W. H. Klink
T. Ton-That

Holomorphic induction and the tensor product decomposition of irreducible representations of compact groups. I. $SU(n)$ groups

Mme Y. CHOQUET-BRUHAT
Section A.

Physique théorique.

Holomorphic Induction and the Tensor Product Decomposition of Irreducible Representations of Compact Groups. I. SU(n) Groups

by

W. H. KLINK
Department of Physics and Astronomy

and

T. TON-THAT
Department of Mathematics
The University of Iowa.
Iowa City, Iowa 52242, U. S. A.

ABSTRACT. — A formalism for decomposing tensor products of irreducible representations of SU(n) is presented, using holomorphic induction techniques. Irreducible representation spaces of SU(n) are realized as polynomial functions over GL(n, C) group variables and it is shown how to generate invariant spaces labelled by double cosets, with one double coset subspace isomorphic to the original tensor product space. Maps are defined which carry irreducible representations to the tensor product space via the isomorphic double coset space. The entire procedure is compared with the Mackey induced representation theory for semi-direct product groups.

I. INTRODUCTION

One of the important problems in group theory is to decompose the tensor product of irreducible representations of a given group. For compact groups there are a number of different formulas from which one can compute the Clebsch-Gordan series [*1*] (that is, which representations occur in a given tensor product decomposition and with what multiplicity). But the Clebsch-Gordan coefficients are in general more difficult to compute because they depend on a basis, and for a given compact group, there are usually many different sets of bases that are of interest

in various physical problems. For example, in SU(3), the nuclear physicist is often interested in a basis in which the subgroup SO(3) is diagonal, whereas a particle physicist is more interested in a U(2) subgroup decomposition [2]. Each of these bases have different Clebsch-Gordan coefficients associated with a tensor product decomposition. Just because the Clebsch-Gordan problem seems so hopelessly basis dependent means that many mathematicians have ignored it as an interesting mathematical problem.

There are some general formulas that can be written for Clebsch-Gordan coefficients that involve group integration. Let $D^{(m)}_{[k][k']}(g)$ be the (unitary) matrix element of a given compact group G, with g an element of G and (m) an irreducible representation label of G. $[k]$ (and $[k']$) denotes a set of labels needed to specify a basis for the representation (m). Then it is possible to write the Clebsch-Gordan coefficients C as

$$C^{*(m)(m')(m'')}_{[k][k'][k'']} C^{(m)(m')(m'')}_{[\bar{k}][\bar{k}'][\bar{k}'']} = \int_G dg\, D^{(m)*}_{[k][\bar{k}]}(g)\, D^{(m')}_{[k'][\bar{k}']}(g)\, D^{(m'')}_{[k''][\bar{k}'']}(g)$$

where $\int dg$ means Haar measure integration. One of the major problems with this formula is that it is usually very difficult to carry out the group integration explicitly. The matrix elements $D^{(m)}(g)$ are complicated in themselves, and when coupled with the group integration make the problem from a practical sense intractable. It is possible to simplify the integration somewhat by splitting the group integration and matrix elements into parts depending only on a subgroup H and on G/H. If the Clebsch-Gordan coefficients for H are known, then it may be possible to integrate the remaining part over G/H; this procedure has been used to compute Clebsch-Gordan coefficients for SU(3), when the Clebsch-Gordan coefficients for SU(2) are known [3]. But the above formula ignores the problem of multiplicity, which means that in the tensor product decomposition of two irreducible representations, a given representation may occur more than once. So often all one can do is compute the Clebsch-Gordan coefficients for very special representations, without any attempt being made to find a general procedure.

Many of these objections are eliminated when the groups in question have irreducible representations that can be written as induced representations [4]; in this case the Mackey subgroup theorem can be used to rewrite the tensor product representation as a reducible representation labelled by double cosets. Then in each double coset subspace one can attempt a further reduction into irreducible constituents. This procedure has the advantage of being basis independent, so that only at the end of the tensor product reduction does one have to choose a basis to actually compute the Clebsch-Gordan coefficients. Also, the double cosets can be used to break the multiplicity in a tensor product decomposition.

There is unfortunately no induced representation theory of the Mackey type for the compact semisimple groups. But there is a so-called holomorphic induction procedure [5], in which the irreducible representations of the compact groups are realized as the restriction of irreducible holomorphic representations of their complexification. That is, one forms representations of the complexification of a given compact group using holomorphic functions which form finite dimensional vector spaces. These representations remain irreducible under the restriction of the complexified group to the group itself. The goal of this series of papers is to take parts of the Mackey theory and apply it to holomorphically realized representations of the compact groups. The relevant holomorphic functions which are actually polynomials in group variables for $GL(n, C)$, will be functions that transform in a prescribed fashion. One of the great advantages in using such holomorphic realizations is that the inner product becomes essentially differentiation of polynomials of the group variables. In fact, one of the long-term goals of this series of papers is to express the various possible basis functions of a given compact group as a polynomial function of the group variables, after which computing Clebsch-Gordan coefficients becomes an exercise in carefully differentiating polynomials; that is, it becomes a problem for a computer.

As already noted, another obstacle to actually computing Clebsch-Gordan coefficients concerns the multiplicity problem. We will show in Section III how maps, defined with the help of the highest weight Clebsch-Gordan coefficients, when combined with double coset maps, can be used to resolve at least some of the multiplicity problem for $SU(n)$, and in Section IV we show how the multiplicity problem associated with the eight-dimensional representation of $SU(3)$ is resolved by double coset labels.

The goal of this paper is to set up a general formalism for the $SU(n)$ groups. All results will be basis independent, except in Section IV where we show how to compute some Clebsch-Gordan coefficients for $SU(3)$, using the results of the general theory developed in earlier sections.

The main strategy of Sections II and III is as follows : First, use is made of the fact that the finite dimensional irreducible representations of $SL(n, C)$, when restricted to elements of $SU(n)$, remain irreducible. Further the « differentiation » inner product becomes unitary for $SU(n)$ elements. Then extending the Mackey procedure, double cosets are introduced in Section III, and it is shown that, unlike the more familiar Mackey theory, for the $SU(n)$ groups, there is one double coset element which generates a subspace isomorphic to the original tensor product space. The double cosets can be chosen to be elements of the permutation group S_n, and it is shown that all of the other elements of S_n generate invariant subspaces. Then a number of maps are defined which carry irreducible representations to these invariant subspaces. Once these maps are known (and shown to intertwine), the Clebsch-Gordan coefficients become the overlap between

tensor products basis functions suitably mapped to a subspace, and an irreducible basis function mapped to the same subspace. Since all these subspaces are realized as polynomial functions over group elements, transforming suitably with respect to subgroups, the inner product is always the same for all these subspaces, namely the « differentiation » inner product. This is to be contrasted with realizing the representations as polynomials over $SL(n, C)/H$ [6] where H is some subgroup of $SL(n, C)$. In this case the different double coset subspaces involve different subgroups H of $SL(n, C)$, so that the inner product is different for each double coset subspace (and seems quite difficult to find in general).

Thus, while the procedure for obtaining representations and Clebsch-Gordan coefficients for the compact groups is new, some of the results obtained using the holomorphic induction technique must of course agree with results obtained using other techniques. For example in Section III we compute the highest weight Clebsch-Gordan coefficients for $SU(n)$ in terms of norms of basis functions. Such coefficients have been computed —at least implicitly—using other techniques [7].

II. GENERAL THEORY

In this section, notation will be established and a number of theorems quoted that are relevant to decomposing tensor products. Let $G = GL(n, C)$; B will denote the subgroup of lower triangular matrices of G, Z will denote the subgroup of upper triangular (with ones on the diagonal) matrices of G, and D ($= B_0$) the diagonal subgroup of G. Let W_0 be the subgroup of B of matrices with one on the diagonal. For $1 \leq r \leq n$ let $\Delta_{j_1 \dots j_r}^{i_1 \dots i_r}(g)$ denote the minor of the matrix g formed from the rows $i_1 \dots i_r$ and columns $j_1 \dots j_r$; the principal minors $\Delta_{1 \dots r}^{1 \dots r}(g)$ are denoted simply by $\Delta_r(g)$. Then the Gauss decomposition on G leads to the following: For every $g \in G$ such that the principal minors of g are nonzero

$$g = bz \qquad b \in B, d \in D(= B_0)$$
$$= dwz \qquad w \in W_0, z \in Z$$

$$b_{rs}(g) = \frac{\Delta_{1 \dots s-1,s}^{1 \dots s-1,r}(g)}{\Delta_{s-1}(g)}, \qquad b_{r1}(g) = g_{r1}$$

$$d_s(g) = b_{ss}(g) \tag{II-1}$$
$$= \frac{\Delta_s(g)}{\Delta_{s-1}(g)}$$

$$d_1(g) = b_{11}(g) = g_{11}, \qquad r \geq s, \qquad s = 2 \dots n$$

$$w_{rs}(g) = \frac{b_{rs}(g)}{d_r(g)}, \qquad r \geq s, \qquad s = 1 \dots n$$

$$z_{rs}(g) = \frac{\Delta_{1 \dots r-1,r}^{1 \dots r-1,s}(g)}{\Delta_r(g)}, \qquad s > r, \qquad r = 1 \dots n.$$

Let $U = U(n)$ denote the unitary group and T be the torus, that is the diagonal subgroup of U. The irreducible representations of T are all one dimensional (i. e., characters) and form a multiplicative group $\hat{T}$, the dual group of T. A character π of T is given by

$$\pi(e^{i\theta_1}, e^{i\theta_2}, \ldots e^{i\theta_n}) = e^{i(m_1\theta_1 + \ldots + m_n\theta_n)}$$

where the n-tuple $(m_1, \ldots m_n)$ is denoted by (m) and belongs to the vector lattice Z^n. Thus, there is a one-to-one correspondence between $\hat{T}$ and Z^n. Hence, the lexicographic order on Z^n induces an ordering on $\hat{T}$. Now let R be an irreducible representation of U and $R \mid T$ be its restriction to T; then we have the following decomposition of $R \mid T$ into irreducible sub-representations:

$$R \mid T = \sum_{\pi \in \hat{T}} [R \mid T : \pi]\pi$$

where $[R \mid T : \pi]$ denotes the multiplicity of π in $R \mid T$. In the above expression those characters π with $[R \mid T : \pi] > 0$ are called the *weights* of R. In this context, it is then clear that one can define a *highest weight* for R with respect to the lexicographic order on $\hat{T}$. In fact, it is well known [8] that every irreducible representation R of $U(n)$ is uniquely determined by its highest weight which always occurs in $R \mid T$ with multiplicity 1.

Now let $\pi \in \hat{T}$; then $\pi = \pi^{(m)}$ extends uniquely to a holomorphic character on B, which we shall denote by the same symbol π. Thus

$$\pi^{(m)}(b) = b_{11}^{m_1} \ldots b_{nn}^{m_n} \tag{II-2}$$

for all $b \in B$, and $(m) = (m_1, \ldots m_n) \in Z^n$. In this context, a character $\pi^{(m)}$ is called *dominant* if

$$m_1 \geq m_2 \geq \ldots \geq m_n.$$

Let $\text{Hol}(G, \pi)$ denote the space of all holomorphic functions f on G which also satisfy $f(bg) = \pi(b)f(g)$ for all $(b, g) \in B \times G$. Let R_π denote the holomorphically induced representation (i. e., $(R_\pi(g_0)f)(g) = f(gg_0)$ for all $f \in \text{Hol}(G, \pi)$ and $g, g_0 \in G$). In this context we have the following

THEOREM (Borel-Weil) [9]. — *The space* $\text{Hol}(G, \pi)$ *is nonzero if and only if π is dominant; and in this case R_π is irreducible.*

By combining the Borel-Weil theorem with Weyl's « unitarian trick » [10] one can deduce that, in the restriction of R_π to U, $R_\pi \mid U$ remains irreducible and its highest weight is π.

Now let $SL = SL(n, C)$ and $SU = SU(n)$. Then it is well known that the restriction of every irreducible unitary representation of U to SU is an irreducible unitary representation of SU. Conversely, every irreducible unitary representation of SU can be obtained by restricting an irreducible

unitary representation of U to SU. Moreover, if R and R' are two irreducible representations of U with highest weight (m) and (m') then R | SU and R' | SU are unitarily equivalent if and only $(m) = (m') + (\bar{k})$ for some $(\bar{k}) = (k, k, \ldots, k) \in Z^n$. These facts lead to the following conclusions: Firstly, to study the irreducible unitary representations of SU, it suffices to consider the irreducible unitary representations of U with highest weight $\pi^{(m)}$ where $(m) = (m_1, m_2, \ldots, m_{n-1}, 0)$ with $m_1 \geq \ldots \geq m_{n-1} \geq 0$. Secondly, there exists a one-to-one correspondence between the set of all equivalence classes of holomorphic representations of SL and the set of all equivalence classes of irreducible unitary representations of SU.

Now consider $\pi^{(m)}$ with $m = (m_1, \ldots, m_{n-1}, 0)$ and $m_1 \geq \ldots \geq m_{n-1} \geq 0$. Since G is a dense and open submanifold of $C^{n \times n}$, every holomorphic function on G can be extended by analytic continuation on $C^{n \times n}$. Moreover, the condition $f(bg) = \pi^{(m)}(b)f(g) = b_{11}^{m_1} \ldots b_{n-1,n-1}^{m_{n-1}} f(g)$ implies immediately that such an f must be a polynomial function on $C^{n \times n}$. This leads to another characterization for $\text{Hol}(G, \pi^{(m)})$ which we shall make use of whenever convenient:

$$\text{Hol}(G, \pi^{(m)}) \equiv V^{(m)} = \{ f : C^{n \times n} \rightarrow C : f \text{ polynomial}, $$
$$f(by) = \pi^{(m)}(b)f(y) \} \quad \text{(II-3)}$$

for all $(b, y) \in B \times C^{n \times n}$. Then the representation $R^{(m)}$ of SL on $V^{(m)}$ defined by $(R^{(m)}(\gamma)f)(y) = f(y\gamma), f \in V^{(m)}, \gamma \in SL$ is irreducible with highest weight $(m_n, \ldots, m_{n-1}, 0)$.

We shall equip the ring of all complex valued polynomial functions with an inner product defined in the following fashion: If

$$q(y) = \Sigma c(k_{11}, \ldots, k_{nn}) y_{11}^{k_{11}} \ldots y_{nn}^{k_{nn}}, \qquad y \in C^{n \times n}$$

we define

$$\tilde{q}(y) = \Sigma \overline{c(k_{11}, \ldots, k_{nn})} y_{11}^{k_{11}} \ldots y_{nn}^{k_{nn}}.$$

Then if p is any polynomial let $p(D)$ denote the differential operator obtained from $p(y)$ by replacing the entries y_{ij} by the partial derivatives $\partial/\partial y_{ij}$. Now define

$$(p, q) = (p(D) \tilde{q}(0); \quad \text{(II-4)}$$

then it is well known [11] that $(.,.)$ is indeed an inner product such that

$$(R^{(m)}(\gamma)f_1, R^{(m)}(\gamma)f_2) = (f_1, f_2) \quad \text{(II-5)}$$

for all $\gamma \in SU$ and all $f_1, f_2 \in V^{(m)}$; i. e., $R^{(m)} | SU$ is irreducible and unitary. Note that this type of inner product has also appeared in the boson calculus [12].

III. **TENSOR PRODUCTS**
OF HOLOMORPHIC IRREDUCIBLE REPRESENTATIONS
OF SL(n, C)

Let N(T) denote the normalizer of T; then the Weyl group W(G) of G (or of U) may be defined as the factor group N(T)/T and can be identified with the symmetric group S_n. By the Iwasawa decomposition of G we have G = BU, and to the pair (G, U) we associate a Tits system [5]; this leads to the so-called Bruhat decomposition for G, G = $U_{p \in W(G)} BpB$ (disjoint union).

Now let $p_0, p_1, \ldots p_{|w|-1}$ ($| w |$ being the order of the Weyl group W(G)) be an enumeration of W(G); then W(G) acts on the vector lattice $Z^n \approx \hat{T}$ via $(p_i(m)) \to p_i.(m) = (m_{p_i(1)}^{-1}, m_{p_i(2)}^{-1}, \ldots, m_{p_i(n)}^{-1})$ where $(m) = (m_1, \ldots, m_n)$ and p_i is identified with an element of S_n. Let Δ be the root system for the Lie algebra g of G relative to a Cartan subalgebra of g. Let $u = t \oplus p$ be a Cartan decomposition of U, the Lie algebra of U, and let a be a maximal Abelian subalgebra of p. Let Λ be the restricted root system of u relative to a, let Λ^+ and Λ^- be the corresponding positive restricted roots and negative restricted roots. Then there exists a unique element $p_0 \in$ W(G) such that $p_0 \Lambda^+ = \Lambda^-$ [5]. For U

$$p_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $B_i = \{ b \in B : p_i b p_i^{-1} \in B \}$ and $W_i = \{ w \in B : p_i w p_i^{-1} \in Z \}$. Then clearly $W_i \cap B_i = \{ e \}$ and $B = B_i W_i$; moreover, all elements $g \in$ G with nonvanishing principal minors can be uniquely written as $g = b_i w_i z$, where $b_i \in B_i$, $w_i \in W_i$, and $z \in Z$. In particular, $B_0 = D$ and W_0 is just the unipotent subgroup of B defined in eq. II-1.

Now consider the tensor product of two irreducible holomorphic representations $(R^{(m')}, V^{(m')})$ and $(R^{(m'')}, V^{(m'')})$. Set

$$
\begin{aligned}
V^{(m') \otimes (m'')} &= \text{Hol}\,(G, (m') \otimes (m'')) \\
&= \{ F : G \times G \to C : F \text{ is holomorphic, } F(b'g', b''g'') \\
&\qquad\qquad = \pi^{(m')}(b')\pi^{(m'')}(b'')F(g', g'') \} \\
b', b'' &\in B, g', g'' \in G .
\end{aligned}
\tag{III-1}
$$

For each $i (0 \leq i \leq | w | - 1)$ let $H^{(m') + p_i.(m'')}$ denote the space of all holomorphic functions f on G which also satisfy the condition

$$f(b_i g) = \pi^{(m') + p_i.(m'')}(b_i) f(g)$$

for all $(b_i, g) \in B_i \times G$. Now define a map Φ_i from $V^{(m') \otimes (m'')}$ into $H^{(m') + p_i.(m'')}$ by

$$(\Phi_i F)(g) = F(g, p_i g) \quad \forall g \in G, F \in V^{(m') \otimes (m'')} .
\tag{III-2}$$

Φ_i is clearly holomorphic and for all $b_i \in B_i$ we have

$$
\begin{aligned}
(\Phi_i F)(b_i g) &= F(b_i g, p_i b_i g) \\
&= F(b_i g, p_i b_i p_i^{-1} p_i g) \\
&= \pi^{(m')}(b_i) \pi^{(m'')}(p_i b_i p_i^{-1})(\Phi_i F)(g) \\
&= \pi^{(m') + p_i.(m'')}(b_i)(\Phi_i F)(g) .
\end{aligned}
\tag{III-3}
$$

The pair $(\Phi_0, H^{(m') + (p_0.m'')})$ is particularly interesting and will be seen to play a special role in the tensor product problem. Indeed, we have the following

THEOREM 1. — *The mapping Φ_0 defined by $(\Phi_0 F)(g) = F(g, p_0 g)$, $\forall g \in G$, $F \in V^{(m') \otimes (m'')}$ is an isomorphism (of G-modules) of $V^{(m') \otimes (m'')}$ into the « dense double coset space » $H^{(m') + p_0.(m'')}$.*

Proof. — By a well known theorem [5] $B p_0 B$ is the unique double coset in G whose dimension is equal to that of G, so that $B p_0 B$ is a dense open submanifold of G. From the Bruhat decomposition of G it follows immediately that

$$
\begin{aligned}
G \times G &= \bigcup_{p_i \in W(G)} (B \times B)(e, p_i) G \\
B \times B \backslash G \times G / G &\approx B \backslash G / B \\
&\approx W(G)
\end{aligned}
\tag{III-4}
$$

(here we identify G with the diagonal subgroup in $G \times G$ generated by the elements (g, g), $g \in G$). From the above remarks one can easily deduce that with the exception of a set of measure zero (finite union of submanifolds of lower dimension) every element $(g', g'') \in G \times G$ can be uniquely represented as

$$
\begin{aligned}
g' &= b' w z, & z &= z(g'), & b' &= b(g') w^{-1} \\
g'' &= b'' p_0 w z, & p_0 w p_0^{-1} &= z(g'' z^{-1}(g') p_0^{-1}), \\
& & b'' &= b(g'' z^{-1}(g') p_0^{-1})
\end{aligned}
\tag{III-5}
$$

where b', b'' are elements of B, w is in W_0, z is in Z, and $z(g)$, $b(g)$ are given in eq. II-1.

Suppose $F \in V^{(m') \otimes (m'')}$ is such that $\Phi_0 F = 0$, that is, $F(g, p_0 g) = 0$ for all $g \in G$, then in particular, for $g = wz$ as in (III-5) $F(wz, p_0 wz) = 0$. Because $\pi^{(m')}(b') \pi^{(m'')}(b'') F(wz, p_0 wz) = F(b'wz, b''p_0 wz) = F(g', g'')$ it follows that $F = 0$. Thus Φ_0 is injective. Since it is clear that Φ_0 is an intertwining operator, Φ_0 is seen to be a (G-module) isomorphism.

In general, Φ_0 is not surjective, however, we can define a restricted inverse map

$$
\Psi_0 : \Phi_0(V^{(m') \otimes (m'')}) \to V^{(m') \otimes (m'')}
$$

by

$$
(\Psi_0 f)(g', g'') = \pi^{(m')}(b') \pi^{(m'')}(b'') f(wz)
\tag{III-6}
$$

where b', b'', w and z are related to (g', g'') as in equation (III-5). Strictly speaking $\Psi_0 f$ is just defined for every (g', g'') in $G \times G$ such that the decomposition (III-5) holds, but since all the decompositions involved are holomorphic and the set of all such elements (g', g'') is open and dense in $G \times G$, one can extend the domain of definition of $\Psi_0 f$ to the entire manifold $G \times G$ by analytic continuation. In fact, since all the maps and representations we consider in this section are holomorphic, to show a certain property of these maps it suffices to verify that they satisfy this property on a dense open submanifold of their domain of definition. This is a fact that will always be tacitly assumed in these sections. Obviously Ψ_0 is linear and well defined (the fact that

$$(\Psi_0 f)(\beta'g', \beta''g'') = \pi^{(m)}(\beta')\pi^{(m)}(\beta'')\Psi_0 f(g', g'')$$

for all β', β'' in B follows immediately from the uniqueness of the decomposition (III-5)). In equation (III-5) if we let $g' = g$ and $g'' = p_0 g$, then one can easily show that b' and b'' must belong to D and $b'' = p_0 b' p_0^{-1}$, so that $b' = d$ where d appears in the Gauss decompositon of g. Thus

$$\begin{aligned}
(\Phi_0 \Psi_0 f)(g) &= \Psi_0 f(g, p_0 g) \\
&= \pi^{(m')}(d)\pi^{(m'')}(p_0 d p_0^{-1})f(wz) \\
&= \pi^{(m') + (p_0.m'')}(d)f(wz) \\
&= f(dwz) \\
&= f(g).
\end{aligned} \tag{III-7}$$

So, $\Phi_0 \Psi_0$ is the identity map on $\Phi_0(V^{(m') \otimes (m'')})$ and

$$\begin{aligned}
(\Psi_0 \Phi_0 F)(g', g'') &= \pi^{(m')}(b')\pi^{(m'')}(b'')\Phi_0 F(wz) \\
&= \pi^{(m')}(b')\pi^{(m'')}(b'')F(wz, p_0 wz) \\
&= F(b'wz, b''p_0 wz) = F(g', g'').
\end{aligned} \tag{III-8}$$

This shows that $\Psi_0 \Phi_0$ is the identity map on $V^{(m') \otimes (m'')}$. The preceding discussion leads to the following important conclusion: To show that a function f in $H^{(m') + p_0.(m'')}$ belongs to the image of $V^{(m') \otimes (m'')}$ under Φ_0, one only needs to verify that $\Psi_0 f$ is well defined, i. e., $\Psi_0 f$ is polynomial in (g', g''). As we shall see, this criterion will be very useful in subsequent subsections. ∎

To proceed further in our problem of decomposing the tensor product $V^{(m') \otimes (m'')}$, it is necessary to find maps that send elements of an irreducible space $V^{(m)}$ (where (m) includes those representations that occur in the Clebsch-Gordan series of $(m') \otimes (m'')$) into $V^{(m') \otimes (m'')}$. By Theorem 1, we see that this can be done by finding a map directly from an irreducible representation space $V^{(m)}$ to $H^{(m') + p_0.(m'')}$ (such a map will be designated as $\Omega^{(m)}$).

In trying to get a handle on the Ω maps, the identity double coset plays an important role, so we first turn to an analysis of the space $H^{(m') + (m'')}$.

For $f \in H^{(m')+(m'')}$, it is clear that $f(bg) = \pi^{(m')+(m'')}(b)f(g)$, $b \in B$, thus the identity double coset generates an irreducible representation, with highest weight equal to $(m') + (m'')$; hence, $H^{(m')+(m'')} \equiv V^{(m')+(m'')}$.

We wish to find a map back to $V^{(m')\otimes(m'')}$ (called Ψ_e) that intertwines. Let $\{ h_{[k]}^{(m)}(g) \}$ denote a set of basis functions satisfying

(1)   span $\{ h_{[k]}^{(m)} \} = V^{(m)}$

(2) $(h_{[k]}^{(m)}, h_{[k']}^{(m)}) = 0$, $[k] \neq [k']$

(3) $h_{[k]}^{(m)}(bg) = \pi^{(m)}(b)h_{[k]}^{(m)}(g)$, $h_{[k]}^{(m)}$ holomorphic in G.

(4) $(R_{(\gamma)}^{(m)}h_{[k]}^{(m)}(g)) = h_{[k]}^{(m)}(g\gamma)$

$$= \sum_{[\bar{k}]} h_{[\bar{k}]}^{(m)}(g)D_{[\bar{k}][k]}^{(m)}(\gamma), \qquad \gamma \in SU \qquad \text{(III-9)}$$

(An example of such a basis is constructed in our paper, « Orthogonal Polynomial Bases for Holomorphically Induced Representations of the General Linear Groups » [13].) $[k]$ denotes a set of labels needed for $h_{[k]}^{(m)}$ to span the space $V^{(m)}$; since $V^{(m)}$ is an irreducible space, right translation of $h_{[k]}^{(m)}$ by $\gamma \in SU$ produces a linear combination of $h_{[k]}^{(m)}$ denoted by $D_{[k][\bar{k}]}^{(m)}(\gamma)$. Thus, $D_{[k][\bar{k}]}^{(m)}(\gamma)$ is a matrix element of SU relative to the basis $[k]$. See Ref. [13] for more details.

The actual realization of the map Ψ_e can be precisely described as follows:

THEOREM 2. — *If* $(m) = (m') + (m'')$ *then the irreducible representation space* $V^{(m)}$ *is injected isomorphically onto the highest weight submodule of the tensor product* $V^{(m')\otimes(m'')}$ *via the mapping*

$$(\Psi_e h_{[k]}^{(m)})(g', g'') = \sum_{[k']+[\overline{k''}]=[k]} C_{[k][k'][k'']}^{(m)(m')(m'')}h_{[k']}^{(m')}(g')h_{[k'']}^{(m'')}(g'') \qquad \text{(III-10)}$$

where in Eq. (III-10) *the summation* $[k'] + [k'']$ *means pattern addition; that is, the addition of an element in* $[k']$ *with the corresponding element in* $[k'']$, *and the constants* $C_{[k'][k'][k'']}^{(m)(m'')(m'')}$ *are the (non-normalized) Clebsch-Gordan coefficients occurring in the highest weight submodule and computed (in Lemma 3) with respect to the bases* $\{ h_{[k]}^{(m)} \}$, $\{ h_{[k']}^{(m')} \}$, *and* $\{ h_{[k'']}^{(m'')} \}$. *The map* Ψ_e *extends by linearity to all elements of* $V^{(m)}$ *and satisfies* $\Phi_e\Psi_e = \text{Id}_V(m)$; $\text{Id}_V(m)$ *being the identity map on* $V^{(m)}$.

Proof. — The fact that $\Psi_e h_{[k]}^{(m)}$ is in $V^{(m')\otimes(m'')}$ can be seen from computing $(\Psi_e h_{[k]}^{(m)})(b'g', b''g'') = \pi^{(m')}(b')\pi^{(m'')}(b'')(\Psi_e h_{[k]}^{(m)})(g', g'')$, $b', b'' \in B$. Let us show that Ψ_e intertwines. That is we demand that

$$(\rho(\gamma)\Psi_e h_{[k]}^{(m)})(g', g'') = (\Psi_e R^{(m)}(\gamma)h_{[k]}^{(m)})(g', g''), \qquad \gamma \in SU. \qquad \text{(III-11)}$$

But

$$\left(\rho(\gamma)\Psi_e h_{[k]}^{(m)}\right)(g', g'') = \left(\Psi_e h_{[k]}^{(m)}\right)(g'\gamma, g''\gamma)$$

$$= \sum_{[k']+[k'']=[k]} C_{[k][k'][k'']}^{(m)(m')(m'')}$$
$$\times h_{[k']}^{(m')}(g'\gamma)h^{(m'')}(g''\gamma)$$

$$= \sum_{[k']+[k'']=[k]} \sum_{[\overline{k}']\,[\overline{k}'']} C_{[k][k'][k'']}^{(m)(m')(m'')}$$
$$\times h_{[\overline{k}']}^{(m')}(g')D_{[\overline{k}'][k']}^{(m')}(\gamma)$$
$$\times h_{[\overline{k}'']}^{(m'')}(g'')D_{[\overline{k}''][k'']}^{(m'')}(\gamma) \qquad \text{(III-12)}$$

whereas

$$\left(\Psi_e R^{(m)}(\gamma)h_{[k]}^{(m)}\right)(g', g'') = \sum_{[\overline{k}]} \sum_{[\overline{k}']+[\overline{k}'']=[\overline{k}]} C_{[\overline{k}][\overline{k}'][\overline{k}'']}^{(m)(m')(m'')}$$
$$\times h_{[\overline{k}']}^{(m')}(g')h_{[\overline{k}'']}^{(m'')}(g'')D_{[\overline{k}][k]}^{(m)}(\gamma). \qquad \text{(III-13)}$$

Now (III-12) can be rewritten as

$$\sum_{[\overline{k}]}\sum_{[\overline{k}']+[\overline{k}'']=[\overline{k}]}\left(\sum_{[k']+[k'']=[k]} C_{[k][k'][k'']}^{(m)(m')(m'')}D_{[\overline{k}'][k']}^{(m')}(\gamma)\,D_{[\overline{k}''][k'']}^{(m'')}(\gamma)\right)h_{[\overline{k}']}^{(m')}(g')h_{[\overline{k}'']}^{(m'')}(g'')$$

so that Ψ_e intertwines if

$$C_{[\overline{k}][\overline{k}'][\overline{k}'']}^{(m)(m')(m'')}D_{[\overline{k}][k]}^{(m)}(\gamma) = \sum_{[k']+[k'']=[k]} C_{[k][k'][k'']}^{(m)(m')(m'')} D_{[\overline{k}'][k']}^{(m')}(\gamma)D_{[\overline{k}''][k'']}^{(m'')}(\gamma) \qquad \text{(III-14)}$$

But Eq. (III-14) is just a Clebsch-Gordan relation, so Ψ_e is indeed an intertwining operator. Since the representation $R^{(m)}$ is irreducible, to prove that Ψ_e is injective, it suffices to show that Ψ_e is not identically zero. For this purpose we remark that in Eq. (III-10) if we chose $[k]$ to be the highest weight $[k_{\text{Max}}] \equiv [k_{\text{M}}]$ for $R^{(m)}$ then this equation reduces to

$$\Psi_e h_{[k_{\text{M}}]}^{(m)}(g', g'') = C_{[k_{\text{M}}][k_{\text{M}}'][k_{\text{M}}'']}^{(m)(m')(m'')}h_{[k_{\text{M}}']}^{(m')}(g')h_{[k_{\text{M}}'']}^{(m'')}(g'').$$

But the condition $\Phi_e\Psi_e h_{[k_{\text{M}}]}^{(m)} = h_{[k_{\text{M}}]}^{(m)}$ implies that $C_{[k_{\text{M}}][k_{\text{M}}'][k_{\text{M}}'']}^{(m)(m')(m'')} = 1$. Thus $\Psi_e h_{[k_{\text{M}}]}^{(m)} \neq 0$ and the proof of the theorem is also completed. ∎

LEMMA 3. — *Let* $\Psi_e h_{[k]}^{(m)}$ *and* $C_{[k][k'][k'']}^{(m)(m')(m'')}$ *be given by* Eq. (III-10) *of Theorem* 2. *Then*

$$\left(\| \Psi_e h_{[k]}^{(m)} \|\right)^{-2} = \sum_{[k']+[k'']=[k]} \left(\| h_{[k']}^{(m')} \| \| h_{[k'']}^{(m'')} \|\right)^{-2} \qquad \text{(III-15)}$$

and

$$C_{[k][k'][k'']}^{(m)(m')(m'')} = \left(\parallel h_{[k']}^{(m')} \parallel \parallel h_{[k'']}^{(m'')} \parallel \right)^{-2} \left(\parallel \Psi_e h_{[k]}^{(m)} \parallel \right)^2$$

where in Eq. (III-15) the norms are computed with respect to the inner product given by Eq. (II-4).

Proof. — Let us write

$$h_{[k']}^{(m')} \otimes h_{[k'']}^{(m'')} = \sum_{(m),[k]} K_{[k][k'][k'']}^{(m)(m')(m'')} \eta \Lambda_\eta^{(m)} h_{[k]}^{(m)} \qquad (III-16)$$

where in Eq. (III-16) the $\Lambda_\eta^{(m)} h_{[k]}^{(m)}$ are the basis elements equivalent to $h_{[k]}^{(m)}$ that occur in the orthogonal direct sum decomposition of $V^{(m') \otimes (m'')}$, the η's specify different copies of the same representation (m), and the K's are constants depending on the indicated variables. The summation ranges over all irreducible representations (m) that occur in the Clebsch-Gordan series of $(m') \otimes (m'')$, and over all tableaux $[k]$ of each representation (m). Moreover, we also assume that $\Lambda^{(m'+m'')}$ which occurs with multiplicity 1 is equal to Ψ_e. By applying Φ_e to Eq. (III-16) we obtain

$$h_{[k']+[k'']}^{(m'+m'')} = \sum_{[k]=[k']+[k'']} K_{[k][k'][k'']}^{(m'+m'')(m')(m'')} h_{[k]}^{(m'+m'')} .$$

The last equation imples immediately that

$$K_{[k][k'][k'']}^{(m'+m'')(m')(m'')} = \delta_{[k],[k']+[k'']} \qquad (III-17)$$

where in Eq. (III-17) the right-handed side is the Kronecker delta symbol.

Now let $d_{(m)}, d_{(m')}, d_{(m'')}$ be the degrees of $R^{(m)}$, $R^{(m')}$, and $R^{(m'')}$, respectively. Let H be the column matrix representing the $d_{(m')} \cdot d_{(m'')}$ basis elements $h_{[k']}^{(m')} \otimes h_{[k'']}^{(m'')}$ in a certain order, let ΛH be the column matrix representing the basis elements $\Lambda_\eta^{(m)} h_{[k]}^{(m)}$ ordered in such a way that the first $d_{(m)}$ elements are $\Psi_e h_{[k_M]}^{(m'+m'')}, \ldots, \Psi_e h_{[k_M]}^{(m'+m'')}$ which belong to the highest weight submodule $\Psi_e(V^{(m'+m'')})$. Let E_1 (resp. E_2) be the diagonal matrix with diagonal entries equal to $\left(\parallel h_{[k']}^{(m')} \parallel \parallel h_{[k'']}^{(m'')} \parallel \right)^{-1}$ (resp. to $\left(\parallel \Lambda_\eta^{(m)} h_{[k]}^{(m)} \parallel \right)^{-1}$) and corresponding to the basis elements $h_{[k]}^{(m')} \otimes h_{[k'']}^{(m'')}$ (resp. $\Lambda_\eta^{(m)} h_{[k]}^{(m)}$) ordered as above. It follows that we have the identity

$$E_1 H = ME_2(\Lambda H). \qquad (III-18)$$

The matrix M which represents the changes of bases of two orthonormal bases $E_1 H$ and $E_2(\Lambda H)$ is unitary; in fact, M may be assumed to be orthogonal, that is, $M^t = M^{-1}$. If K denotes the matrix with coefficients $K_{[k][k'][k'']}^{(m)(m')(m'')} \eta$ then it follows from Eq. (III-16) and Eq. (III-18) that

$$K = E_1^{-1} M E_2 \qquad \text{or} \qquad E_1 K E_2^{-1} = M$$

so that

$$K^{-1} = E_2^{-1} M^{-1} E_1 .$$

But

$$M^{-1} = M^t = E_2^{-1} K^t E_1 ;$$

thus

$$K^{-1} = (E_2)^{-2} K^t (E_1)^2 .$$

Now, it follows from Eq. (III-17) that K can be represented in block form as

$$d_{(m')} d_{(m'')} \left\{ \left[\underbrace{\boxed{K_1}}_{d_{(m'+m'')}} \; \vdots \; K_2 \right] \right.$$

where K_1 is a matrix with entries $\delta_{[k],[k']+[k'']}$. Thus, by comparing all the above equalities we obtain

$$\Lambda H = (E_2)^{-2} \left[\begin{array}{c} K_1^t \\ \hline K_2^t \end{array} \right] (E_1)^2 H . \qquad \text{(III-19)}$$

The first $d_{(m')+(m'')}$ rows of Eq. (III-19) gives

$$\Psi_e h_{[k]}^{(m'+m'')} = \left(\| \Psi_e h_{[k]}^{(m'+m'')} \|^2 \right) \sum_{[k']+[k'']=[k]} \left(\| h_{[k']}^{(m')} \| \| h_{[k'']}^{(m'')} \| \right)^{-2} h_{[k']}^{(m')} \otimes h_{[k'']}^{(m'')} . \qquad \text{(III-20)}$$

Equation (III-20) together with Eq. (III-10) implies that

$$C_{[k][k'][k'']}^{(m)(m')(m'')} = \left(\| h_{[k']}^{(m')} \| \| h_{[k'']}^{(m'')} \| \right)^{-2} \left(\| \Psi_e h_{[k]}^{(m)} \| \right)^2 .$$

It remains to compute $\| \Psi_e h_{[k]}^{(m)} \|^2$. For this purpose, we rewrite Eq. (III-20) as follows.

$$\Psi_e h_{[k]}^{(m'+m'')} \left(\| \Psi_e h_{[k]}^{(m'+m'')} \| \right)^{-1} = \left(\| \Psi_e h_{[k]}^{(m'+m'')} \| \right) \times$$

$$\sum_{[k']+[k'']=[k]} \left(\| h_{[k']}^{(m')} \| \| h_{[k'']}^{(m'')} \| \right)^{-1} \left(\| h_{[k']}^{(m')} \|^{-1} h_{[k']}^{(m')} \right) \otimes \left(\| h_{[k'']}^{(m'')} \|^{-1} h_{[k'']}^{(m'')} \right). \qquad \text{(III-21)}$$

By taking the inner product of each member of the above equation with itself and by observing that the orthogonal basis element involved are now normalized, we deduce that

$$1 = \left(\| \Psi_e h_{[k]}^{(m'+m'')} \| \right)^2 \sum_{[k']+[k'']=[k]} \left(\| h_{[k']}^{(m')} \| \| h_{[k'']}^{(m'')} \| \right)^{-2}$$

which implies the desired expression for $\left(\,\|\,\Psi_e h_{[k]}^{(m)}\,\|\,\right)^2$. The proof of Lemma 3 is now achieved. ∎

REMARK 4. — Besides its use for the construction of the map Ψ_e, Lemma 3 is of great interest in its own right, for it gives us a very elegant procedure to compute the Clebsch-Gordan coefficients of the highest submodule of $V^{(m')\otimes(m'')}$ in a very simple and compact form. Indeed, if we denote by $\langle\,(m')[k']\,;\,(m'')[k'']\,|\,(m'+m'')[k]\,\rangle$ the Wigner coefficients (that is the normalized Clebsch-Gordan coefficients) of the highest weight sub-module, then Eq. (III-21) gives

$$
\langle\,(m')[k']\,;(m'')[k'']\,|\,(m'+m'')[k]\,\rangle = \|\,\Psi_e h_{[k]}^{(m'+m'')}\,\|\,\left(\,\|\,h_{[k']}^{(m')}\,\|\,\|\,h_{[k'']}^{(m'')}\,\|\,\right)^{-1}
$$
$$
= \left[\sum_{[k']+[k'']=[k]}\left(\,\|\,h_{[k']}^{(m')}\,\|\,\|\,h_{[k'']}^{(m'')}\,\|\,\right)^{-2}\right]^{-1/2}
$$
$$
\times\left(\,\|\,h_{[k']}^{(m')}\,\|\,\|\,h_{[k'']}^{(m'')}\,\|\,\right)^{-1}. \qquad \text{(III-22)}
$$

To conclude this section we show how the composition of Ψ_e and Φ_i maps lead to maps Ω carrying an irreducible representation space $V^{(m)}$ into $H^{(m')+p_0(m'')}$. First it is straightforward to compute the highest weight Clebsch-Gordan coefficients for n fold tensor products by generalizing the notion of a double coset map to an n fold tensor product space. Let F be an element of $V^{(m_1)}\otimes V^{(m_2)}\otimes\ldots\otimes V^{(m_n)}$, and define $\Phi_{p_1 p_2\ldots p_n}$ by

$$
(\Phi_{p_1 p_2\ldots p_n}F)(g) \equiv F(p_1 g, p_2 g \ldots p_n g).
$$

It then follows that $\Phi_{e,e,\ldots,e}$ is a map from the n fold tensor product space to an irreducible representations space $V^{(m)}$, $(m) = (m_1) + (m_2) + \ldots (m_n)$. The Clebsch-Gordan coefficients for this highest weight, written $C_{kk_1 k_2\ldots k_n}^{mm_1 m_2\ldots m_n}$ are multiplicity free and can be computed along the lines given for highest weight Clebsch-Gordan coefficients of twofold tensor products. Then an inverse map Ψ_e carrying elements from V to $V^{(m_1)}\otimes V^{(m_2)}\otimes\ldots\otimes V^{(m_n)}$ is defined analogously to Ψ_e for twofold tensor products, Eq. (III-10), as

$$
(\Psi_e h_{[k]}^{(m)})(g_1\ldots g_n) = \Sigma C_{k\,k_1\ldots k_n}^{mm_1\ldots m_n} h_{[k_1]}^{(m_1)}(g_1)\ldots h_{[k_n]}^{(m_n)}(g_n).
$$

Then Ω is defined as the composition $\Phi_{p_1\ldots p_n}\Psi_e$, where $(p_1\ldots p_n)$ are per-mutations chosen so that $(\Omega h_{[k]}^{(m)}(g))$ transforms to the left as $(m') + p_0.(m'')$ with respect to the diagonal subgroups. It is clear that many different Ω maps can be formed for different choices of $(p_1\ldots p_n)$ and Ψ_e. In a following paper we will show that there always are linearly independent Ω maps (for a given irreducible representations (m)) that break the multiplicity occurring in a two-fold tensor product decomposition. An example of this procedure will be given for SU(3) at the end of Section IV.

IV. AN EXAMPLE:
THE EIGHT-DIMENSIONAL REPRESENTATION
OF SU(3)

The goal of this section is to take one tensor product in SU(3) and show how it decomposes using techniques and constructions developed in the previous sections. We choose the eight-dimensional representation and consider the tensor product $8 \otimes 8$. This example is physically important and is illuminating because the Clebsch-Gordan series of $8 \otimes 8$ contains the eight-dimensional representation twice. The Clebsch-Gordan coefficients of this tensor product are of course well known [3], so we will not go into the computational aspects of the problem. Rather, we shall describe the mechanism of the decomposition and propose a « canonical » method for solving the multiplicity problem. A generalization of this method to $SU(n)$ will be discussed in succeding papers.

To begin we note that a representation of SU(3) is labelled by

$$(m) = (m_1 - l, m_2 - l, m_3 - l), m_1 \geq m_2 \geq m_3 ;$$

since all representations differing by values of l are equivalent, it is sufficient to set $l = m_3$, so that any representation of SU(3) can be labelled by $(m_1, m_2, 0), m_1 \geq m_2 \geq 0$. In this notation the eight-dimensional representation of the previous paragraph becomes (2, 1, 0). With the Gauss decomposition of Section II, we have $g = b(g)z(g) \, g \in GL(3, \mathbb{C})$ and a representation of SU(3) acts on the space $V^{(m)}$ of holomorphic functions transforming as $f(bg) = \pi^{(m)}(b)f(g)$, $f \in V^{(m)}$, $b \in B$, the 3×3 lower triangular matrices; $\pi^{(m)}(b) = b_{11}^{m_1}b_{22}^{m_2}$.

The tensor product space $V^{(m') \otimes (m'')}$ is injected into $H^{(m') + p_0.(m'')}$ under the map $(\Phi_0 F)(g) = F(g, p_0 g)$, $F \in V^{(m') \otimes (m'')}$. Here p_0 is an element of the Weyl group S_3, of the form

$$p_0 = (13)(2) \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Besides p_0, there are 5 other permutations which generate invariant subspaces under the map $(\Phi_i F)(g) = F(g, p_i g)$, $p_i \in S_3$. In particular, the identity permutation leads to the irreducible representation space $V^{(m') + (m'')}$. We consider now the reduction of the tensor product

$$8 \otimes 8 = 1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27$$
$$(2,1,0) \otimes (2,1,0) = (0,0,0) \oplus (2,1,0) \oplus (2,1,0) + (3,0,0) \oplus (3,3,0) \oplus (4,2,0)$$

where the top numbers give the dimensions of the representations. The identity double coset generates an irreducible representation

$$(m') + (m'') = (2,1,0) + (2,1,0) = (4,2,0).$$

The dense double coset is labelled $H^{(2,2,2)}$; here it is possible to make use of the fact that $|g|$, the determinant of $g \in GL(3, C)$, is invariant for $SU(3)$. Thus the identity irreducible representations $(0,0,0)$ can be mapped to the dense double coset via

$$(\Omega f)(g) = |g|^2 f(g), \qquad f \in V^{(0,0,0)},$$

and the map Ψ_0 carries Ωf into $V^{(2,1,0)\otimes(2,1,0)}$. The highest weight sub-module $V^{(4,2,0)}$ is mapped into $V^{(2,1,0)\otimes(2,1,0)}$ by the map Ψ_e described in Theorem 2. For the irreducible representations $(3,0,0)$ and $(3,3,0)$, we need the generation of Theorem 2 to threefold tensor products (see Remark 3). First, we realize the representation $(3,0,0)$ as the highest weight submodule of $V^{(1,0,0)} \otimes V^{(1,0,0)} \otimes V^{(1,0,0)}$ via the mapping

$$\Psi_e h_{[k]}^{(3,0,0)}(g', g'', g''') = \sum_{[k']+[k'']+[k''']=[k]} C_{[k][k'][k''][k''']}^{(3,0,0),(1,0,0),(1,0,0),(1,0,0)}$$
$$\times\; h_{[k']}^{(1,0,0)}(g') h_{[k'']}^{(1,0,0)}(g'') h_{[k''']}^{(1,0,0)}(g''')$$

where the C coefficients are computed by the generalization of the method described in Lemma 3. Define now a map $\Omega : \Psi_e(V^{(3,0,0)}) \rightarrow H^{(2,2,2)}$ by

$$(\Omega\Psi_e h_{[k]}^{(3,0,0)})(g) = |g|\, \Psi_e h_{[k]}^{(3,0,0)}(g, p_{(123)}g, p_{(132)}g)\,.$$

Again, by applying Ψ_0 to the resulting image of $h_{[k]}^{(3,0,0)}$ in $H^{(2,2,2)}$, $V^{(3,0,0)}$ is sent into $V^{(2,1,0)\otimes(2,1,0)}$. Similarly the representation $(3,3,0)$ is realized as the highest weight submodule of $V^{(1,1,0)} \otimes V^{(1,1,0)} \otimes V^{(1,1,0)}$, which fixes the map sending $V^{(3,3,0)}$ into $V^{(2,1,0)\otimes(2,1,0)}$.

Now let us focus our attention on the representation $(2,1,0)$ which appears twice in the tensor product $(2,1,0) \otimes (2,1,0)$. From Ref. [13] it follows that our canonical orthogonal basis for $V^{(2,1,0)}$ consists of the following vectors:

$$h_{\left[\begin{smallmatrix}2&1\\&2\end{smallmatrix}\right]}^{(210)}(g) = g_{11}\Delta_2(g)\;, \qquad h_{\left[\begin{smallmatrix}2&1\\&1\end{smallmatrix}\right]}^{(210)}(g) = g_{12}\Delta_2(g)$$

$$h_{\left[\begin{smallmatrix}2&0\\&2\end{smallmatrix}\right]}^{(210)}(g) = g_{11}\Delta_{13}^{12}(g), \qquad h_{\left[\begin{smallmatrix}2&0\\&1\end{smallmatrix}\right]}^{(210)}(g) = \frac{1}{2}\left(g_{12}\Delta_{13}^{12}(g) + g_{11}\Delta_{23}^{12}(g)\right)$$

$$h_{\left[\begin{smallmatrix}2&0\\&0\end{smallmatrix}\right]}^{(210)}(g) = g_{12}\Delta_{23}^{12}(g), \qquad h_{\left[\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right]}^{(210)}(g) = g_{13}\Delta_2(g)$$

$$h_{\left[\begin{smallmatrix}1&0\\&1\end{smallmatrix}\right]}^{(210)}(g) = g_{13}\Delta_{13}^{12}(g), \qquad h_{\left[\begin{smallmatrix}1&0\\&0\end{smallmatrix}\right]}^{(210)}(g) = g_{13}\Delta_{23}^{12}(g)\,. \qquad \text{(IV-1)}$$

For the notation used in Eq. (IV-1), see Section III. By the method described in Section III, we can easily see that there are essentially two different maps for injecting $V^{(2,1,0)}$ into $H^{(2,2,2)}$, the others being linear combinations of the above two. The two maps can be defined as follows:

$$\Omega_{(p_0,e)} : V^{(2,1,0)} \rightarrow H^{(2,2,2)}$$
$$\Omega_{(e,p_0)} : V^{(2,1,0)} \rightarrow H^{(2,2,2)},$$

with

$$\left(\Omega_{(p_0,e)}h_{[k]}^{(2\,1\,0)}\right)(g) = |g| \sum_{[k']+[k'']=[k]} C_{[k]}^{(2\,1\,0)}{}_{[k']}^{(1\,1\,0)}{}_{[k'']}^{(1\,0\,0)} h_{[k']}^{(1\,1\,0)}(p_0g)h_{[k'']}^{(1\,0\,0)}(g)$$

$$\left(\Omega_{(e,p_0)}h_{[k]}^{(2\,1\,0)}\right)(g) = |g| \sum_{[k']+[k'']=[k]} C_{[k]}^{(2\,1\,0)}{}_{[k']}^{(1\,1\,0)}{}_{[k'']}^{(1\,0\,0)} h_{[k']}^{(1\,1\,0)}(g)h_{[k'']}^{(1\,0\,0)}(p_0g)$$

where the C coefficients are easily computed using Lemma 3. We will show that the above two maps generate equivalent but linearly independent irreducible subspaces of $V^{(2\,1\,0)\otimes(2\,1\,0)}$. To see this we compute the images of the highest weight vector $h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\2\end{smallmatrix}\right]}^{(2\,1\,0)} \equiv h_{[k_M]}^{(2\,1\,0)}$ under $\Omega_{(p_0,e)}$ and $\Omega_{(e,p_0)}$. This gives

$$\left(\Omega_{(p_0,e)}h_{[k_M]}^{(2\,1\,0)}\right)(g) = |g|\,g_{11}\Delta_{12}^{23}(g)$$

$$\left(\Omega_{(e,p_0)}h_{[k_M]}^{(2\,1\,0)}\right)(g) = |g|\,g_{31}\Delta_2(g)$$

so that

$$\left(\Psi_0\Omega_{(p_0,e)}h_{[k_M]}^{(2\,1\,0)}\right)(g',g'') = -h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g'')$$

$$+ \frac{1}{2}h_{\left[\begin{smallmatrix}2\,1\,0\\1\,1\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g'') + h_{\left[\begin{smallmatrix}2\,1\,0\\2\,0\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g'') - h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\1\,1\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g'')$$

$$\left(\Psi_0\Omega_{(e,p_0)}h_{[k_M]}^{(2\,1\,0)}\right)(g',g'') = -h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\2\,0\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g'')$$

$$+ h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\1\,1\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g'') + h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\2\,0\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g'') - h_{\left[\begin{smallmatrix}2\,1\,0\\1\,1\\1\end{smallmatrix}\right]}^{(2\,1\,0)}(g')h_{\left[\begin{smallmatrix}2\,1\,0\\2\,1\\2\end{smallmatrix}\right]}^{(2\,1\,0)}(g''). \quad \text{(IV-2)}$$

Using Eq. (IV-2), an easy computation shows that

$$\Phi_{(e,p_{(123)})}\Psi_0\Omega_{(p_0,e)}h_{[k_M]}^{(2\,1\,0)} = 0$$

$$\Phi_{(e,p_{(132)})}\Psi_0\Omega_{(p_0,e)}h_{[k_M]}^{(2\,1\,0)} = -g_{21}\Delta_2(g)|g|$$

$$\Phi_{(e,p_{(123)})}\Psi_0\Omega_{(e,p_0)}h_{[k_M]}^{(2\,1\,0)} = g_{11}\Delta_{23}^{13}(g)|g|$$

$$\Phi_{(e,p_{(132)})}\Psi_0\Omega_{(e,p_0)}h_{[k_M]}^{(2\,1\,0)} = 0. \quad \text{(IV-3)}$$

Using Eq. (IV-3) and the fact that both $\Psi_0\Omega_{(p_0,e)}(V^{(2,1,0)})$ and $\Psi_0\Omega_{(e,p_0)}(V^{(2\,1\,0)})$ are irreducible submodules of $V^{(2\,1\,0)\otimes(2\,1\,0)}$, we see that $\Psi_0\Omega_{(p_0,e)}(V^{(2\,1\,0)})$ lies in the kernel of $\Phi_{(e,p_{(123)})}$ while $\Psi_0\Omega_{(e,p_0)}(V^{(2\,1\,0)})$ lies in the kernel of $\Phi_{(e,p_{(132)})}$, so that the two maps generate a sixteen-dimensional invariant direct sum subspace of $V^{(2\,1\,0)\otimes(2\,1\,0)}$.

This entire procedure generalizes to an $(n+1)$ fold multiplicity of $V^{(2n,n,0)}$ occurring in the tensor product decomposition of $V^{(2n,n,0)\otimes(2n,n,0)}$.

Here the relevant maps are

$$\left(\Psi_e h_{[k]}^{(2n,n,0)}\right)(g_1g_2\cdots g_{2n}) =$$

$$\sum_{[k_1]+\cdots+[k_{2n}]=[k]} C_{[k]}^{(2n,n,0);\;\overbrace{(1\,1\,0)\ldots}^{n\text{ times}};\;\overbrace{(1\,0\,0)\ldots(1\,0\,0)}^{n\text{ times}}}_{[k_1]\;\ldots\;[k_n]\;[k_{n+1}]\;[k_{2n}]} h_{[k_1]}^{(1\,1\,0)}(g_1)\cdots h_{[k_n]}^{(1\,1\,0)}(g_n)$$

$$\times\; h_{[k_{n+1}]}^{(1\,0\,0)}(g_{n+1})\cdots h_{[k_{2n}]}^{(1\,0\,0)}(g_{2n})$$

where $C_{[k]}^{(2n,n,0),(110)...(100)}{}_{[k_1]}{}^{...}{}_{[k_{2n}]}$ is the highest weight Clebsch-Gordan coefficient of $(110) \otimes \ldots \otimes (110) \otimes (100) \ldots (100) = (2n, n, 0)$, and is easily computed from the generalization of Lemma 3 because there is no multiplicity involved, and the double coset maps $\Phi_{(p_1...p_{2n})}$, defined in Section III. These maps define a set of $(n + 1)$ Ω maps $\Omega_{(e...e;p_0...p_0)} \cdots \Omega_{(p_0...p_0;e,...e)}$ that are linearly independent, with the property that all other Ω maps with different permutations are linear combinations of these Ω maps. For example, for $n = 3$, the multiplicity of $(6,3,0)$ occurring in the tensor product decomposition of $(6,3,0) \otimes (6,3,0)$ is 4, and the four linearly independent Ω maps are $\Omega_{(e,e,e;p_0p_0p_0)}$, $\Omega_{(p_0,e,e,;e,p_0p_0)}$, $\Omega_{(p_0p_0e;e,e,p_0)}$ and $\Omega_{(p_0p_0p_0;e,e,e)}$.

Thus, we have exhibited a set of Ω maps that explicitly break the multiplicity for certain representations of SU(3). It remains to show that such Ω maps break all the multiplicity for arbitrary tensor products of SU(n) in a *canonical* fashion. This issue will be examined in a following paper and compared with the null space concept for breaking the multiplicity in a canonical fashion introduced by Biedenharn and Louck [14].

V. CONCLUSION

We have shown how the Borel Weil holomorphic induction theory may be used to realize all of the representations of SU(n) as polynomials over GL(n, C). The goal in succeeding papers will be to use these polynomial representations to attack the multiplicity problem and the problem of computing Clebsch-Gordan and Racah coefficients for all the compact groups. In this paper we have defined two kinds of maps, the double coset maps and the inverse identity double coset maps, that will play a key role in carrying out this program. If G is the complexification of a compact group and B a subgroup, then the double coset map with respect to the subgroup H of G is given by $(\Phi_D f)(h) = f(g_0 h)$, where g_0 is a double coset representative of B\G/H and $f(g)$ is a polynomial function over G. One of the double coset representatives can always be chosen to be the identity element and generates the identity double coset map, $(\Phi_e f)(h) = f(h)$. For the spaces of interest in this paper, the identity double coset map generates an irreducible representation of H. The map carrying this irreducible representation back to the original space is called the inverse identity double coset map, Ψ_e. As shown in Theorem 2 it becomes a unique map by demanding that it intertwines.

One of the main uses of these maps is to study the reduction of tensor products. In such an application G becomes the direct product of the complexification of a compact group, while B is the direct product of Borel subgroups, and H is the diagonal subgroup (Eq. (III-4)). In this paper we have focused attention on GL(n, C) so that the double coset space

$$B \times B\backslash GL(n, C) \times GL(n, C)/GL(n, C)$$

is isomorphic to Sn, the symmetric group. The double coset maps generate spaces of polynomials over G that—except for the identity double coset—are not in general irreducible. In fact one of the main results of this paper is to show that there always is one double coset space—called H_0—that « contains » the original tensor product space.

Such a result is very different from that obtained by Mackey for induced representations of semidirect product groups. In Mackey's theory the representation space of G breaks up into an orthogonal direct sum of subspaces in which the subspaces are labeled by the double cosets. The subgroup theorem then asserts that each such subspace is equivalent to a representation space of H induced by a subgroup of H. While our double coset maps are modelled after Mackey's subgroup theorem and the double coset spaces are mutually orthogonal, there is no direct sum decomposition of the original tensor product space as there is in Mackey's theory. However, in Mackey's theory no double coset subspace is isomorphic to the original tensor product space, whereas in our theory we have shown that there is always one unique double coset subspace that is isomorphic to the original tensor product space. The difference between these theories lies in the kinds of spaces being used. In the Mackey theory the spaces are Hilbert spaces over G/B, and it is possible to decompose such spaces into an orthogonal direct sum of subspaces labeled by double cosets by setting elements of the space equal to zero when a group element is not in a given double coset. In contrast our representation spaces are polynomial spaces and it is not possible to get double coset subspaces by setting the polynomial functions equal to zero for certain values of the group variables. Thus the dense double coset space is the analogue in our theory of the direct sum decomposition in Mackey's theory.

Further, multiplicity is dealt with in Mackey's theory by using the double coset labels. If the double coset spaces contain a given representation more than once, then the double cosets will not suffice to deal with the multiplicity problem. In contrast an Ω map carrying an irreducible representation into the dense double coset H_0 is the composition of an inverse identity double coset map and some double coset map, so that it is the double coset label in conjunction with an irreducible representation label that breaks the multiplicity. The extent to which this resolves the multiplicity problem will be dealt with in a succeeding paper.

Now the double coset maps involve essentially only permutation group elements for $GL(n, C)$ (see Eq. (III-4)). In contrast the inverse identity double coset map is defined in terms of the highest weight Clebsch-Gordan coefficients. We have shown in the lemma of Section III that these highest weight Clebsch-Gordan coefficients are essentially given by the norms of the basis functions $h_{[k]}^{(m)}(g)$, (Eq. (III-9)). It is thus important to be able to compute the basis functions explicitly; this we do in the following paper [13]. It is interesting to note that our construction of orthogonal polynomial basis

functions depends only on the two maps defined in this paper, the double coset and inverse identity double coset maps. The procedure is an inductive one and uses the double cosets $B\backslash GL(n, C)/GL(n - 1, C)$.

Finally it is not hard to show that irreducible representations of $SO(n)$ and $Sp(2n)$ can be realized via the holomorphic induction procedure as polynomials over group variables. In fact the representations of these groups are constructed with the help of the double cosets

$$B\backslash GL(n, C)/SO(n, C) \quad \text{and} \quad B\backslash GL(2n, C)/Sp(2n, C).$$

Once these polynomial representations are given, the tensor product decomposition, multiplicity breaking, and construction of orthogonal polynomial bases can be carried out using the same type of maps introduced in this paper.

From these remarks it should be clear that the decomposition of an irreducible representation of a group into irreducible representations of a subgroup, is *not* particularly basis dependent. Specifying a chain of subgroups specifies an inductive procedure for constructing orthogonal basis functions for the irreducible representations of a group. What we will show in this series of papers is that the two types of maps introduced in this paper provide just the necessary tools for constructing representations and reducing tensor products.

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