ther. Phys.*, t. **16** (6), 1977, p. 465.
- [3] G. G. ASGEKAR and T. H. DATE, *Prog. of Ther. Phys.*, t. **59** (6), 1978.
- [4] M. H. DE BRAY, *Physique Mathématique. C. R. Acad. Sc.*, Paris, t. **274**, 1972, p. 809.
- [5] T. G. COWLING, *Reports on Progress in Phys.*, XXV, 1962, p. 244.
- [6] T. H. DATE, *J. Shivaji Univ.*, **5** (10), 1972, p. 123.
- [7] T. H. DATE, *Indian J. Pure Appl. Math.*, t. **4** (718), 1973, p. 612.
- [8] J. EHLERS and W. KUNDT, *Gravitation* (Ed. L. Witten), Wiley, New York, 1962.
- [9] G. F. R. ELLIS, *General Relativity and Cosmology*, Ed. B. K. Sachs, Academic Press, New York, 1971, p. 104.
- [10] E. N. GLASS, *J. Math. Phys.*, t. **16** (12), 1975, p. 2361.
- [11] P. J. GREENBERG, *Astrophysics J.*, t. **164**, 1971, p. 569.
- [12] W. KUNDT and M. TRUMPER, *Abh. Akad. Wiss and Lit. Mainz. Mat. Nat. Kl.*, t. **12**, 1962.
- [13] A. LICHTNEROWICZ, *Relativistic Hydrodynamics and Magnetohydrodynamics*, W. A. Benjamin Inc., New York, 1967.

- [14] D. P. MASON, *General Relativity and Gravitation*, t. 5 (4), 1973, p. 429.
- [15] G. A. MAUGIN, *Ann. Inst. Henri Poincaré*, XVI (3), 1972, p. 133.
- [16] E. N. PARKER, *Astrophysics J.*, 157, 1969, p. 1129.
- [17] A. K. RAYCHAUDHURI, *Phys. Rev.*, 98, 1955, p. 1123.
- [18] P. SZEKERES, *Ph. D. Thesis*, Univ. of London, 1964.
- [19] P. R. WILSON, *Solar Phys.*, t. 3, 1968, p. 243.
- [20] P. YODZIS, *Phys. Rev.*, t. 3 (12), 1971, p. 2941.

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