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Coherent and quasi-free states

by

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ABSTRACT. — The convex hull of all quasi-free states, including the gauge transformed ones is not dense in the set of regular states.

RÉSUMÉ. — Nous prouvons que l'enveloppe convexe des états quasi-libres n'est pas dense dans l'ensemble des états réguliers.

I. INTRODUCTION

It is an important property of the coherent states that they generate linearly all normal states. This has been proven for finitely many [1] as well as infinitely many degrees of freedom [2]. It means that every normal state can be represented by a linear combination of coherent states. This is the so called P-representation of density operators, which has been usefull in quantum optics [3].

Yet, it is not true that coherent states generate all normal states in a convex way [2].

In C*-algebraic language, this can be specified as follows: let $\overline{\Delta(\mathbf{H}, \sigma)}$ denote the CCR-C*-algebra build on the symplectic space (\mathbf{H}, σ) [4]. $\overline{\Delta(\mathbf{H}, \sigma)}$ is the C*-closure of the *-algebra Δ generated by $\{ \mathbf{W}(f) \mid f \in \mathbf{H} \}$ where the generators satisfy

$$\mathbf{W}(f)\mathbf{W}(g) = e^{-i\sigma(f,g)}\mathbf{W}(f + g)$$

and $\mathbf{W}(f)^* = \mathbf{W}(-f)$.

We suppose σ non-degenerate.

The gauge-transformations $\{\tau_g \mid g \in \mathbf{H}\}$ are defined by

$$\begin{aligned}\tau_g(\mathbf{W}(f)) &= \mathbf{W}(g)\mathbf{W}(f)\mathbf{W}(-g) \\ &= e^{-2i\sigma(g,f)}\mathbf{W}(f)\end{aligned}$$

A state ω on $\overline{\Delta(\mathbf{H}, \sigma)}$ is called regular if the map

$$\lambda \in \mathbf{R} \rightarrow \omega(\mathbf{W}(f + \lambda g)) \quad \text{is continuous for all } f, g \in \mathbf{H}.$$

The set of regular states is denoted by \mathbf{R} , and the set of states on $\overline{\Delta(\mathbf{H}, \sigma)}$ by \mathbf{S} .

If \mathbf{J} is an operator on \mathbf{H} with the properties

$$\mathbf{J}^2 = 1, \quad \mathbf{J}^+ = -\mathbf{J}$$

($+$ is the adjoint with respect to σ) and $s_j(f, g) = -\sigma(\mathbf{J}f, f) > 0$ for all $f \neq 0$ then $\omega_j(\mathbf{W}(f)) = e^{-1/2s_j(f,f)}$ defines a pure, quasi-free state or a Fockstate on $\overline{\Delta(\mathbf{H}, \sigma)}$.

The properties on coherent states now read: the positive, normalized elements of the linear hull of $\{\omega_j \cdot \tau_f \mid f \in \mathbf{H}\}$ are norm dense in \mathbf{S} [2, th. II.4] and the convex hull of $\{\omega_j \cdot \tau_f \mid f \in \mathbf{H}\}$ is not norm dense in \mathbf{S} [2, th. II.5]. We generalize the latter result in the following sense:

THEOREM 1. — The convex hull \mathbf{Q} of quasi-free states is not norm dense in \mathbf{S} .

§ 2. *The proof.* — We first recall some facts:

* the quasi-free states are the states ω such that

$$\omega = \omega_s \circ \tau_g$$

and

$$\omega_s(\mathbf{W}(f)) = e^{-1/2s(f,f)}$$

with

$$|\sigma(f, g)|^2 \leq s(f, f)s(g, g)(1) \quad [5]$$

where s is a positive bilinear map on \mathbf{H} .

* a map $\tau: \overline{\Delta(\mathbf{H}, \sigma)} \rightarrow \overline{\Delta(\mathbf{H}, \sigma)}$ is positive if $\tau(x^*x) \geq 0$ for all $x \in \overline{\Delta(\mathbf{H}, \sigma)}$.
* \mathbf{R} is weak*-dense in \mathbf{S} .

LEMMA 2. — If \mathbf{P} is a weak*-dense subset of \mathbf{S} and \mathbf{T} is a map of Δ into Δ which satisfies

$$\mathbf{T}(1) = 1$$

$(\omega \cdot \mathbf{T})(x^*x) \geq 0$ for all $\omega \in \mathbf{P}$ and $x \in \Delta$ then \mathbf{T} has a positive extension to $\overline{\Delta(\mathbf{H}, \sigma)}$.

Proof. — $\omega \circ \mathbf{T}$ has an extension $\omega_{\mathbf{T}}$ to $\overline{\Delta(\mathbf{H}, \sigma)}$ which is a state [4] (remark

that this is true for the CCR-algebra, but wrong in general) For $x = x^* \in \Delta$

$$0 \leq \|x\| - x \in \Delta$$

so

$$0 \leq \omega_T(\|x\| - x) = (\omega \circ T)(\|x\| - x) \\ = \omega(\|x\| - T(x)) \quad \text{for all } \omega \in P$$

since P is dense in S

$$\|x\| \geq T(x)$$

and

$$\|T(x)\| \leq \|x\| \quad \text{for } x = x^* \in \Delta$$

For general $x \in \Delta$:

$$\|T(x)\| \leq \left\| \frac{T(x + x^*)}{2} \right\| + \left\| T\left(\frac{x - x^*}{2i}\right) \right\| \\ \leq \left\| \frac{x + x^*}{2} \right\| + \left\| \frac{x - x^*}{2i} \right\| \leq 2\|x\|$$

So T is bounded and has an extension to $\overline{\Delta(H, \sigma)}$ which will be positive. ■

LEMMA 3. — Let T be defined on Δ as

$$T(W(f)) = W(2f)$$

If Q is normdense in R , then T has a positive extension to $\overline{\Delta(H, \sigma)}$.

Proof. — If Q is norm dense in R , then it is weak*-dense in S . Moreover

$$((\omega_s \circ \tau_g) \circ T)(W(f)) = (\omega_{4s} \circ \tau_{2g})(W(f))$$

and if s satisfies (1) also $4s$ satisfies (1) so, T satisfies the conditions of lemma 2, namely $(\omega \circ T)(x^*x) \geq 0$ for all ω in a w^* -dense subset of S and for all $x \in \Delta$. ■

Proof of theorem 1. — If T defined in lemma 3 is positive, then it satisfies the inequality:

$$(2) \quad T(x^*x + xx^*) \geq T(x^*)T(x) + T(x)T(x^*) \quad [6]$$

for all $x \in \overline{\Delta(H, \sigma)}$.

Take $x = aW(f) + bW(g)$ then (2) becomes

$$0 \leq 2 \operatorname{Re} \bar{b}a [\cos \sigma(g, f) - \cos 4\sigma(g, f)]\omega(W(2(f - g)))$$

for any $\omega \in S$ and all $a, b \in \mathbb{C}$, $f, g \in H$ so

$$|\sigma(g, f)| = 4|\sigma(g, f)| \quad \text{for all } f, g \in H$$

which is false since σ is non-zero.

So T is not positive and consequently Q is not normdense in R . ■

DISCUSSION

In many applications one takes as first approximation of a state its associated quasi-free state, determined by its two-point function (see e. g. Hartree-Fock approximation, random phase approximation). The theorem points out that this procedure might be highly questionable.

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