ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 32, nº 3 (1980), p. 249-256
<http://www.numdam.org/item?id=AIHPA_1980__32_3_249_0>

Ann. Inst. Henri Poincaré,
Vol. XXXII, n° 3, 1980, p. 249-256.

Section A :
Physique théorique.

Zeta-Function and analytic renormalization

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ABSTRACT. — It is shown that ζ-function renormalization is equivalent to a form of analytic regularization with a prescription for taking the finite part.

I. INTRODUCTION

One of the techniques which has been developed [*1*, *2*] to calculate the index of a differential operator on a compact manifold uses generalizations of the θ-function and its Mellin transform, the ζ-function [*3*]. Analogous techniques have recently been applied to the problem of calculating effective Lagrangians to the one-loop level [*4*, *5*, *6*]. However the use of the ζ-function itself in this problem, as a method of analytically continuing a divergent series is considerably older [*7*]. We would like here to remark that this ζ-function renormalization is equivalent to a form of analytic regularization [*8*] with a prescription for taking the finite part.

In Section II, to fix notation, we briefly recall, using the ζ-function technique, the calculation of Coleman and Weinberg [*9*] of the one-loop contribution to the effective potential of a self-interacting scalar field. In Section III we analytically regularize the two divergent one-loop graphs and show that a simple prescription for taking the finite part leads to the same result as that found in Section II. In Section IV we briefly and superficially discuss the problem of gauge-invariance when the ζ-function technique is applied to QED. The classical gauge-invariant results of Schwinger [*10*] can be obtained using ζ-function renormalization, but in general the latter is not useful as a gauge-invariant renormalization scheme.

II

In the Euclidean domain, the Lagrangian for a self-interacting scalar field may be written

$$\mathscr{L} = \frac{1}{2}\phi(\Box + m^2)\phi + \frac{\lambda\phi^4}{4!}. \tag{II.1}$$

We have chosen

$$\Box\phi = -\partial_\lambda\partial^\lambda\phi,$$

so that $\Box$ is a positive operator and we shall work in a general dimension d in order to use later dimensional regularization.

We suppose that ϕ is defined in a box of volume $V = L^3$ at a temperature β^{-1} so that the spectra of the two operators P′ and P which we shall define below are discrete, but we shall suppose that L and β are sufficiently large that we may neglect all effects of order L^- or β^{-1}.

Let J be an external source and Z[J] the partition function. The classical field ϕ_c is given by

$$\phi_c = -\frac{\delta W}{\delta J}; \qquad W[J] = -\log Z[J]. \tag{II.2}$$

The effective action, or free energy $\Gamma[\phi_c]$ is defined by the Legendre transformation

$$\Gamma[\phi_c] = W[J] + \int J\phi_c. \tag{II.3}$$

The first-order quantum fluctuations around ϕ_c are determined by the eigenvectors of the operator $m^2(P' + 1)$ where

$$P' \equiv \frac{\Box}{m^2} + \frac{\lambda\phi_c^2}{2m^2}, \tag{II.4}$$

and the one-loop expression for the effective action is [*11*]

$$\Gamma[\phi_c] = S[\phi_c] + \frac{1}{2}\operatorname{Tr}\log\left(\frac{P'+1}{P+1}\right). \tag{II.5}$$

S is the classical action, and $P = \Box/m^2$.

Let $\nu'_n[\phi_c]$ (ν_n) be the eigenvalues of P′ (P). Then we have

$$\Gamma = S + \frac{1}{2}\sum_0^\infty [\log(1 + \nu'_n) - \log(1 + \nu_n)]. \tag{II.6}$$

In writing (II.5) and (II.6) it is implicitly supposed that ϕ_c is sufficiently smooth and small that the two spectra $\{\nu'_n\}$ and $\{\nu_n\}$ are in one-to-one correspondance.

The expression (II.6) gives Γ in terms of a divergent series. A finite value may be assigned to Γ by way of analytic continuation using a generalization of the Riemann ζ-function. One defines for $s \in \mathbb{C}$

$$\zeta'(s) = \sum_{0}^{\infty} (1 + \nu'_n)^{-s}, \tag{II.7}$$

and similarly $\zeta(s)$ in terms of $\{ \nu_n \}$. These series converge for

$$\operatorname{Re} s > \overline{\lim} \left(\frac{\log n}{\log \nu'_n} \right) = \overline{\lim} \left(\frac{\log n}{\log \nu_n} \right) = \frac{d}{2} \tag{II.8}$$

and therefore define analytic functions in this region. See, for example, reference [*3*] for a discussion of Dirichlet series and their convergence properties.

The functions ζ' and ζ possess analytic continuations to a neighbourhood of $S = 0$ [*13*]. One defines therefore the effective action by $\Gamma = \Gamma(0)$, where

$$\Gamma(s) = S - \frac{1}{2}\left(\frac{d\zeta'}{ds} - \frac{d\zeta}{ds} \right). \tag{II.9}$$

In order to effectively compute $\zeta'(s)$ in terms of ϕ_c and its derivatives, use must be made of the Mellin transform of a generalization of the θ-function. Define

$$\theta'(t) = \sum_{0}^{\infty} e^{-\nu'_n t}, \tag{II.10}$$

and similarly $\theta(t)$ in terms of $\{ \nu_n \}$. Then

$$\zeta'(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \theta'(t). \tag{II.11}$$

The function $\theta'(t)$ may be computed in terms of the Fourier transform or symbol $\sigma(P')$ of P' by the formula [*2*]

$$\theta'(t) = \int \sigma(e^{-tP'}) \frac{d^dk}{(2\pi)^{d/2}} \frac{d^dx}{(2\pi)^{d/2}}. \tag{II.12}$$

The symbol of P' is given by

$$\sigma(P') = \frac{k^2}{m^2} + \frac{\lambda \phi_c^2}{2m^2}. \tag{II.13}$$

The difficult part of the calculation in general is to calculate $\sigma(e^{-tP'})$ in terms of $e^{-t\sigma(P')}$. However, here we are only interested in the effective potential contribution to the effective action Γ. We may therefore neglect derivatives of ϕ_c and we have the equality

$$\sigma(e^{-tP'}) = e^{-t\sigma(P')}. \tag{II.14}$$

The preceeding six formulae, (II.9) to (II.14), now give $\Gamma(s)$ in terms of a known quantity (II.13). For example, from (II.12, 13, 14)

$$\theta'(t) = \frac{m^d}{(2t)^{d/2}} \int e^{-t\lambda\phi_c^2/2m^2} \frac{d^dx}{(2\pi)^{d/2}}. \tag{II.15}$$

From (II.11, 15)

$$\zeta'(s) = \frac{m^d}{2^{d/2}} \frac{\Gamma(s-d/2)}{\Gamma(s)} \int (1 + \lambda\phi_c^2/2m^2)^{d/2-s} \frac{d^dx}{(2\pi)^{d/2}}. \tag{II.16}$$

From (II.9, 16) one can calculate the effective potential. For $d = 4$ we find [9]

$$\begin{aligned} V[\phi] &= \frac{m_R^2}{2}\phi^2 + \frac{\lambda_R\phi^4}{4!} \\ &+ \frac{1}{64\pi^2}\left\{\left(m_R^2 + \frac{\lambda_R\phi^2}{2}\right)^2 \log\left(1 + \frac{\lambda_R\phi^2}{2m_R^2}\right) - \frac{\lambda_R m_R^2}{2}\phi^2 - \frac{3}{8}\lambda_R^2\phi^4\right\}. \end{aligned} \tag{II.17}$$

where we have replaced m by m_R, defined by

$$m_R^2 \equiv \left.\frac{d^2V}{d\phi^2}\right|_{\phi=0} = m^2\left(1 - \frac{\lambda}{32\pi^2}\right), \tag{II.18}$$

and λ by λ_R :

$$\lambda_R \equiv \left.\frac{d^4V}{d\phi^4}\right|_{\phi=0} = \lambda. \tag{II.19}$$

The correction to m^2 in (II.18) comes from the tadpole and we shall see in Section IV that quite independent of the effect of higher order graphs, there is no reason to ascribe a physical significance to the value $\lambda = 32\pi^2$.

The mass and coupling constant renormalizations are ultra-violet finite but the limit $m \to 0$ is still infra-red singular. If we shift the field

$$\phi^2 \to \phi^2 - 2m^2/\lambda,$$

and define the coupling constant λ_R by

$$\lambda_R \equiv \left.\frac{d^4V}{d\phi^4}\right|_{\phi=M} = \lambda\left(1 + \frac{\lambda}{4\pi^2} + \frac{3\lambda}{32\pi^2}\log\frac{\lambda M^2}{2m^2}\right), \tag{II.20}$$

then [9]

$$V[\phi] = \frac{\lambda_R\phi^4}{4!} + \frac{\lambda_R^2\phi^4}{256\pi^2}\left(\log\frac{\phi^2}{M^2} - \frac{25}{6}\right) + o(m^2).$$

λ_R is singular as m tends to zero.

III

Before discussing analytic renormalization, it is of interest to compare the ζ-function result (II.18), for example, with the corresponding result using dimensional regularization. To the lowest order in λ, from (II.16) we have

$$\zeta'(s) - \zeta(s) = \frac{-\lambda}{2} \frac{m^{d-2}}{2^{d/2}} \frac{\Gamma(1 + s - d/2)}{\Gamma(s)} \int \phi^2 \frac{d^d x}{(2\pi)^{d/2}}. \qquad \text{(III.1)}$$

From (II.9), this yields a contribution $\mathrm{I}(d; s)$ to the effective potential given by

$$\mathrm{I}(d; s) = \frac{\lambda\phi^2}{4} \frac{m^{d-2}}{2^{d/2}} \frac{1}{(2\pi)^{d/2}} \frac{d}{ds}\left(\frac{\Gamma(1 + s - d/2)}{\Gamma(s)}\right)\Bigg|_{s=0}. \qquad \text{(III.2)}$$

If we keep $d \neq 4$ and set $s = 0$ we obtain the dimensionally regularized contribution of the tadpole to the effective potential and the limit $d \to 4$ is singular:

$$\lim_{s \to 0} \mathrm{I}(d; s) = \frac{\lambda\phi^2}{4} \frac{m^{d-2}}{2^{d/2}} \frac{1}{(2\pi)^{d/2}} \Gamma(1 - d/2) = \frac{\lambda\phi^2}{4} \int \frac{1}{k^2 + m^2} \frac{d^d k}{(2\pi)^d}. \qquad \text{(III.3)}$$

On the other hand, if we set $d = 4$ and then take the limit $s \to 0$ we obtain the mass correction in (II.18):

$$\lim_{s \to 0} \lim_{d \to 4} \mathrm{I}(d; s) = -\frac{\lambda\phi^2}{32\pi^2} \frac{m^2}{2}. \qquad \text{(III.4)}$$

The two one-loop divergent graphs in the theory (II.1) are the tadpole and the contribution to the 4-point function. The former involves the divergent integral

$$\mathrm{I} = \int \frac{d^4 k}{k^2 + m^2}. \qquad \text{(III.5)}$$

Analytic regularization consists in replacing the propagator by

$$\frac{m^{2s}\Gamma(s + 1)}{(k^2 + m^2)^{1+s}} = m^{2s} \int_0^\infty t^s e^{-t(k^2 + m^2)} dt. \qquad \text{(III.6)}$$

The integral I is then replaced by

$$\mathrm{I}(s) = m^{2s} \int_0^\infty \int t^s e^{-t(k^2 + m^2)} d^4 k\, dt = m^2 \pi^2 \Gamma(s - 1).$$

ζ-function renormalization consists in then defining the renormalized value of the integral (III.5) as

$$\mathrm{I} = \frac{d}{ds}\left(\frac{\mathrm{I}(s)}{\Gamma(s)}\right)_{s=0} = -m^2\pi^2 \qquad \text{(III.7)}$$

which yields the mass correction in (II.18). If $\mathrm{I}(s)$ were finite at $s = 0$, the prescription (III.7) would define I as $\mathrm{I}(0)$.

The one-loop contribution to the 4-point function contains the divergent integral

$$I = \int \frac{d^4k}{(k^2 + m^2)((k - q)^2 + m^2)}. \qquad \text{(III.8)}$$

A slightly modified version of analytic regularization consists in replacing I by

$$\begin{aligned} I(s) &= m^{2s} \int_0^\infty (a_1 + a_r)^s e^{-m^2(a_1 + a_2)} \int e^{-(k^2 a_1 + (k - q)^2 a_2)} d^4k da_1 da_2 \\ &= \pi^2 \Gamma(s) \int_0^1 (1 + \lambda(1 - \lambda) q^2/m^2)^{-s} d\lambda . \end{aligned}$$

ζ-function renormalization consists in then defining the renormalized value of the integral (III.8) as

$$I = \frac{d}{ds}\left(\frac{I(s)}{\Gamma(s)}\right)_{s=0} = -\pi^2 \int_0^1 \log (1 + \lambda(1 - \lambda) q^2/m^2) d\lambda . \qquad \text{(III.9)}$$

This yields, for $q^2 = 0$, the equality (II.19), to within one-loop corrections.

IV

We now turn to QED and the problem of gauge-invariance. Schwinger has given a gauge-invariant derivation [*10*] of the calculation of Euler and Heisenberg [*7*] of the effect of electron-positron quantum fluctuations around a constant classical electromagnetic field. We shall very briefly recall this result using the ζ-function technique since it differs little from Schwinger's.

In the Euclidean domain with $d = 4$ we may write

$$\frac{1}{2} \gamma_{[\mu}\gamma_{\nu]} = \begin{pmatrix} \sigma^-_{\mu\nu} & 0 \\ 0 & \sigma^+_{\mu\nu} \end{pmatrix},$$

where $(\sigma^-_{\mu\nu})$ $\sigma^+_{\mu\nu}$ is (anti-) self-dual. Decompose the electromagnetic field tensor $F_{\mu\nu}$ into self-dual and anti-self-dual parts and let $a^\pm$ be their norms:

$$F^\pm_{\lambda\mu} = \frac{1}{2}(F_{\lambda\mu} \pm \overset{*}{F}_{\lambda\mu}) ; \qquad (a^\pm)^2 = F^\pm_{\lambda\mu} F^{\pm\lambda\mu} . \qquad \text{(IV.1)}$$

Define the operators $P^\pm$ by

$$P^\pm = \frac{\square}{m^2} - \frac{2ie}{m^2} A^\lambda \partial_\lambda + \frac{e^2 A^2}{m^2} - \frac{ie}{2m^2} \sigma^{\pm\lambda\mu} F^\pm_{\lambda\mu} \qquad \text{(IV.2)}$$

in the Lorentz gauge $\partial_\lambda A^\lambda = 0$. Let $\nu^\pm_n$ be the eigenvalues of $P^\pm$ and ν_n the corresponding free eigenvalues:

$$\nu_n = \lim_{a\pm \to 0} \nu^\pm_n .$$

One can show that $v_n^+ = v_n^- \equiv v_n'$; each eigenvalue has multiplicity 2. Then, as in Section II, the effective action is

$$\Gamma = S - \sum_0^\infty [\log(1 + v_n') - \log(1 + v_n)] \tag{IV.3}$$

The minus sign is due to the fact that there are now fermions in the loops.

As in Section II, one defines the generalized ζ-function, $\zeta'(s)$ and the generalized θ-function, $\theta'(s)$ and one calculates the latter by the formulae corresponding to (II.12). Now, however, the potential A_λ is never constant and one has no longer the equality corresponding to (II.14). One has instead in the case where $F_{\lambda\mu}$ is constant the formula [10]

$$\operatorname{Tr} \int \sigma(e^{-tP\pm}) \frac{d^4k}{(2\pi)^2} = \frac{e^2}{8} [(a^+)^2 - (a^-)^2] \frac{\cosh\left(\frac{tea^+}{m^2}\right) + \cosh\left(\frac{tea^-}{m^2}\right)}{\cosh\left(\frac{tea^+}{m^2}\right) - \cosh\left(\frac{tea^-}{m^2}\right)} \tag{IV.4}$$

Using this and the steps outlined is Section II one obtains for example, when the magnetic field vanishes, that is, when

$$a^+ = a^- = a,$$

the following expression for the effective Lagrangian:

$$\mathscr{L} = \frac{1}{4} F^2 + \frac{d}{ds} \left\{ \frac{m^4}{8\pi^2} \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-3} \left(\frac{tea}{m^2} \coth\left(\frac{tea}{m^2}\right) - 1 \right) dt \right\}_{s=0} . \tag{IV.5}$$

Rotating back to Minkowski space, $a = i(E + i\varepsilon)$ yields a ζ-function regularized expression for a particular case of Schwinger's formula (3.44) [10]. See also [12].

This result is gauge invariant. However if one attempts to develop a set of rules to regularize even the simple Feynman graphs analogous to those discussed in Section II, one finds that gauge-invariance is broken. For example consider the one-loop photon self-energy contribution. It contains the divergent integral (in the Euclidean domain)

$$I_{\mu\nu}(q) = \frac{i\alpha}{4\pi^3} \int \frac{\operatorname{Tr}(\gamma_\mu(i\not{k} - m)\gamma_\nu((i\not{k} - i\not{q}) - m))}{(k^2 + m^2)((k - q)^2 + m^2)} d^4k$$

If one attempts to regularize this integral as we have regularized (III.8), one finds that the renormalized value, given by an expression of the form (III.9) when written in the form

$$I_{\mu\nu}(q) = \frac{2i\alpha}{\pi}(q_\mu q_\nu - q^2\delta_{\mu\nu})I + \frac{i\alpha}{\pi}\delta_{\mu\nu}D$$

has $$D = -(m^2 + q^2/6) \neq 0.$$

The reason for this may be seen by considering the field theory which gives rise to the regularized Feynman integrals. The regularized values of the two integrals which we discussed in Section II may be thought of as the ordinary Feynman integrals given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\phi(\Box + m^2)^{1+s}\phi + \frac{\lambda\phi^4}{4!} \tag{IV.6}$$

That is, it is possible to change the propagator without changing the vertex. Any attempt to find a similar higher-order Lagrangian generalizing the QED Lagrangian, must be for example of the form

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \mathrm{D}_\mu + m)^{1+s}\psi \tag{IV.7}$$

if gauge-invariance is to be maintained. That is, it is not possible to change the propagator without also changing the vertex. For non-integral values of s, the Lagrangian (IV.7) gives rise to a non-local interaction.

To conclude, we would like to remark that the tadpole contribution to the 2-point function using Lagrangian (IV.6) is finite if $s = 2$. The most physical theory therefore with $d = 4$ of which the result (III.4) is an analytic continuation is a 6th-order theory.

ACKNOWLEDGMENT

The author would like to thank J. L. Richard and Larry Trueman for enlightening discussions.

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(Manuscrit reçu le 8 novembre 1979)