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Incompressible heat-conducting relativistic fluid (*)

by

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ABSTRACT. — We consider a heat-conducting relativistic fluid in the case extreme in which the incompressibility conditions hold. We deduce some important consequences after we have written the field equations in a simple particular form by introducing two four-vectors which determine the complete field.

1. INTRODUCTION

Let us consider a perfect heat-conducting relativistic fluid. By following [1], the field equations can be written:

$$
(1) \qquad \begin{cases} \nabla_\alpha T^{\alpha\beta} = 0, \qquad T^{\alpha\beta} = rfu^\alpha u^\beta - pg^{\alpha\beta} \\ \nabla_\alpha(ru^\alpha - q^\alpha) = 0 \\ rdf = rTdS + dp \\ rf = \rho + p \\ p = RrT. \end{cases}
$$

where $T^{\alpha\beta}$ is the energy tensor, r the proper material density (number of particles), p the pressure, f the index of the fluid, ρ the proper energy density: $\rho = T^{\alpha\beta}u_\alpha u_\beta$, T and S are the proper temperature of the fluid and its proper specific entropy, $g_{\alpha\beta}$ is the metric tensor (with signature +, −,

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$-, -)$, u_α the unitary four-velocity: $u^\alpha u_\alpha = 1$, q_α the heat flux vector. Together with the equations (1) we consider the following heat equation [2]

$$\frac{q_\beta}{\chi} = u^\alpha(\partial_\beta q_\alpha - \partial_\alpha q_\beta) + \frac{2\mathrm{R}r}{f} u^\alpha \{ \partial_\beta(\mathrm{T}u_\alpha) - \partial_\alpha(\mathrm{T}u_\beta) \}, \tag{2}$$

which automatically implies $q^\alpha u_\alpha = 0$.

At first we observe that, differently from the Boillat's results [3] (on account of the different scheme used), the system (1)-(2) remains hyperbolic also when the characteristic velocities λ_1 and λ_2 are equal.

In fact if $\lambda_1 = \lambda_2$, we have [2]

$$\frac{2p}{rf} = \frac{\mathrm{R}}{\mathrm{TS}'_\mathrm{T}} (= \lambda_1^2 = \lambda_2^2), \tag{3}$$

and in general we find

$$\begin{cases} \delta r \neq 0, \qquad \delta\mathrm{T} \neq 0 \\ 2p\varphi_\alpha \delta u^\alpha = -\mathcal{U}\mathrm{R}(\mathrm{T}\delta r + r\delta\mathrm{T}) \\ 2\mathrm{T}\varphi_\alpha \delta q^\alpha = \mathcal{U}(\mathrm{T}\delta r - r\delta\mathrm{T}). \end{cases} \tag{4}$$

All the discontinuities are determined in terms of δr and $\delta\mathrm{T}$ which are completely free. It follows that the wave

$$(rf - 2p)\mathcal{U}^2 + 2p\mathrm{G} = 0 \tag{5}$$

is double.

Here $\mathcal{U} = u^\alpha \varphi_\alpha$, $\mathrm{G} = g^{\alpha\beta}\varphi_\alpha\varphi_\beta$, $\varphi_\alpha \equiv \dfrac{\partial\varphi}{\partial x^\alpha}$, $\varphi(x^\alpha)$ being the solutions of the wave equation.

2. INCOMPRESSIBLE FLUID [4]

For an incompressible fluid $\lambda_1^2 = \lambda_2^2 = 1$ and equation (3) gives

$$\begin{cases} rf = 2p \\ \mathrm{S}'_\mathrm{T} = \dfrac{\mathrm{R}}{\mathrm{T}}. \end{cases} \tag{6}$$

which, taking account of equations (1), permits to write the heat equation as:

$$\frac{q_\beta}{\chi} = u^\alpha(\partial_\beta q_\alpha - \partial_\alpha q_\beta) + \frac{r}{\mathrm{T}} u^\alpha \{ \partial_\beta(\mathrm{T}u_\alpha) - \partial_\alpha(\mathrm{T}u_\beta) \}, \tag{7}$$

while the stream lines system becomes

$$ru^\alpha \{ \partial_\alpha(\mathrm{T}u_\beta) - \partial_\beta(\mathrm{T}u_\alpha) \} + \mathrm{T}u^\alpha \{ \partial_\alpha(ru_\beta) - \partial_\beta(ru_\alpha) \} = 0. \tag{8}$$

The equation (7), by virtue of equation (8), can be rewritten as

$$\frac{q_\beta}{\chi} = u^\alpha \{ \partial_\alpha(ru_\beta - q_\beta) - \partial_\beta(ru_\alpha - q_\alpha) \}. \tag{9}$$

Finally the continuity equation

(10) $$\nabla_\alpha(rfu^\alpha) - u^\alpha \partial_\alpha p = 0$$

may be written

(11) $$r\nabla_\alpha(\mathrm{T}u^\alpha) + \mathrm{T}\nabla_\alpha(ru^\alpha) = 0\,.$$

3. THE TWO FIELD FOUR-VECTORS v^α AND w^α

At this stage let us introduce the two four vectors

(12) $$v^\alpha = \mathrm{T}u^\alpha, \quad w^\alpha = ru^\alpha - q^\alpha\,.$$

It is immediately seen that all the field variables are known in terms of v^α and w^α. In fact we have:

(13) $$\begin{cases} \mathrm{T} = (v^\alpha v_\alpha)^{\frac{1}{2}} \\ r = \dfrac{v^\alpha w_\alpha}{(v^\beta v_\beta)^{\frac{1}{2}}} = \dfrac{v^\alpha w_\alpha}{\mathrm{T}} \\ u_\alpha = \dfrac{v_\alpha}{(v^\beta v_\beta)^{\frac{1}{2}}} = \dfrac{v_\alpha}{\mathrm{T}} \\ q_\alpha = \dfrac{v^\alpha w_\alpha}{v^\beta v_\beta} v_\alpha - w_\alpha = ru_\alpha - w_\alpha \end{cases}$$

After this, if we put

(14) $$\bar{v}_\alpha = \frac{v^\gamma w_\gamma}{v^\beta v_\beta} v_\alpha = ru_\alpha\,,$$

the field equations for an incompressible fluid may be simply written as

(15) $$\begin{cases} r\nabla_\alpha v^\alpha + \mathrm{T}\nabla_\alpha \bar{v}^\alpha = 0 \\ \bar{v}^\alpha(\partial_\alpha v_\beta - \partial_\beta v_\alpha) + v^\alpha(\partial_\alpha \bar{v}_\beta - \partial_\beta \bar{v}_\alpha) = 0 \\ \nabla_\alpha w^\alpha = 0 \\ v^\alpha(\partial_\alpha w_\beta - \partial_\beta w_\alpha) = \dfrac{\mathrm{T}}{\chi} q_\beta \end{cases}$$

4. DISCONTINUITIES

The discontinuities are immediately evaluable. By setting

(16) $$\mathrm{V} = v^\alpha \varphi_\alpha\,, \ \mathrm{W} = w^\alpha \varphi_\alpha\,, \ \bar{\mathrm{V}} = \bar{v}^\alpha \varphi_\alpha\,,$$

the equations (15) gives:

(17) $$\begin{cases} r\varphi_\alpha \delta v^\alpha + \mathrm{T}\varphi_\alpha \delta \bar{v}^\alpha = 0 \\ \bar{\mathrm{V}}\delta v_\beta - \varphi_\beta \bar{v}_\alpha \delta v^\alpha + \mathrm{V}\delta \bar{v}_\beta - \varphi_\beta v_\alpha \delta \bar{v}^\alpha = 0 \\ \varphi_\alpha \delta w^\alpha = 0 \\ \mathrm{V}\delta w_\beta - \varphi_\beta v_\alpha \delta w^\alpha = 0 \end{cases}$$

By contracting the equations (17_2) and (17_4) with φ_β and taking into account equation (17_3), we have

$$G(\bar{v}_\alpha \delta v^\alpha + v_\alpha \delta \bar{v}^\alpha) = 0, \tag{18}$$

$$G v_\alpha \delta w^\alpha = 0. \tag{19}$$

If $G \neq 0$, from equation (19) we find $v_\alpha \delta w^\alpha = 0$, so that equation ($17_4$) gives

$$V \delta w_\beta = 0. \tag{20}$$

We have two possibilities

(*a*) $$V = 0, \qquad \delta w_\beta \neq 0$$

and

(*b*) $$V \neq 0, \qquad \delta w_\beta = 0.$$

In the case (*b*), from equations (17) and (18) we conclude that $\delta w_\beta = \delta v_\beta = 0$ and we have not discontinuities.

This excluded, the (*a*) holds and from equations (17)-(19) we deduce the following four independent equations for the eight components of δv_α and δw_α:

$$\begin{cases} \bar{v}_\alpha \delta v^\alpha + v_\alpha \delta \bar{v}^\alpha = 0 \\ r\varphi_\alpha \delta v^\alpha + T\varphi_\alpha \delta \bar{v}^\alpha = 0 \\ v_\alpha \delta w^\alpha = 0 \\ \varphi_\alpha \delta w^\alpha = 0. \end{cases} \tag{21}$$

There are four degree of freedom, so that the wave $V = 0$ (which corresponds to the material waves $\mathcal{U} = 0$) has multiplicity four. Finally when $G = 0$, we have

$$\begin{cases} \delta w_\alpha = k\varphi_\alpha \\ \delta v_\alpha + T\delta \bar{v}_\alpha = h\varphi_\alpha \end{cases} \tag{22}$$

where k and h are arbitrary parameters and, of course, $\delta \bar{v}_\alpha$ can be given in terms of δv_α and δw_α.

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