ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 35, nº 3 (1981), p. 195-224

<http://www.numdam.org/item?id=AIHPA_1981__35_3_195_0>

Ann. Inst. Henri Poincaré,
Vol. XXXV, n° 3, 1981, p. 195-224.

Section A :
Physique théorique.

Partial inner product spaces of entire functions

by

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ABSTRACT. — We investigate various partial inner product space generalizations of Bargmann's Hilbert space of entire functions as used in the coherent state representation of Quantum Mechanics. In particular, we exhibit a hierarchy of nested Hilbert spaces, the smallest of them being Bargmann's original scale.

RÉSUMÉ. — Nous étudions différents espaces à produit interne partiel qui généralisent l'espace hilbertien de fonctions entières introduit par Bargmann et utilisé dans la représentation « états cohérents » de la mécanique quantique. Nous obtenons, en particulier, une hiérarchie d'espaces « nestés » (au sens de Grossmann), le plus petit d'entre eux étant l'échelle originelle de Bargmann.

1. INTRODUCTION

A partial inner product (PIP) space is a vector space equipped with a nondegenerate Hermitian form $\langle \cdot | \cdot \rangle$, defined on particular pairs of vectors, the so-called compatible vectors. A systematic study of such objects may be found in a series of papers by A. Grossmann and one of us [*1*]-[*4*]; these papers will be quoted below as I-IV respectively. Here

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we will restrict ourselves to the case where the form $\langle \cdot | \cdot \rangle$ is positive definite, and we will assume, moreover, the existence of a central Hilbert space. In most cases, the latter property follows in fact from positivity, as shown in IV.

Such PIP-spaces are a natural generalization of Hilbert spaces. Of course the latter are all unitarily equivalent (we consider only infinite dimensional separable spaces), but their concrete realizations may vary considerably. Three types are most often considered: the space l^2 of square integrable sequences, the space $L^2(X, \mu)$ of square integrable functions over a measure space (X, μ) and the Hilbert space $\mathfrak{H}(\Omega)$ of functions $f : \mathbb{C} \to \mathbb{C}$, analytic in a domain $\Omega \subset \mathbb{C}$ and square integrable on Ω with respect to some measure. A well-known example is Bargmann's space $\mathfrak{F}$ of all entire functions square integrable for the Gaussian measure [*5*] [*6*]. Now l^2 leads to a natural PIP-space structure on the space ω of all complex sequences, and $L^2(X, \mu)$ leads to the PIP-space $L^1_{loc}(X, \mu)$ of all μ-locally integrable functions on X. Both structures have been discussed extensively in the papers I-IV. In this paper we study the corresponding PIP-space generalization of Bargmann's space $\mathfrak{F}$.

The motivations for such an analysis are multiple. First, Bargmann's approach yields a very elegant description of tempered distributions, representing Schwartz's space $\mathcal{S}'$ as a continuous scale of Hilbert spaces (there is also a Banach space realization), which is itself a very simple example of PIP-space. This realization has been extended by Grossmann [*7*] to cover some classes of nontempered distributions as well. Here we are able to go much farther in that direction.

A second motivation lies in the existence of Bargmann's integral transform, which plays a fundamental rôle in the so-called Weyl quantization. The latter establishes a correspondence between (classical) functions on phase space and (quantum) operators on a Hilbert space. The main problem is to identify the type of functions that are mapped on a given class of operators, and *vice versa*. What is needed is a detailed parametrization of such classes of functions or operators, and the concept of PIP-space is ideally suited for that purpose. The beautiful results obtained recently by Daubechies [*8*]-[*10*] and Daubechies and Grossmann [*11*] are an eloquent proof of that statement.

A third area where a PIP-space analysis looks promising is the problem of representations of complex canonical transformations, in the framework of the phase space approach to Quantum Mechanics. Here again we refer to recent work by Daubechies [*8*]-[*10*].

The paper is organized as follows. In Sections 2 and 3, we study the space $\mathfrak{Z}$ of all entire functions. The result is that no global PIP-space structure on $\mathfrak{Z}$ is fully satisfactory. Roughly speaking, the growth at infinity of the functions must be somehow restricted if one is to get a nontrivial and useful structure. This is best achieved by considering, instead of $\mathfrak{Z}$

itself, a lattice of Hilbert (assaying) subspaces of $\mathfrak{Z}$, i. e. building a nested Hilbert space [12] in $\mathfrak{Z}$, centered around $\mathfrak{F}$. Candidates for such subspaces are analyzed in Section 4, namely weighted L^2-spaces of entire functions. Typically such a space, denoted $\mathfrak{F}(\rho)$, consists of all entire functions of $z \in \mathbb{C}$, square integrable with respect to the measure $\exp[-\rho(z)]d\mu(z)$, where μ is the (normalized) Gaussian measure on $\mathbb{C}$ and ρ is a real-valued, measurable function, bounded on every compact subset of $\mathbb{C}$. Following Grossmann [13], who introduced them first, we call such a function ρ a (logarithmic) weight. The problem is to find adequate conditions on the weights ρ such that the spaces $\mathfrak{F}(\rho)$ have all the properties needed for generating a nested Hilbert space. Once this information is obtained, we proceed to build in Section 5 a hierarchy of four possible lattices; the smallest of these is Bargmann's original scale, corresponding to tempered distributions. Four appendices conclude the paper, including a discussion of some pathological examples and an account of the solution to Stieltjes' moment problem, which is used in the text.

2. SPACES OF ENTIRE FUNCTIONS

Our starting point is the space $\mathfrak{F}$ of Bargmann [5] [6] (also variously attributed to Fock, Segal or Fischer). Let $\mathfrak{Z}$ be the space of all entire (holomorphic) functions of one complex variable $z \in \mathbb{C}$. Denote by μ the normalized Gaussian measure on the complex plane (we always identify $\mathbb{C} \equiv \mathbb{R}^2$):

$$d\mu(z) = \pi^{-1} \exp(-|z|^2)dz$$

with $z = x + iy$ and $dz \equiv dxdy$. The space $\mathfrak{F}$ is then defined as the intersection $\mathfrak{Z} \cap L^2(\mathbb{C}, \mu)$:

$$\mathfrak{F} = \left\{ f \in \mathfrak{Z} \,\middle|\, \int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty \right\} \tag{2.1}$$

With the corresponding inner product:

$$\langle f | g \rangle = \int_{\mathbb{C}} \overline{f(z)} g(z) d\mu(z) \tag{2.2}$$

$\mathfrak{F}$ is a Hilbert space, i. e. it is complete and, in fact, a closed subspace of $L^2(\mathbb{C}, \mu)$. An orthonormal basis in $\mathfrak{F}$ is given by the functions:

$$u_n(z) \equiv z^n(n!)^{-1/2} \qquad n = 0, 1, 2 \ldots \tag{2.3}$$

Expanding two elements $f, g \in \mathfrak{F}$ in that basis brings the inner product into the equivalent form:

$$\langle f | g \rangle = \sum_{n=0}^{\infty} \overline{f_n} g_n n! = \sum_{n=0}^{\infty} \overline{\alpha_n} \beta_n \tag{2.4}$$

where we have put:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \alpha_n u_n(z), \qquad \alpha_n = f_n(n\,!)^{1/2}$$

$$g(z) = \sum_{n=0}^{\infty} g_n z^n = \sum_{n=0}^{\infty} \beta_n u_n(z), \qquad \beta_n = g_n(n\,!)^{1/2}.$$

Furthermore, $\mathfrak{F}$ contains a total set consisting of the so-called *principal vectors* $\{ e_w, w \in \mathbb{C} \}$, defined by the relation:

$$f(w) = \langle e_w \mid f \rangle, \qquad \forall f \in \mathfrak{F}. \tag{2.5}$$

They are simply the familiar coherent states:

$$e_w(z) = e^{\overline{w}z} \tag{2.6}$$

In other words, $\mathfrak{F}$ is a Hilbert space with *reproducing kernel* [*14*], namely:

$$\mathrm{K}(w, z) = \exp(w\overline{z})$$

$$f(w) = \int_{\mathbb{C}} e^{w\overline{z}} f(z) d\mu(z) \tag{2.7}$$

Since our aim is to embed (as central Hilbert space) $\mathfrak{F}$ in a PIP-space, we shall first have a closer look at the obvious candidate, namely the space $\mathfrak{Z}$.

As is well-known [*15*]-[*17*] the space $\mathfrak{Z} \equiv \mathfrak{H}(\mathbb{C})$ of all entire functions is a nuclear Fréchet space (hence reflexive and Montel) for the topology of uniform convergence on compact sets, i. e. for the norms:

$$\| f \|_k = \max_{|z|=k} | f(z) |, \qquad k = 1, 2, 3 \ldots$$

Accordingly, the anti-dual $\mathfrak{Z}^\times$, i. e. the space of all antilinear continuous functionals on $\mathfrak{Z}$ (the so-called anti-analytic functionals), with its strong dual topology, is a complete, nonmetrizable, nuclear (DF)-space, in particular it is also reflexive and Montel (see also the Appendix of IV for the terminology).

We will need also the space *Exp* of all entire functions of exponential type:

$$f \in Exp \Leftrightarrow \exists a \geqslant 0, \qquad c > 0 \qquad \text{s. t.} \qquad | f(z) | \leqslant c e^{a|z|}, \qquad \forall z \in \mathbb{C}.$$

This space has a natural topology, strictly finer than that induced by $\mathfrak{Z}$, as a union (inductive limit) of Fréchet spaces, for which it is a complete space of type (DF). Thus one has:

$$Exp \subset \mathfrak{F} \subset \mathfrak{Z} \tag{2.8}$$

where each space is complete in its own topology, but dense in the next one. Notice that $\{ e_w, w \in \mathbb{C} \}$ is a total set of $\mathfrak{Z}$ contained in *Exp*. Conversely every function of exponential type may be majorized by a finite linear combination of elements e_w.

It is a standard result that $\mathfrak{Z}^\times$, with its strong topology, is isomorphic to the space *Exp* with its natural topology. The isomorphism is given by the Fourier-Borel (or Laplace) transform $\mu \mapsto \hat{\mu}$ defined as:

$$\hat{\mu}(w) = \langle \mu, e_w \rangle \tag{2.9}$$

where the r. h. s. is the value taken by $\mu \in \mathfrak{Z}^\times$ on the element $e_w \in \mathfrak{Z}$. Comparing this relation with (2.5), (2.7), we see that the restriction of the Fourier-Borel transform to $\mathfrak{F}$ is simply the identity operator.

Since $\mathfrak{Z}$ is reflexive, we have also $(Exp)^\times \simeq \mathfrak{Z}$. Since both spaces *Exp*, $\mathfrak{Z}$ are nuclear and complete, and dual of each other, it follows that the triplet (2.8) is a Rigged Hilbert Space in the sense of Gel'fand and Vilenkin [*18*].

It is illuminating to translate these results in the language of sequences. Indeed, by identification of an entire function with its Taylor coefficients, we may realize $\mathfrak{Z}$ as a space of complex sequences:

$$f(z) \equiv \sum_{n=0}^{\infty} f_n z^n \in \mathfrak{Z} \Leftrightarrow \forall k = 1, 2, \ldots, \quad \sum_{n=0}^{\infty} k^n \, | f_n | < \infty$$
$$\Leftrightarrow \limsup_{n\to\infty} | f_n |^{1/n} = 0 \tag{2.10}$$

Similarly for $Exp \simeq \mathfrak{Z}^\times$:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \in Exp \Leftrightarrow \exists k > 0 \quad \text{s. t.} \sup_n k^{-n} n \, ! \, | f_n | < \infty \tag{2.11}$$

Then the antiduality between *Exp* and $\mathfrak{Z}$ is given by the sesquilinear form

$$\langle f \,|\, g \rangle = \sum_{n=0}^{\infty} \overline{f_n} g_n n \, !, \qquad f \simeq (f_n) \in Exp, \qquad g \simeq (g_n) \in \mathfrak{Z}$$

that is, precisely, the partial inner product which is inherited from the space ω of all sequences, and extends the inner product of $\mathfrak{F}$, as given in (2.4).

3. COMPATIBILITIES ON SPACES OF ENTIRE FUNCTIONS

Our aim is to embed Bargmann's space $\mathfrak{F}$, as central Hilbert space, in a partial inner product space. A first step in this direction was made by

Bargmann himself [6], with the scale of Hilbert spaces $\{\mathfrak{F}^\rho, -\infty<\rho<\infty\}$. This approach proves to be a convenient substitute for the standard (Schwartz) formalism of the theory of distributions.

Since $\mathfrak{F}$ is the intersection of $L^2(\mathbb{C}, \mu)$ and $\mathfrak{Z}$, it inherits two natural notions of linear compatibility (see III). The first one, to be denoted by $\#_1$, is the usual compatibility on spaces of measurable functions:

$$f \#_1 g \Leftrightarrow \int_{\mathbb{C}} | f(z)g(z) | \, d\mu(z) < \infty \tag{3.1}$$

The other comes from the identification, made above, of $\mathfrak{F}$ and $\mathfrak{Z}$ as spaces of sequences:

$$f \#_2 g \Leftrightarrow \sum_{n=0}^{\infty} | f_n g_n | \, n\,! < \infty \tag{3.2}$$

with

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} g_n z^n$$

As shown by Bargmann [6] the two compatibilities $\#_1$ and $\#_2$ coincide and yield the same (partial) inner product on $\mathfrak{F}$ and, more generally, on $\mathfrak{E}' \equiv \bigcup_{\rho\in\mathbb{R}} \mathfrak{F}^\rho$. However, this is no longer true is general.

Proposition 3.1. — The two linear compatibilities $\#_1$ and $\#_2$ are not comparable on $\mathfrak{Z}$.

Proof. — It is sufficient to consider exponential functions

$$f(z) = \exp\left(\frac{1}{2}\gamma z^2\right) \qquad \text{and} \qquad g(z) = \exp\left(\frac{1}{2}\delta z^2\right).$$

One has indeed [5]:

$$\begin{aligned} f \#_1 g \quad &\text{iff} \quad |\gamma + \delta| < 2 \\ f \#_2 g \quad &\text{iff} \quad |\gamma\delta| < 1\,. \end{aligned}$$

If we choose $\gamma_1 = 1, \delta_1 = i$, then $f_1 \#_1 g_1$, but f_1 and g_1 are not $\#_2$-compatible. On the other hand, for $\gamma_2 = 2$, $\delta_2 = 1/4$, $f_2 \#_2 g_2$, but f_2 and g_2 are not $\#_1$-compatible. Hence $\#_1$ and $\#_2$ are not comparable on $\mathfrak{Z}$. □

Thus we have to study the two compatibilities separately. Let us begin with $\#_1$ as given by (3.1). This is the restriction to $\mathfrak{Z}$ of the natural compatibility $\#$ on the space $V = L^1_{loc}(\mathbb{C}, \mu)$ of locally integrable functions $f(z, \bar{z}) \equiv f(x, y)$ on $\mathbb{C} \equiv \mathbb{R}^2$. In the corresponding PIP-space structure, we have as usual $V^\# = L^\infty_{comp}(\mathbb{C}, \mu)$, the essentially bounded functions of compact support, and the central Hilbert space is $L^2(\mathbb{C}, \mu)$:

$$L^\infty_{comp}(\mathbb{C}, \mu) \subset L^2(\mathbb{C}, \mu) \subset L^1_{loc}(\mathbb{C}, \mu) \tag{3.3}$$

Taking the intersection of each space with $\mathfrak{Z}$, we get:

$$\{0\} \subset \mathfrak{F} \subset \mathfrak{Z} \tag{3.4}$$

Indeed every entire function is locally integrable, even locally bounded ($\mathfrak{Z} \subset L^\infty_{loc} \subset L^1_{loc}$), but it may have an arbitrary growth at infinity. Thus any function $f \in L^1_{loc}$ compatible with every $g \in \mathfrak{Z}$ must have compact support: $\mathfrak{Z}^\# = L^1_{comp}$. Therefore we have:

PROPOSITION 3.2. — Let $\#$ be the natural compatibility on $L^1_{loc}(\mathbb{C}, \mu)$, $\#_1$ its restriction to $\mathfrak{Z}$. Then $\mathfrak{Z}^{\#_1} = \mathfrak{Z}^\# \cap \mathfrak{Z} = \{0\}$. □

Remark. — It is easy to prove the proposition directly. One may, for instance, show that $f \#_1 (\exp az^2)$ for all $a \in \mathbb{C}$ implies $f = 0$.

$\mathfrak{F}$ is a closed subspace of $L^2(\mathbb{C}, \mu)$. As for every Hilbert space of analytic functions [*14*], the corresponding projection operator is given by the reproducing kernel; thus for any $f \in L^2$, its projection on $\mathfrak{F}$ is:

$$(Pf)(w) = \langle e_w | f \rangle \tag{3.5}$$

(in other words, P extends the Fourier-Borel transform to $L^2(\mathbb{C}, \mu)$.) Then the relation $\mathfrak{Z}^{\#_1} = \{0\}$ means that P, although it is an orthogonal projection in L^2, does *not* extend to a projection operator in the PIP-space $(L^1_{loc}, \#)$; thus there is no way of obtaining a PIP-space structure on $\mathfrak{Z}$ by projecting the one on L^1_{loc}.

A possible answer to this difficulty is to restrict the analysis from $\mathfrak{Z}$ to the subspace $\mathfrak{Y} \equiv (Exp)^{\#_1}$. Of course one has:

$$Exp \subset \mathfrak{F} \subset (Exp)^{\#_1} \subset \mathfrak{Z}$$

This subspace $\mathfrak{Y}$ has many interesting properties. For instance:

i) Since *Exp* is generated by the principal vectors $\{e_a\}$ (Sect. 2), it follows that $(Exp)^{\#_1} = \bigcap_{a \in \mathbb{C}} \{e_a\}^{\#_1}$, i. e. $f \in \mathfrak{Y}$ iff it is $\#_1$-compatible with every e_a.

ii) As a consequence, $\mathfrak{Y}$ is exactly the trace on $\mathfrak{Z}$ of the domain (in L^1_{loc}) of the operator P which projects L^2 onto $\mathfrak{F}$: $\mathfrak{Y} = D(P) \cap \mathfrak{Z}$.

iii) Correspondingly, $\mathfrak{Y}$ is the subspace of $\mathfrak{Z}$ on which the (inverse) Fourier-Borel transform reduces to the identity, i. e. the set of functions f which have the reproduction property: $f(z) = \langle e_z | f \rangle$. This fact has been noticed by Krée and Rączka (Ref. [*16*], Lemma 4.6) under the following form: for any $T \in (Exp)^\times$ such that $\hat{T} \in (Exp)^{\#_1}$, one has $\langle T, h \rangle = \langle h | \hat{T} \rangle$ for every $h \in Exp$.

iv) As suggested by Grossmann [*19*], $\mathfrak{Y}$ seems the natural space for studying the representations of complex canonical transformations.

Unfortunately $\mathfrak{Y} = (Exp)^{\#_1}$ is still too large. Here again the compatibilities $\#_1$ and $\#_2$ are not comparable; indeed the argument of Proposition 3.1 still applies, since all functions $\exp\left(\frac{1}{2}\gamma z^2\right)$ belong to $\mathfrak{Y} = (Exp)^{\#_1}$ for $|\gamma| < 2$. Thus there seems to be little hope of building a decent PIP-space structure on $\mathfrak{Y}$.

We turn now to the compatibility $\#_2$ inherited from ω. As could be guessed from the topological structure described in Sect. 2, $\mathfrak{Z}$ and *Exp*, identified with the corresponding spaces of sequences, form an involutive pair of assaying subsets under $\#_2$.

PROPOSITION 3.3. — Let $\mathfrak{Z}$, resp. *Exp*, denote the spaces of all entire functions, resp. of entire functions of exponential type, both considered as spaces of sequences. Let $\#_2$ be the natural compatibility on complex sequences. Then one has:

$$\mathfrak{Z}^{\#_2} = Exp, \qquad (Exp)^{\#_2} = \mathfrak{Z}.$$

Proof. — Given $g \in \mathfrak{Z}$, let $g \#_2 f$ for every $f \sim (f_n) \in \mathfrak{Z}$, i. e.

$$\sum_{n=0}^{\infty} |f_n||g_n| n! < \infty .$$

Choose an arbitrary positive number K and write $\tilde{f}_n = K^n f_n$, $\tilde{g}_n = K^{-n} g_n n!$. By (2.10), $\sum_{n=0}^{\infty} |\tilde{f}_n| < \infty$. Thus $\sum_{n=0}^{\infty} |\tilde{f}_n||\tilde{g}_n| < \infty$, for every sequence $(\tilde{f}_n) \in l^1$. It follows that $(\tilde{g}_n) \in (l^1)^{\#_2} = l^\infty$. Since this result holds for every $K > 0$, it follows by (2.11) that $g \in Exp$. This proves that $\mathfrak{Z}^{\#_2} \subset Exp$. The converse is obvious. The other assertion if proven in the same way. □

Remark. — Here again a direct proof, i. e. without recourse to the involutive pair (l^1, l^∞), is easy to give; like the one above, it amounts to an application of the principle of uniform boundedness.

At this point we face a dilemma. On the one hand, the compatibility $\#_2$ is the natural one on $\mathfrak{Z}$, making it a nondegenerate PIP-space, with the structure inherited from ω. On the other hand, it is the compatibility $\#_1$ that we have to extend from $\mathfrak{F}$ to some larger space $\mathfrak{L} \subset \mathfrak{Z}$, if we want to consider entire functions themselves, not their power expansions. The way out is clear: we have to find a space $\mathfrak{L}$, strictly smaller than $\mathfrak{Z}$, on which the two compatibilities $\#_1$ and $\#_2$ coincide. This goal is best achieved by constructing directly a lattice of assaying subsets, i. e. an indexed PIP-space in the sense of IV. In the next section, we will analyze in detail a class of such assaying subsets, which are in fact Hilbert spaces.

4. HILBERTIAN ASSAYING SUBSPACES OF ENTIRE FUNCTIONS

As a guide for achieving the program just outlined, we return to the PIP-space $L^1_{loc}(\mathbb{C}, \mu)$, as described in III. More precisely we consider the rich subset of all weighted L^2-spaces $\{ L^2(r), r$ measurable, $r > 0$, r and $r^{-1} \in L^1_{loc} \}$ (see also App. B). The weight $r \equiv 1$ corresponds to the central Hilbert space $L^2(\mathbb{C}, \mu)$ and the involution is $[L^2(r)]^{\#} = L^2(r^{-1}) = [L^2(r)]^{\times}$. By restriction from L^1_{loc} to $\mathfrak{Z}$ we get a family of spaces $L^2(r) \cap \mathfrak{Z}$, out of which we will build the nested Hilbert spaces we are looking for. Additional conditions on the weight functions will be needed, since the restriction $L^1_{loc} \to \mathfrak{Z}$ is not a PIP-space projection.

Let ρ be a μ-measurable (which is the same as Lebesgue measurable) real-valued function on $\mathbb{C}$; we consider on $\mathfrak{Z}$ the following norm:

$$\| f \|_\rho^2 = \int | f(z) |^2 e^{-\rho(z)} d\mu(z) \tag{4.1}$$

and the corresponding inner product:

$$\langle f, g \rangle_\rho = \int \overline{f(z)} g(z) e^{-\rho(z)} d\mu(z) \tag{4.2}$$

Then we define $\mathfrak{F}(\rho)$ (1) as the vector space of all $f \in \mathfrak{Z}$ such that $\| f \|_\rho < \infty$, i. e. $\mathfrak{F}(\rho) = L^2(e^\rho) \cap \mathfrak{Z}$. Notice that Bargmann's space $\mathfrak{F}^\rho$ is obtained for the choice $\rho(z) = \log (1 + | z |^2)^{-\rho}$ $(\rho \in \mathbb{R})$ and $\mathfrak{F}(0) \equiv \mathfrak{F}$.

We require this space $\mathfrak{F}(\rho)$ to satisfy four conditions, all of them satisfied for $\mathfrak{F}(0) \equiv \mathfrak{F}$ and every $\mathfrak{F}^\rho$.

i) $\mathfrak{F}(\rho)$ is a Hilbert space, i. e. it is complete. This is not automatic, since the $\| \cdot \|_\rho$-limit of a sequence of entire functions need not be analytic.

ii) The set $\mathfrak{P}$ of all polynomials is dense in $\mathfrak{F}(\rho)$. If the weight function ρ grows too fast at infinity, it might force $\mathfrak{F}(\rho)$ to be trivial, i. e. finite dimensional or $\{ 0 \}$. On the other hand, the corresponding space $\mathfrak{F}(-\rho)$ might be too large. Condition *ii)* will thus be a restriction on the growth of ρ; it has the further advantage that the set of monomials $\{ u_m(z), m=0, 1, 2 \dots \}$ given in Eq. (2.3) will be a basis in $\mathfrak{F}(\rho)$. In case ρ is radial, i. e. ρ depends only on $| z |$, this basis will be orthonormal and this will make the compatibility $\#_2$ easy to handle.

iii) $\mathfrak{F}(\rho)^\times$ is isomorphic to $\mathfrak{F}(-\rho)$. This is in fact the crucial condition,

(1) This space is often denoted $A^2(| z |^2 - \rho(z))$ in the mathematical literature (see e. g. Hantler [*20*]). Its use in the present context was first suggested by Grossmann [*13*], who called ρ a *logarithmic weight*.

which again restricts the growth of ρ. For a radial weight ρ, condition *iii*) will imply that $\#_1$ and $\#_2$ coincide on $\mathfrak{F}(\rho)$.

iv) $e_a \in \mathfrak{F}(\rho)\,, \qquad \forall a \in \mathbb{C}\,.$

Together with *ii*), this last condition implies of course that $\{ e_a \}$ generates a dense subspace of $\mathfrak{F}(\rho)$. From this follows that every operator A in the resulting PIP-space is an integral operator with kernel $A(z, w) = \langle e_z \mid Ae_w \rangle$.

We will now study the four conditions *i*)-*iv*) successively.

4.1. **Completeness of $\mathcal{F}(\rho)$.**

PROPOSITION 4.1. — Let $\rho : \mathbb{C} \to \mathbb{R}$ be measurable and essentially bounded on every compact set, i. e. $\rho \in L^\infty_{loc}(\mathbb{C}, \mu)$. Then $\mathfrak{F}(\rho)$ is complete in the norm $\| \,.\, \|_\rho$, it is a Hilbert space.

Proof. — Under the conditions stated, the proof of Bargmann (Ref. [*6*], Sect. 3.2), which is the standard proof for Hilbert spaces of analytic functions, applies literally. □

Notice that $\rho \in L^\infty_{loc}$ implies $e^{\pm\rho} \in L^\infty_{loc}$, so that both $\mathfrak{F}(\rho)$ and $\mathfrak{F}(-\rho)$ are complete, although they might not be dual of each other with respect to the form $\langle \cdot \mid \cdot \rangle$, unless condition *iii*) is satisfied. In fact, the family of all $\mathfrak{F}(\rho)$, with $\rho \in L^\infty_{loc}$, is rich in $\mathfrak{Z}$, in the following sense (see III): whenever $f, g \in \mathfrak{Z}$ and $f \#_1 g$, there exists a locally bounded weight ρ such that $f \in \mathfrak{F}(\rho)$ and $g \in \mathfrak{F}(-\rho)$. This result follows from the proof given in III, Sect. 4.B and the fact that $\mathfrak{Z} \subset L^\infty_{loc}$.

4.2. **Density of polynomials.**

We consider first the weaker statement $\mathfrak{P} \subset \mathfrak{F}(\rho)$, which already guarantees that $\mathfrak{F}(\rho)$ is nontrivial.

If we define, in analogy with ref. [*6*]:

$$\eta_m^{(\rho)} = \int | u_m(z) |^2 e^{-\rho(z)} d\mu(z) = \| u_m \|_\rho^2 \tag{4.3}$$

then that statement is conveniently rephrased as:

$$\eta_m^{(\rho)} < \infty\,, \qquad m = 0, 1, 2 \,\ldots \tag{4.4}$$

Given (4.4), the density of $\mathfrak{P}$ in $\mathfrak{F}(\rho)$ is a classical problem for Hilbert spaces of analytic functions (see Hantler [*20*] for instance) for which no general solution is known. Only sufficient conditions exist, in the following cases:

a) *subharmonic weights:*

$\varphi \equiv | z |^2 - \rho$ is a subharmonic function (i. e. $\Delta\varphi \geqslant 0$ everywhere in $\mathbb{C}$ in a distribution sense) and $| \rho(z) | \leqslant c(1 + | z |^2)^{1/2}$ for some $c > 0$.

b) *harmonic weights:*

$\rho = 2 \log | h |$, where either h is an exponential polynomial, i. e. $h(z) = \sum_{k=1}^{k} p_k(z) e^{a_k z}$ ($a_k \in \mathbb{C}$, p_k polynomials) or h is an entire function of order less than two without zeros (such weights ρ are obviously harmonic).

c) *radial weights:* $\rho(z) = \rho(|z|)$.

In this case, which is the most interesting, Bargmann's proof (Ref. [*6*], Sect. 3.6) applies literally.

Remark. — Cases *b*) and *c*) almost exclude each other, see Appendix A for a proof.

If the weight ρ is radial, the space $\mathfrak{F}(\rho)$ possesses the following orthonormal basis:

$$u_m^{(\rho)}(z) = (\eta_m^{(\rho)})^{-1/2} u_m(z), \qquad m = 0, 1, 2 \ldots \tag{4.5}$$

where the coefficient $\eta_m^{(\rho)}$ defined in Eq. (4.3) may also be expressed as:

$$\eta_m^{(\rho)} = \frac{1}{m!} \int_0^\infty t^m e^{-\rho(t)-t} dt \tag{4.6}$$

As a consequence, all the spaces $\mathfrak{F}(\rho)$, with ρ radial, may be simultaneously realized as weighted l^2 sequence spaces. One has indeed, for

$$f = \sum_{m=0}^{\infty} \alpha_m u_m, \qquad g = \sum_{m=0}^{\infty} \beta_m u_m:$$

$$\| f \|_\rho^2 = \sum_{m=0}^{\infty} \eta_m^{(\rho)} | \alpha_m |^2 \tag{4.7.a}$$

$$\langle f, g \rangle_\rho = \sum_{m=0}^{\infty} \eta_m^{(\rho)} \overline{\alpha_m} \beta_m \tag{4.7.b}$$

Thus $\mathfrak{F}(\rho) \simeq l^2(\overline{\eta^{(\rho)}})$, where $\overline{\eta^{(\rho)}} \equiv ([\eta_m^{(\rho)}]^{-1})_{m=0}^{\infty}$.

4.3. **Duality.**

Given $\mathfrak{F}(\rho)$, what are the relations between $\mathfrak{F}(-\rho)$, $\mathfrak{F}(\rho)^\times$, $\mathfrak{F}(\rho)^{\#_1}$ and $\mathfrak{F}(\rho)^{\#_2}$?

PROPOSITION 4.2. — Let ρ be bounded on every compact set. Then one has:

$$\mathfrak{F}(-\rho) = \mathfrak{F}(\rho)^{\#_1} \subseteq \mathfrak{F}(\rho)^\times \tag{4.8}$$

If ρ is radial, one has in addition:

$$\mathfrak{F}(\rho)^{\times} = \mathfrak{F}(\rho)^{\#_2} \tag{4.9}$$

Proof. — The inclusion $\mathfrak{F}(-\rho) \subseteq \mathfrak{F}(\rho)^{\times}$ follows from the Schwarz inequality:

$$|\langle f \mid g \rangle| \leqslant \| f \|_{\rho} \| g \|_{-\rho}, \qquad f \in \mathfrak{F}(\rho), \quad g \in \mathfrak{F}(-\rho).$$

The equality $\mathfrak{F}(-\rho) = \mathfrak{F}(\rho)^{\#_1}$ follows from the corresponding relation in L^1_{loc} : $L^2(e^{-\rho}) = [L^2(e^{\rho})]^{\#}$; indeed, since ρ is locally bounded, only the behavior at infinity matters, i. e. $\mathfrak{F}(\rho)^{\#} = [L^2(e^{\rho})]^{\#} = L^2(e^{-\rho})$ (compatibility in L^1_{loc}) and therefore $\mathfrak{F}(\rho)^{\#_1} = \mathfrak{F}(\rho)^{\#} \cap \mathfrak{Z} = L^2(e^{-\rho}) \cap \mathfrak{Z} = \mathfrak{F}(-\rho)$. If ρ is radial, then $\mathfrak{F}(\rho)$ may be realized as $l^2(\eta^{(\rho)})$, and therefore

$$\mathfrak{F}(\rho)^{\times} \simeq l^2(\eta^{(\rho)}) = [l^2(\overline{\eta^{(\rho)}})]^{\#_2}. \qquad \square$$

Remark. — It is amusing to notice that the two relations (4.8), (4.9) are the same as those proven for $\mathfrak{Z}$ in Propositions 3.2 and 3.3:

$$\{0\} = \mathfrak{Z}^{\#_1} \subset \mathfrak{Z}^{\times} = \mathfrak{Z}^{\#_2} = Exp.$$

Eqs. (4.8), (4.9) actually imply these, if one remembers that $\mathfrak{Z} = \bigcup_{\rho} \mathfrak{F}(\rho)$ where ρ ranges over all locally bounded functions, and therefore

$$\mathfrak{Z}^{\#_1} = \bigcap_{\rho} \mathfrak{F}(\rho) = \{0\}.$$

We emphasize that the inclusion in Eq. (4.8) may be strict. We will give explicit examples below (for radial weights). Of course this pathology does not arise in the space L^1_{loc} : $L^2(e^{\rho})$ is always the antidual of $L^2(e^{-\rho})$. The standard proof is to notice that both spaces are unitarily equivalent to L^2, the mapping being multiplication by $\exp\left(\frac{1}{2}\rho\right)$. But this argument fails for spaces of analytic functions, since ρ is real-valued.

One way to rescue it is to consider harmonic weights, $\rho(z) = 2 \operatorname{Re} h(z)$ for some entire function h (any function harmonic in the whole plane is of this form). Then indeed multiplication by $\exp(\pm h)$ maps $\mathfrak{F}$ unitarily onto $\mathfrak{F}(\pm \rho)$. Let now L be an element of the antidual $\mathfrak{F}(\rho)^{\times}$. By the Riesz lemma, the action of L is given by a unique element $g' \in \mathfrak{F}(\rho)$:

$$L(f) = \langle f, g' \rangle_{\rho}, \qquad \forall f \in \mathfrak{F}(\rho).$$

By the unitary equivalence, there is a unique element $g \in \mathfrak{F}(-\rho)$ such that $g'(z) = e^{2h(z)}g(z)$. Hence we have:

$$\begin{aligned} L(f) &= \langle f, g' \rangle_{\rho} \\ &= \int g'(z)\overline{f(z)}e^{-2 \operatorname{Re} h(z)} d\mu(z) \\ &= \int g(z)\overline{f(z)}e^{2i \operatorname{Im} h(z)} d\mu(z) \end{aligned}$$

and this expression cannot be of the form $\langle f \mid \hat{g} \rangle$ for some $\hat{g} \in \mathfrak{F}(-\rho)$. Hence, for a harmonic weight ρ, the antidual $\mathfrak{F}(\rho)^\times$ can *never* be identified with $\mathfrak{F}(-\rho)$.

For an arbitrary weight ρ it seems difficult to derive necessary and sufficient conditions for the equality $\mathfrak{F}(-\rho) = \mathfrak{F}(\rho)^\times$. Things are different, however, if ρ is radial.

PROPOSITION 4.3. — Let ρ be a radial weight. Then $\mathfrak{F}(-\rho) = \mathfrak{F}(\rho)^\times$ iff there exists a positive constant $c(\rho)$ such that

$$1 \leqslant \eta_m^{(\rho)}\eta_m^{(-\rho)} \leqslant c(\rho), \qquad \text{for all} \quad m = 0, 1, 2, \ldots \tag{4.10}$$

Proof. — The argument is straightforward, and entirely analogous to Bargmann's (Ref. [*6*], Sect. 3.12). For radial ρ, we have the identification $\mathfrak{F}(\rho) \simeq l^2(\eta^{(\rho)})$, and therefore $\mathfrak{F}(-\rho) \simeq l^2(\eta^{(-\rho)})$, $\mathfrak{F}(\rho)^\times \simeq l^2(\eta^{(\rho)})$. Thus the required equality holds iff $l^2(\eta^{(-\rho)}) = l^2(\eta^{(\rho)})$. As discussed in Appendix B, this in turn is equivalent to the existence of positive constants $d(\rho)$, $c(\rho)$ such that

$$d(\rho) \leqslant \eta_m^{(\rho)}\eta_m^{(-\rho)} \leqslant c(\rho)\,.$$

Finally, we may set $d(\rho) = 1$, by the Schwarz inequality:

$$1 = \| u_m \|^2 \leqslant \| u_m \|_\rho \; \| u_m \|_{-\rho} = (\eta_m^{(\rho)}\eta_m^{(-\rho)})^{1/2}\,. \qquad \square$$

The usefulness of condition (4.10) is nicely illustrated by the following families of weights:

$$\rho_{\gamma,\tau}(z) = \tau \mid z \mid^{2\gamma}, \qquad \tau \in \mathbb{R}, \quad 0 < \gamma < 1$$
$$\rho_\beta(z) = \beta \mid z \mid^2, \qquad -1 < \beta < 1, \quad \beta \neq 0\,.$$

In the first case, the behaviour of the corresponding coefficient $\eta_m^{(\gamma,\tau)}$ has been evaluated explicitly by Daubechies [*8*]. The result is that the weights $\rho_{\gamma,\tau}$ satisfy the duality condition (4.10) for $0 < \gamma \leqslant \frac{1}{2}$ but they don't for $\frac{1}{2} < \gamma < 1$. In the second case, one finds immediately, for $\mid \beta \mid < 1$, $\eta_m^{(\beta)} = (1+\beta)^{-m-1}$. Thus $\eta_m^{(\beta)}\eta_m^{(-\beta)} = (1-\beta^2)^{-m-1}$ which is unbounded as $m \to \infty$, and therefore the weights ρ_β never satisfy the duality condition. A more detailed discussion of these two classes of weights will be found in Appendix C.

Starting from a space $\mathfrak{F}(\rho)$ of entire functions, one gets another one $\mathfrak{F}(-\rho)$ by the involution $\#_1$. But it is not obvious that $\mathfrak{F}(\rho)^\times = \mathfrak{F}(\rho)^{\#_2}$ can be realized in the same way. In other words, given the corresponding sequence $([\eta_m^{(\rho)}]^{-1})$, is there a unique weight $\bar{\rho}$ such that $\eta_m^{(\bar{\rho})} = [\eta_m^{(\rho)}]^{-1}$? In view of Eq. (4.6), this is a Stieltjes moment problem [*21*]: the numbers $\mu_m^{(\rho)} = m\,!\,\eta_m^{(\rho)}$ are the moments on $[0, \infty)$ of the function $\exp[-\rho(t)-t]$.

So the question is: given the numbers $\mu_m^{(\bar{\rho})} = m\,!\,[\eta_m^{(\rho)}]^{-1}$, does the moment problem

$$\mu_m^{(\bar{\rho})} = \int_0^\infty t^m d\psi(t)\,, \qquad m = 0, 1, 2, \ldots$$

admit a unique solution of the form $d\psi(t) = \exp\,[-\bar{\rho}(t) - t]dt$? Sufficient conditions for the existence of a unique solution ψ (albeit not necessarily of the form indicated) are well-known, and for convenience we have collected them in Appendix D.

First we consider the space $\mathfrak{F}(\beta)^\times$. The corresponding moments are $\mu_n = n\,!\,(1+\beta)^{n+1}$, and in this case (see App. D) the moment problem has a unique solution, which coincide necessarily with $\mathfrak{F}\left(\frac{-\beta}{1+\beta}\right)$, i. e. $\bar{\rho}_\beta(z) = -\frac{\beta}{1+\beta}\,|\,z\,|^2$.

In the general case, we obtain a sufficient condition by comparing ρ with $\rho_{\pm\beta}$, $0 < \beta < 1$.

PROPOSITION 4.4. — Let ρ be a radial weight such that $|\,\rho(z)\,| \leqslant \beta\,|\,z\,|^2$ for $0 < \beta < 1$. Then there exists a unique non-negative, non-decreasing function φ such that:

$$[\eta_m^{(\rho)}]^{-1} = \frac{1}{m\,!}\int_0^\infty t^m d\varphi(t)\,, \qquad m = 0, 1, 2, \ldots$$

Proof. — The growth condition on ρ implies the inequalities:

$$(1+\beta)^{m+1} \geqslant [\eta_m^{(\rho)}]^{-1} \geqslant (1-\beta)^{m+1}\,.$$

Then the assertion follows for the moments $\mu_m = m\,!\,[\eta_m^{(\rho)}]^{-1}$ exactly as in App. D. The lower bound $\mu_m \geqslant m\,!\,(1-\beta)^{m+1}$ implies the existence of a solution, by Eq. (D.2), and the upper bound $\mu_m \leqslant m\,!\,(1+\beta)^{m+1}$ implies uniqueness of that solution, by Eq. (D.3). □

Of course we are mostly interested in cases where the duality condition *iii*) does hold, and for that purpose the relations (4.10) give a useful criterion. However their validity depends on a delicate balance between the growth properties of $\eta_m^{(\rho)}$ and $\eta_m^{(-\rho)}$, so that a simple growth condition of the type $|\,\rho(z)\,| \leqslant \tau\,|\,z\,|$ does not suffice. Yet it is useful to notice that only the asymptotic behavior in m of $\eta_m^{(\rho)}$ actually matters. This information may be exploited as follows. We split the integral in (4.6) into two pieces:

$$m\,!\,\eta_m^{(\rho)} = \int_0^1 t^m e^{-\rho(t)-t}dt + \int_1^\infty t^m e^{-\rho(t)-t}dt$$

The first term comes from integration over the (compact) unit disk $|\,z\,| \leqslant 1$, on which ρ is bounded:

$$|\,\rho(z)\,| \leqslant c\,, \qquad \text{whenever} \qquad |\,z\,| \leqslant 1\,,$$

Hence we get the estimation:

$$\frac{e^{-c}}{m!}\int_0^1 t^m e^{-t}dt \leqslant \frac{1}{m!}\int_0^1 t^m e^{-\rho(t)-t}dt \leqslant \frac{e^c}{m!}\int_0^1 t^m e^{-t}dt .$$

Using the expansion:

$$\frac{1}{m!}\int_0^1 t^m e^{-t}dt = e^{-1}\left[\frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \dots\right]$$

we conclude that the first term in $\eta_m^{(\rho)}$ goes asymptotically as const $[(m+1)!]^{-1}$, i. e. becomes fast negligible.

As for the second term, the integral is actually a monotone increasing function of m. Thus we get finally:

$$\eta_m^{(\rho)} \underset{m\to\infty}{\simeq} \frac{\text{const}}{(m+1)!} + \frac{1}{m!}\langle \rho \rangle_m \tag{4.11}$$

where $\langle \rho \rangle_m$ is monotone increasing as $m \to \infty$. We will exploit this estimate in Sect. 5 below.

4.4. **Principal vectors.**

There remains finally to examine condition *iv*), that is: $e_a \in \mathfrak{F}(\rho)$, for all $a \in \mathbb{C}$.

First, we have:

PROPOSITION 4.5. — Let ρ be such that e_a belongs to $\mathfrak{F}(\rho)$, for every $a \in \mathbb{C}$; then $Exp \subset \mathfrak{F}(\rho)$, and the embedding in continuous.

Proof. — The first assertion results from the fact that every function of exponential type $f \in Exp$ may be majorized by a finite linear combination of elements e_a. As for continuity of the embedding, we notice that the natural topology of Exp is defined by the set of norms

$$\| f \|_m = \sup_z | f(z) | e^{-m|z|}, \qquad m = 0, 1, 2, \dots .$$

Let $f \in Exp$ be such that $\| f \|_m < \infty$. Then we have:

$$\begin{aligned} \| f \|_\rho^2 &= \int | f(z) |^2 e^{-\rho(z)} d\mu(z) \\ &\leqslant (\sup_z | f(z) |^2 e^{-2m|z|}) . \int e^{2m|z|-\rho(z)} d\mu(z) = \mathrm{K}_m \| f \|_m^2 \end{aligned}$$

where K_m is finite by the assumption on ρ. □

It is easy to give sufficient conditions for this situation to happen.

PROPOSITION 4.6. — Let ρ be any weight such that $|\rho(z)| \leqslant \beta |z|^2$ with $0 < \beta < 1$. Then all principal vectors e_a, $a \in \mathbb{C}$, are contained in $\mathfrak{F}(\rho)$.

Proof. — A straightforward estimate:

$$\| e_a \|_\rho^2 = \pi^{-1} \int | e^{az} |^2 e^{-\rho(z)} e^{-|z|} dz$$
$$\leqslant \pi^{-1} \int e^{K|z|} e^{-(1-\beta)|z|^2} dz < \infty . \qquad \square$$

The conditions of Prop. 4.6 are, of course, verified for all the weights $\rho_{\gamma,\tau}$, ρ_β described in Sect. 4.3 above.

Assume now the conditions *i)* and *ii)* to be satisfied. It follows that *Exp* is dense in $\mathfrak{F}(\rho)$. If V is any PIP-space made out of such subspaces $\mathfrak{F}(\rho)$, one has:

$$Exp \subset \bigcap_\rho \mathfrak{F}(\rho) = V^{\#} \subset \mathfrak{F} \subset V = \bigcup_\rho \mathfrak{F}(\rho) \subset \mathfrak{Z}$$

where all embeddings are continuous and have dense range. Thus every operator in Op(V) has a restriction to *Exp* which is continuous into $\mathfrak{Z} \simeq (Exp)^\times$. Since $Exp \equiv Exp\ (\mathbb{C})$ is nuclear, it verifies the kernel theorem of Schwartz [*15*] [*17*] [*18*]. Therefore $A \in \mathscr{L}(Exp, (Exp)^\times)$ may be identified with an antianalytic functional in two variables:

$$\mathscr{L}\ (Exp\ (\mathbb{C}),\ [Exp\ (\mathbb{C})]^\times) \simeq [Exp\ (\mathbb{C} \times \overline{\mathbb{C}})]^\times$$

which in turn, by the Fourier-Borel isomorphism $(Exp)^\times \simeq \mathfrak{Z}$ is an entire, sesquiholomorphic function in two variables, namely $A(z, w) = \langle e_z | Ae_w \rangle$ (analytic in z, antianalytic in w). In other words, every operator $A \in Op(V)$ is the extension to its maximal domain (see II) of an integral operator over *Exp*, with kernel $A(z, w)$. This map $A \mapsto A(z, w)$ is of paramount importance in the study of the so-called Weyl quantization [*8*]-[*11*].

Remark. — If ρ is nonradial, condition *iv)* does not imply *ii)*; $\mathfrak{F}(\rho)$ is infinite dimensional, since it contains *Exp*, and, *a fortiori*, all powers z^m, but the polynomials need not be dense in $\mathfrak{F}(\rho)$.

5. Nested Hilbert spaces of entire functions.

We have now at our disposal the building blocks for an indexed PIP-space around $\mathfrak{F}$, namely the subspaces $\mathfrak{F}(\rho)$. They are assaying both for $\#_1$ and $\#_2$ if the duality condition is satisfied. First we examine the relationship between subspaces $\mathfrak{F}(\rho)$ and weight functions $\rho(z)$.

DEFINITION 5.1. — Let ρ_1, ρ_2 be two locally bounded real-valued functions on $\mathbb{C}$. We say that ρ_1 and ρ_2 are *equivalent weights* ($\rho_1 \approx \rho_2$) whenever $\mathfrak{F}(\rho_1) = \mathfrak{F}(\rho_2)$ as vector spaces.

Of course, $\approx$ is an equivalence relation. It is characterized by the following easy lemma, whose proof is left to the reader.

LEMMA 5.2. — Let $\rho_1, \rho_2 \in L^\infty_{loc}(\mathbb{C}, \mu)$. Then:

i) $\rho_1 \approx \rho_2$ iff there is a positive constant C such that $|\rho_1(z)-\rho_2(z)| \leqslant C$ almost everywhere;

ii) if $\rho_1 \approx \rho_2$, the identity map $\mathfrak{F}(\rho_1) \to \mathfrak{F}(\rho_2)$ is bicontinuous. □

Notice that *i*) may be restated as follows: $\rho_1 \approx \rho_2$ iff there exists two positive constants A, B such that for almost every z:

$$A \leqslant \exp(\rho_1(z) - \rho_2(z)) \leqslant B$$

which is the usual formulation of equivalence of weights. Referring to Appendix B for a detailed discussion, we simply remark that our family $\{\mathfrak{F}(\rho)\}$ is actually indexed by the quotient $L^\infty_{loc}/\approx$, which is an involutive lattice for the pointwise partial order ($\rho_1 \leqslant \rho_2$ iff $\rho_1(z) \leqslant \rho_2(z)$ a. e.), modulo $\approx$. We notice in particular the two (mutually equivalent) obvious relations:

$$\rho_1 \wedge \rho_2 \equiv \inf(\rho_1, \rho_2) \approx -\log(e^{-\rho_1} + e^{-\rho_2})$$
$$\rho_1 \vee \rho_2 \equiv \sup(\rho_1, \rho_2) \approx \log(e^{\rho_1} + e^{\rho_2})$$

Then we may finally state the first main theorem of this Section.

THEOREM 5.3. — Given a real-valued measurable function ρ on $\mathbb{C}$, bounded on compact sets, let $\mathfrak{F}(\rho) = L^2(e^\rho) \cap \mathfrak{Z}$ be the corresponding Hilbert space of entire functions. Let $\tilde{J}$ be the set of all such functions ρ which satisfy the following three conditions:

i) all the principal vectors e_a, $a \in \mathbb{C}$, belong to $\mathfrak{F}(\rho)$;

ii) the antidual $\mathfrak{F}(\rho)^\times$ is isomorphic to $\mathfrak{F}(-\rho)$;

iii) $-\rho$ satisfies *i*), *ii*) whenever ρ does.

Then the family $\mathscr{I} = \{\mathfrak{F}(\rho) \mid \rho \in \tilde{J}\}$ defines a nested Hilbert space with central Hilbert space $\mathfrak{F}$. The compatibility is $\#_1$, the partial inner product is that induced by $\mathfrak{F}$ and the index set is $J = \tilde{J}/\approx$.

Proof. — Given the preceding discussion, the proof is immediate: the set $\tilde{J}$ is an involutive lattice (in particular, if ρ_1 and ρ_2 satisfy the duality condition *ii*), so do $\rho_1 \wedge \rho_2$ and $\rho_1 \vee \rho_2$). So is J, and therefore $\mathscr{I}$, which is indexed by J, is an involutive covering of $V_J = \bigcup_{\rho \in \tilde{J}} \mathfrak{F}(\rho)$, in the sense of III. The rest is obvious. □

This theorem yields a nested Hilbert space, whose assaying subspaces are Hilbert spaces of entire functions and which inherits the lattice structure of L^1_{loc} and of ω at the same time (by *ii*)). In par: cular:

$$\mathfrak{F}(\rho_1 \wedge \rho_2) = \mathfrak{F}(\rho_1) \cap \mathfrak{F}(\rho_2,$$
$$\mathfrak{F}(\rho_1 \vee \rho_2) = \mathfrak{F}(\rho_1) + \mathfrak{F}(\rho_2)$$
$$\mathfrak{F}(-\rho) = [\mathfrak{F}(\rho)]^\times$$

However, as it stands, this result is not very useful, in that the duality condition *ii*) is difficult to verify, except for radial weights, for which the explicit criterion (4.10) may be used. Thus we reformulate the theorem for that particular case.

THEOREM 5.4. — With the same notation as above, denote by $\tilde{\mathrm{I}}$ the set of all weight functions ρ which verify the conditions:

i) ρ is locally bounded and radial: $\rho(z) = \rho(|z|)$;
ii) e_a belongs to $\mathfrak{F}(\rho) \cap \mathfrak{F}(-\rho)$, for any $a \in \mathbb{C}$;
iii) there are positive constants A, B such that:

$$\mathrm{A} \leqslant \eta_m^{(\rho)}\eta_m^{(-\rho)} \leqslant \mathrm{B}, \qquad \text{for all} \quad m = 0, 1, 2 \ldots$$

where

$$\eta_m^{(\rho)} = (m\,!)^{-1}\int_0^\infty t^m e^{-\rho(t)-t}dt$$

Then the family $\mathscr{I} = \{ \mathfrak{F}(\rho), \rho \in \tilde{\mathrm{I}} \}$ is a nested Hilbert space around $\mathfrak{F}$, indexed by $\mathrm{I} \equiv \tilde{\mathrm{I}}/\approx$.

The compatibility coincides with both $\#_1$ and $\#_2$, and the partial inner product with those induced by $\mathfrak{F}$ and l^2 respectively. □

The proof is immediate. Notice that $\rho_1 \approx \rho_2$ is equivalent to $\mathrm{C} \leqslant \eta_m^{(\rho_1)}/\eta_m^{(\rho_2)} \leqslant \mathrm{D}$ for some positive constants C, D and all m, that is $\eta^{(\rho_1)}$ and $\eta^{(\rho_2)}$ are equivalent weight sequences and $l^2(\eta^{(\rho_1)}) = l^2(\eta^{(\rho_2)})$ (See App. B).

The next question, of course, is whether this lattice I is actually not just a chain. We conjecture that it is not, but our proof is incomplete. The first steps run as follows. Given the identification $\mathfrak{F}(\rho) = l^2(\eta^{(\rho)})$, it suffices to consider the weight sequences $(\eta_m^{(\rho)})$. To prove that I is not a chain, it is enough to exhibit a sequence $\eta^{(\rho)}$ which is not comparable to $\eta^{(0)}$ (recall that $\eta_m^{(0)} = 1$, $\forall m$); in other words, a weight sequence $(\eta_m^{(\rho)})$ which is bounded neither below, nor above as $m \to \infty$. We claim that (λ_m) is such a sequence, where:

$$\lambda_m = \exp\left(-\frac{1}{4}\cos m\pi \log m\right) \qquad (m \geqslant 1) \tag{5.1}$$

that is,

$$\lambda_{2k} = (2k)^{-1/4}, \qquad \lambda_{2k+1} = (2k+1)^{1/4} \tag{5.2}$$

Thus, as $m \to \infty$, λ_m oscillates indefinitely between the two bounds $m^{\pm 1/4}$, and (λ_m) is bounded neither below, nor above (it is easy to find other sequences with the same general behaviour, (λ_m) is just a typical example). Next, we have to show that these numbers can actually be identified with a sequence of weights $(\eta_m^{(\rho)})$. First, (5.1) is consistent with the general behaviour (4.11); indeed the corresponding quantity

$$\begin{aligned} m\,!\,\lambda_m &= m\,!\,\exp\left(-\frac{1}{4}\cos m\pi \log m\right) \\ &= \langle \lambda \rangle_m + \mathrm{O}(m^{-1}) \end{aligned}$$

is monotone increasing as $m \to \infty$, for m large enough:

$$\frac{d}{dm} \log (m\,!\,\lambda_m) = \frac{d}{dm} \log (m\,!) + \frac{\pi}{4} \sin m\pi \log m - \frac{1}{4} m^{-1} \cos m\pi .$$

Using the estimate [22]:

$$\frac{d}{dm} \log (m\,!) = \log m + m^{-1} + O(m^{-2}) ,$$

we get:

$$\frac{d}{dm} \log (m\,!\,\lambda_m) = \left(1 + \frac{\pi}{4} \sin m\pi\right) \log m + m^{-1}\left(1 - \frac{1}{4} \cos m\pi\right) + O(m^{-2}),$$

and this quantity is manifestly positive for m large enough. Finally, the values (5.2) imply that the moment problem corresponding to the sequence (λ_m) admits a unique solution, since all conditions (D.2) and (D.3) are satisfied (same argument as in Proposition 4.4). Thus the numbers λ_m may be identified with a sequence of generalized weights:

$$\lambda_m \equiv \eta_m^{(\hat{\rho})} = \frac{1}{m\,!} \int_0^\infty t^m d\hat{\rho}(t) \tag{5.3}$$

For the solution to be complete, however, two more points must be proven. First, since we require a weight function ρ to be locally bounded, we need a measure $\hat{\rho}$ *equivalent* to the Lebesgue measure: $d\hat{\rho}(t) = \exp\,[-\rho(t) - t\,]dt$ with $\rho(t) = \infty$ at most on a set of Lebesgue measure zero (otherwise the space $\mathfrak{F}(\rho)$ would not be complete !). And then of course, the function ρ, if it exists, must satisfy the duality condition (4.10). Concerning the first point, we can transform the Stieltjes moment problem defined by (5.3) into a (uniquely solvable as well) Hamburger moment problem [21]:

$$\mu'_{2m} = m\,!\,\lambda_m = \int_{-\infty}^{+\infty} t^m d\sigma(t) , \qquad \mu'_{2m+1} = 0 ,$$

where $\sigma(t) = \frac{1}{2}(\text{sign } t)\hat{\rho}(t^2)$. Then the measure σ is equivalent to the Lebesgue measure iff it is equivalent to all its translates σ_a, where $\sigma_a(t) = \sigma(t - a)$. But we have not been able to prove this fact. However, relying on our experience with concrete weights, we believe this assertion, as well as the other one about duality, to be true, and thus we state:

CONJECTURE 5.5. — The involutive lattice I is not a chain. □

Of course, as it is often the case with PIP-spaces, it may be difficult to exhibit « all » assaying subspaces. Thus, for practical purposes, it will be sufficient to consider the standard weights

$$\rho_{\gamma,\tau,\sigma}(z) = \log (1 + |\,z\,|^2)^\sigma + \tau\,|\,z\,|^{2\gamma} \qquad 0 \leqslant \gamma \leqslant 1/2 , \quad \tau \in \mathbb{R} , \quad \sigma \in \mathbb{R} .$$

These weights satisfy all the conditions of Theorem 5.4, with

$$-\rho_{\gamma,\tau,\sigma} = \rho_{\gamma,-\tau,-\sigma}.$$

The duality conditions follow from the estimate (Ref. [*8*]; see also App. C) on the corresponding weight $\eta_m^{(\rho)}$:

$$\eta_m^{(\gamma,\tau,\sigma)} \sim \text{const.}\ (1+m)^\sigma \exp \tau m^\gamma \qquad (\text{as } m \to \infty).$$

This estimate also shows that the family $\{ \mathfrak{F}(\rho_{\gamma,\tau,\sigma}) \}$ is actually a chain.

PROPOSITION 5.6. — Let $\{ \rho_{\gamma,\tau,\sigma} \}$ be the family of weights defined above. Then the family of Hilbert spaces $\{ \mathfrak{F}(\rho_{\gamma,\tau,\sigma}) \}$ is a chain, indexed by the set $I_0 \equiv \left[0, \frac{1}{2}\right] \times \mathbb{R} \times \mathbb{R}$, with its lexicographic order $\{ \gamma, \tau, \sigma \}$. □

If we notice that Bargmann's chain (indexed by $\sigma \in I_B \equiv \mathbb{R}$) corresponds to the choice $\tau = 0$, we get a hierarchy of four nested Hilbert spaces increasingly finer (in the sense of III), corresponding to the lattices

$$I_B \subset I_0 \subset I \subset J$$

where the first two are chains, whereas the last two presumably are not.

One could even go further by using interpolation methods [*23*] and refine each of those nested Hilbert spaces, but we won't consider this possibility here.

One last question must be asked. For each of these nested Hilbert spaces, the full vector space is given by

$$V_{I_i} = \bigcup_{\rho \in \tilde{I}_i} \mathfrak{F}(\rho) \qquad (I_i = I_B, I_0, I, J\,;\ I_i = \tilde{I}_i/\approx)$$

Can one identify this vector space concretely? For Bargmann's case $I_i = I_B$, the result is $V_{I_B} \equiv \mathfrak{E}'$, which is isomorphic, under Bargmann's integral transform, to Schwartz' space $\mathscr{S}'$ of tempered distributions [*6*] [*8*] [*11*]. What about the other ones? As far as we know, the question is still open. Of course, V_{I_0} already contains entire functions corresponding to nontempered distributions. In fact, Bargmann's transform and its inverse define continuous embeddings from the Hilbert spaces $H(\bar{\alpha}, \bar{A})$ of Grossmann [*7*] into the subspaces $\mathfrak{F}(\rho_{\gamma,\tau,0})$ of V_{I_0}, and *vice versa*. See Daubechies [*8*] for a detailed analysis.

Finally, coming back to the space $\mathfrak{Y} \equiv (Exp)^{\#1}$, we see that it contains every $\mathfrak{F}(\rho)$ with $\rho \in \tilde{J}$ since the latter contains all e_a, $a \in \mathbb{C}$. Thus we get:

$$V_J = \bigcup_{\rho \in \tilde{J}} \mathfrak{F}(\rho) \subset \mathfrak{Y}$$

As we saw in Sect. 3, $\mathfrak{Y}$ is the trace on $\mathfrak{Z}$ of the domain of the operator P. This suggests another way of selecting suitable weights, using P. Let ρ be such that the restriction of P to $L^2(e^\rho)$ is bounded as a map $L^2(e^\rho) \to L^2(e^\rho)$, and let $\| P \|_{\rho\rho}$ be its norm:

$$\| P \|^2_{\rho\rho} = \sup_{f \in L^2(e^\rho)} \frac{\| Pf \|^2_\rho}{\| f \|^2_\rho} = \sup_{f \in L^2(e^\rho)} \frac{\int | \langle e_z | f \rangle |^2 e^{-\rho(z)} d\mu(z)}{\int | f(z) |^2 e^{-\rho(z)} d\mu(z)}$$

Let $\tilde{I}_P$ be the set of all weights ρ such that:

a) $e_a \in \mathfrak{F}(\rho) \cap \mathfrak{F}(-\rho)$, $\quad \forall a \in \mathbb{C}$;

b) both $\| P \|_{\rho\rho}$ and $\| P \|_{-\rho,-\rho}$ are finite.

Then, using the general theory of projection operators in PIP-spaces [*24*], one can show the following:

i) $\tilde{I}_P$ is an involutive sublattice of the lattice L^∞_{loc} of all weights; the corresponding family $\mathcal{J}_P = \{ L^2(e^\rho), \rho \in \tilde{I}_P \}$ defines a nested Hilbert space W_P, with the structure inherited from $L^1_{loc}(\mathbb{C}, \mu)$;

ii) every $\rho \in \tilde{I}_P$ verifies the duality condition;

iii) the restriction of P to W_P is an orthogonal projection; its image $V_P \equiv PW_P = \bigcup_{\rho \in \tilde{I}_P} \mathfrak{F}(\rho)$ is a nested Hilbert space contained in V_J.

In fact, we conjecture that $V_P = V_J$, that is, condition *b*) above is actually equivalent to the duality condition, but we have been unable to prove it.

In conclusion, we consider the extension of the above analysis to several dimensions. Two classes of weights are tractable:

i) « *globally radial* » *weights*

i. e. $\quad \rho(z) = \rho(|z|), \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \qquad |z|^2 = \sum_{i=1}^{n} |z_i|^2$

In this case the analysis is identical to the one-dimensional case.

ii) « *separately radial* » *weights*

$$\text{i. e.} \qquad \rho(z) = \sum_{i=1}^{n} \rho_i(|z_i|)$$

Noticing that both the Gaussian measure and the monomials $u_m(z)$, $m = (m_1, \ldots, m_n)$, factor:

$$d\mu_n(z) = \prod_{i=1}^{n} d\mu_1(z_i), \qquad u_m(z) = \prod_{i=1}^{n} u_{m_i}(z_i),$$

we see that for such weights the integral giving the weight $\eta_m^{(\rho)}$ also factors, and one gets:

$$\eta_m^{(\rho)} = \prod_{i=1}^{n} \eta_{m_i}^{(\rho_i)} \qquad m = (m_1, \ldots, m_n) .$$

Therefore the whole analysis goes through, and the resulting PIP-spaces are simply tensor products of one-dimensional ones with the result that even in the simple cases I_B, I_0 we get genuine lattices.

ACKNOWLEDGMENTS

It is a pleasure to thank Alex Grossmann and Ingrid Daubechies for many stimulating discussions and communication of unpublished results. In fact, the very idea of considering the spaces $\mathfrak{F}(\rho)$ is due to Grossmann, who proposed it in his Louvain-la-Neuve lectures (1976).

APPENDIX A

HARMONIC VS. RADIAL WEIGHTS

We show that the classes of harmonic, resp. radial weights are almost disjoint, in the following sense:

Proposition A.1. — Let ρ be a harmonic weight: $e^{\rho(z)} = |f(z)|^2$ for some entire function f. Then ρ is radial iff f is a power:

$$f(z) = cz^\alpha, \qquad \alpha \in \mathbb{R}, \qquad c > 0.$$

Proof. — Let $t = |z|^2$. Then ρ radial means

$$f(z)\overline{f(z)} = g(t) \tag{A.1}$$

for some positive function g. Deriving both sides of this equation yields:

$$\frac{df}{dz}\overline{f(z)} = \bar{z}g'^{(t)} \tag{A.2}$$

since $\frac{d}{dz}\overline{f(z)} = \overline{\frac{d}{d\bar{z}}f(z)} = 0$ by analyticity. Dividing (A.2) by (A.1) we get:

$$\frac{d}{dz}\log f(z) = \bar{z}\frac{d}{dt}\log g(t) \tag{A.3}$$

Writing $G(t) = \frac{d}{dt}\log g(t)$, and deriving both sides of (A.3) with respect to $\bar{z}$, we get, again by analyticity:

$$0 = G(t) + tG'(t) \tag{A.4}$$

The general solution of (A.4) is $G(t) = \alpha t^{-1}$, $\alpha \in \mathbb{R}$, which yields $g(t) = ct^\alpha$, with $c > 0$, and therefore $f(z) = c^{1/2}z^\alpha$. □

APPENDIX B

NESTED HILBERT SPACES INDEXED BY WEIGHTS

The simplest examples of nested Hilbert spaces are the family $\{ l^2(r) \}$ of weighted l^2-spaces, in the space ω of all complex sequences, and the family $\{ L^2(r) \}$ of weighted L^2-spaces, in the space $L^1_{loc}(X, \mu)$ of locally integrable functions on some measure space (X, μ). Both spaces have been described repeatedly in the papers I-IV. However, the corresponding index sets have not been identified explicitly. This we will do below.

First the sequence spaces. Given an arbitrary sequence $r \equiv (r_n)$ of positive numbers, the Hilbert space $l^2(r)$ is defined as:

$$l^2(r) = \{ (x_n) \in \omega \mid (x_n r_n^{-1/2}) \in l^2 \}$$

Let $\tilde{I}_s$ be the set of all sequences r of positive numbers. With the pointwise partial order:

$$r \leqslant s \Leftrightarrow r_n \leqslant s_n , \qquad \forall n = 1, 2 \ldots$$

$\tilde{I}_s$ is actually an involutive lattice for the following operations:

$$(r \wedge s)_n = \min \{ r_n, s_n \}$$
$$(r \vee s)_n = \max \{ r_n, s_n \}$$
$$(\bar{r})_n = r_n^{-1}$$

On $\tilde{I}_s$ we define the following equivalence relation:

$$r \sim s \qquad \text{iff} \qquad \exists B, C > 0 \quad \text{such that} \quad B \leqslant \frac{r_n}{s_n} \leqslant C , \qquad \forall n .$$

Then we have the following easy result:

PROPOSITION B.1. — Let $\tilde{I}_s$ $\sim$ be as above. Then:

i) $r \sim s$ iff $l^2(r) = l^2(s)$ as vector spaces;

ii) let $r_1 \sim r_2$, $s_1 \sim s_2$; then $r_1 \leqslant s_1$ iff $r_2 \leqslant s_2$. □

Denote by $I_s = \tilde{I}_s/\sim$ the set of equivalence classes and by $[r]$ the class of r. It follows from *ii)* that I_s is partially ordered by the relation:

$$[r] \leqslant [s] \qquad \text{iff} \qquad \exists B > 0 \text{ such that } r_n \leqslant B s_n \text{ for all } n = 1, 2, \ldots \text{ and any } r \in [r],\ s \in [s] .$$

Furthermore, I_s inherits the lattice structure of $\tilde{I}_s$ as follows:

$$[r] \wedge [s] = [r \wedge s]$$
$$[r] \vee [s] = [r \vee s]$$
$$\overline{[r]} = [\bar{r}]$$

Finally, *i)* means that I_s is precisely the index set of the nested Hilbert space $\{ l^2(r) \}$ (see III, IV):

$$l^2(r) \subseteq l^2(s) \Leftrightarrow [r] \leqslant [s]$$
$$l^2(r \wedge s) = l^2(r) \cap l^2(s)$$
$$l^2(r \vee s) = l^2(r) + l^2(s)$$
$$l^2(\bar{r}) = [l^2(r)]^{\times}$$

We turn now to spaces of locally integrable functions on some σ-finite measure space (X, μ).

Let $\tilde{I}_f$ be the set of all measurable, μ-almost everywhere positive functions r on X, such that both r and r^{-1} are integrable on every compact subset of X. We proceed exactly as above. $\tilde{I}_f$ has a natural partial order:

$$r \leqslant s \Leftrightarrow r(x) \leqslant s(x) \qquad \mu\text{-a. e.}$$

and with respect to that order, $\tilde{I}_f$ is an involutive lattice for the operations:

$$(r \wedge s)(x) = \min \{ r(x), s(x) \}$$
$$(r \vee s)(x) = \max \{ r(x), s(x) \}$$
$$\bar{r}(x) = [r(x)]^{-1}$$

For every $r \in \tilde{I}_f$, we define the associated Hilbert space as:

$$L^2(r) = \{ f \text{ measurable and } f r^{-1/2} \in L^2 \}$$

Again $\tilde{I}_f$ carries a natural equivalence relation:

$$r \sim s \quad \text{iff} \quad \exists B, C > 0 \quad \text{such that} \quad B \leqslant \frac{r(x)}{s(x)} \leqslant C \qquad \mu\text{-a. e.}$$

with the same consequence as above:

Proposition B.2. — With the notations just introduced:

i) $r \sim s$ iff $L^2(r) = L^2(s)$ as vector spaces;

ii) for $r_1 \sim r_2$, $s_1 \sim s_2$, one has $r_1 \leqslant s_1$ iff $r_2 \leqslant s_2$. □

Thus here also the lattice structure passes to the quotient $I_f = \tilde{I}_f/\sim$, partially ordered by the relation:

$$[r] \leqslant [s] \quad \text{iff} \quad \exists B > 0 \text{ such that } r(x) \leqslant Bs(x) \quad \text{a. e.,}$$
$$\text{for any} \quad r \in [r],\ s \in [s].$$

Since $L^2(r) \subseteq L^2(s)$ whenever $[r] \leqslant [s]$, it follows, exactly as for the sequences, that I_f is the index set of the nested Hilbert space $\{ L^2(r) \}$.

Going back to the discussion of logarithmic weights in Section 5 we can state the following equivalences:

$$\rho_1 \approx \rho_2 \Leftrightarrow e^{\rho_1} \sim e^{\rho_2} \Leftrightarrow \eta^{(\rho_1)} \sim \eta^{(\rho_2)} \qquad \text{(for radial weights)}.$$

Another aspect of the index set I_f follows from the fact that, for any $r \in \tilde{I}_f$ the measures $r\mu$ and μ are equivalent: thus I_f can be considered as a set of equivalent measures on X. Also, whenever $L^2(r) \subseteq L^2(s)$, the embedding $E_{sr} : L^2(r) \to L^2(s)$ is continuous and injective, since both spaces are subspaces of $L^1_{loc}(X, \mu)$. In fact this is a particular case of a general property, which might have an independant interest.

Let M be a set of non-negative Radon measures on a locally compact space X. This set M as a natural partial order:

$$\mu_s \leqslant \mu_r \quad \text{iff} \quad \mu_r - \mu_s \quad \text{is non-negative}.$$

Assume M to be directed to the left: for any pair μ_r, $\mu_s \in$ M, there is $\mu_p \in$ M such that $\mu_p \leqslant \mu_r$ and $\mu_p \leqslant \mu_s$. Whenever $\mu_r \geqslant \mu_s$, denote by E_{sr} natural embedding of $L^2(X, \mu_r)$ into $L^2(X, \mu_s)$. Then we can state:

Proposition B.3. — Let M and E_{sr} as above. Then the following are equivalent:

i) All the maps E_{sr} are injective.

ii) All the measures $\mu_r \in$ M are equivalent.

Proof. — Consider a given pair μ_r, $\mu_s \in$ M such that $\mu_r \geqslant \mu_s$. Then μ_s is absolutely continuous with respect to μ_r: since $\mu_r = \mu_s + \nu$, with $\nu \geqslant 0$, $\mu_r(S) = 0$ implies $\mu_s(S) = 0$ for any measurable $S \subset X$. If, in addition, E_{sr} is injective, then μ_r and μ_s are equivalent. It is enough to prove that $\mu_r \prec \mu_s$. Assuming the contrary, there exists a μ_s-null set $T \subset X$

with $\mu_r(\mathrm{T}) \neq 0$, and we may assume that $\mu_r(\mathrm{T}) < \infty$ (see IV, Sect. 5.B.2). Let χ_{T} be the characteristic function of T. Then $\chi_{\mathrm{T}} \in \mathrm{L}^2(\mathrm{X}, \mu_r)$, with $\| \chi_{\mathrm{T}} \|_r = \mu_r(\mathrm{T}) \neq 0$, whereas $\chi_{\mathrm{T}} \in \mathrm{L}^2(\mathrm{X}, \mu_s)$ and $\| \chi_{\mathrm{T}} \|_s = \mu_s(\mathrm{T}) = 0$, and this contradicts the injectivity of E_{sr}.

Next, we prove that, if M is directed to the left, *i*) implies *ii*). Given any pair μ_p, $\mu_q \in \mathrm{M}$, let μ_r be a common predecessor. Then E_{rp} and E_{rq} are injective; thus $\mu_r \sim \mu_p$ and $\mu_r \sim \mu_q$, so that $\mu_p \sim \mu_q$. Conversely, *ii*) implies *i*). Let $\mu_r \geqslant \mu_s$. For $f \in \mathrm{L}^2(\mathrm{X}, \mu_s)$, $f = 0$ means $f(x) = 0$ except on a μ_s-null set, which is the same as a μ_r-null set, and thus $f = 0$ in $\mathrm{L}^2(\mathrm{X}, \mu_r)$, i. e. E_{sr} is injective. □

The same result holds true if M is assumed to be directed to the right, and *a fortiori*, if M is a lattice.

APPENDIX C

EXAMPLES AND COUNTEREXAMPLES TO THE DUALITY CONDITION

In this Appendix we discuss the application of the duality condition (4.10) to some classes of weights:

$$1 \leqslant \eta_m^{(\rho)}\eta_m^{(-\rho)} \leqslant C \qquad \text{for all} \quad m = 0, 1, 2 \ldots$$

Of course only the asymptotic behaviour of $\eta_m^{(\rho)}$ as $m \to \infty$ is needed for the verification of this condition.

First we consider the weights

$$\rho_{\gamma,\tau}(z) = \tau \,|\, z \,|^{2\gamma}, \qquad \tau \in \mathbb{R}, \quad 0 < \gamma < 1 .$$

The asymptotic behaviour of the coefficients $\eta_m^{(\gamma,\tau)}$ has been evaluated by Daubechies [*8*], with the following results:

. $0 < \gamma \leqslant \frac{1}{2}$: $\quad \eta_m^{(\gamma,\tau)} = \text{const.}\,(\exp\,[\tau m^\gamma])\,(1 + O(m^{-1/2}, m^{2\gamma-1}))$

. $\frac{1}{2} < \gamma \leqslant \frac{2}{3}$: $\quad \eta_m^{(\gamma,\tau)} = \text{const.}\left(\exp\left[\tau m^\gamma + \frac{1}{2}\tau^2\gamma^2 m^{2\gamma-1}\right]\right)(1 + O(m^{3\gamma-2}))$

. $\frac{2}{3} < \gamma < 1$: $\quad \eta_m^{(\gamma,\tau)} = \text{const.}\,\exp\left[\tau m^\gamma + \frac{1}{2}\tau^2\gamma^2 m^{2\gamma-1} + O(m^{3\gamma-2})\right]$

Noticing that $-\rho_{\gamma,\tau} = \rho_{\gamma,-\tau}$, we have:

$$\begin{aligned} \eta_m^{(\gamma,\tau)}\eta_m^{(\gamma,-\tau)} &\leqslant \text{const.} && \text{for} \quad 0 < \gamma \leqslant \frac{1}{2} \\ &= \text{const.}\; e^{\tau^2\gamma^2 m^{2\gamma-1}} && \text{for} \quad \frac{1}{2} < \gamma < 1 \end{aligned}$$

Thus the duality condition is satisfied in the first case and fails in the second.

We turn now to the weights:

$$\rho_\beta(z) = \beta \,|\, z \,|^2, \qquad -1 < \beta < 1, \qquad \beta \neq 0$$

An immediate computation gives $\eta_m^{(\beta)} = (1 + \beta)^{-m-1}$, so that $\eta_m^{(\beta)}\eta_m^{(-\beta)} = (1 - \beta^2)^{-m-1}$, which is clearly unbounded as $m \to \infty$. So the duality condition fails for all $\beta \neq 0$, $-1 < \beta < 1$: the lattice involution does not coincide with the topological duality. Writing $\mathfrak{F}(\beta) \equiv \mathfrak{F}(\rho_\beta)$, we have:

$$\mathfrak{F}(\beta)^{\#_1} = \mathfrak{F}(-\beta) \Leftrightarrow \eta_m^{(-\beta)} = (1 - \beta)^{-m-1}$$

$$\mathfrak{F}(\beta)^\times = \mathfrak{F}(\beta)^{\#_2} = \mathfrak{F}\left(\frac{-\beta}{1+\beta}\right) \Leftrightarrow \eta_m^{\left(\frac{-\beta}{1+\beta}\right)} = (1 + \beta)^{m+1}$$

and we have explicitly $\mathfrak{F}(\beta)^\times \supsetneq \mathfrak{F}(-\beta)$. For instance:

$$\mathfrak{F}\left(\frac{1}{n}\right)^{\#_1} = \mathfrak{F}\left(-\frac{1}{n}\right), \qquad \mathfrak{F}\left(\frac{1}{n}\right)^{\#_2} = \mathfrak{F}\left(\frac{1}{n}\right)^\times = \mathfrak{F}\left(-\frac{1}{n+1}\right) \supset \mathfrak{F}\left(-\frac{1}{n}\right).$$

Thus applying alternatively $\#_1$ and $\#_2$, one gets an infinite sequence of spaces, « contracting » progressively towards $\mathfrak{F}(0) \equiv \mathfrak{F}$:

$$\#_2$$

$$\mathfrak{F}\left(\frac{1}{2}\right) \subset \mathfrak{F}\left(\frac{1}{3}\right) \subset \mathfrak{F}\left(\frac{1}{4}\right) \subset \ldots \mathfrak{F} \ldots \subset \mathfrak{F}\left(-\frac{1}{5}\right) \subset \mathfrak{F}\left(-\frac{1}{4}\right) \subset \mathfrak{F}\left(-\frac{1}{3}\right) \ldots$$

$$\#_1$$

This result of course does not contradict the general results of III: it simply reflects the fact that the dual of $\mathfrak{F}(\rho)$ is not always the space one would think naively!

APPENDIX D

THE STIELTJES MOMENT PROBLEM

Given a sequence of non-negative numbers $(\mu_n)_{n=0}^{\infty}$, the Stieltjes moment problem consists in finding a non-negative, non-decreasing function ψ such that its moments over $[0, \infty)$ are precisely $\mu_0, \mu_1, \mu_2 \ldots$:

$$\mu_n = \int_0^{\infty} t^n d\psi(t) \tag{D.1}$$

The integral is of course a Stieltjes integral. Alternatively, $d\psi$ is a positive measure on $[0, \infty)$, finite in the sense that $\psi([0, \infty)) = \mu_0 < \infty$. The solution of this problem is well-known (see for instance the monograph by Shohat and Tamarkin [*21*]. The main results are the following (Theorems 1.3 and 1.11):

a) A necessary and sufficient condition for *existence* of a nontrivial solution (by trivial solution, we mean a piecewise constant ψ with finitely many jumps) is that the following two families of quadratic forms be positive definite:

$$\sum_{n,m=0}^{N} x_n x_m \mu_{n+m} \geqslant 0 \,, \qquad \sum_{n,m=0}^{N} x_n x_m \mu_{n+m+1} \geqslant 0 \tag{D.2}$$

for arbitrary real numbers $x_0, x_1, x_2 \ldots$ and all $N = 0, 1, 2 \ldots$.

b) A sufficient condition for *uniqueness* of the solution is that:

$$\sum_{n=1}^{\infty} (\mu_n)^{-\frac{1}{2n}} = \infty \tag{D.3}$$

However, assuming the problem to have a unique solution ψ, it is rather difficult to find a sufficient condition that ψ be absolutely continuous with respect to Lebesgue measure, possibly with bounded density, i. e. $d\psi(t) = \varphi(t)dt$, with $0 \leqslant \varphi(t) \leqslant C$ (for further informations on this topic, see the monographs of Ahiezer and Krein [*25*] and of Krein and Nudel'man [*26*]).

The typical example of a Stieltjes moment problem that has a unique solution is $\mu_n = n\,!\,C^n$ for some $C > 0$. The verification of conditions (D.2) and (D.3) is then straightforward. For $\delta = 0$ or 1, one has indeed (for any $N = 1, 2 \ldots$):

$$\sum_{n,m=0}^{N} x_n x_m (n + m + \delta)\,!\,C^{n+m+\delta} \geqslant C^{1+\delta}\left(\sum_{n=0}^{N} x_n n\,!\,C^n\right)\left(\sum_{m=0}^{N} x_m m\,!\,C^m\right) \geqslant 0$$

$$\sum_{n=1}^{\infty} (\mu_n)^{-\frac{1}{2n}} = \sum_{n=1}^{\infty} (n\,!)^{-\frac{1}{2n}} C^{-\frac{1}{2}} \simeq (eC^{-1})^{\frac{1}{2}} \sum_{n=1}^{\infty} (2\pi n e^{-1})^{-\frac{1}{4n}} n^{-\frac{1}{2}} = \infty$$

where Stirling's formula has been used.

It should also be remarked that the Stieltjes moment problem (μ_m, ψ) can always be viewed as a Hamburger moment problem (that is on $(-\infty, \infty)$), namely:

$$v_m = \int_{-\infty}^{+\infty} t^m d\sigma(t)$$

where

$$\sigma(t) = \frac{1}{2}\psi(t^2) \quad \text{for} \quad t > 0$$
$$= -\frac{1}{2}\psi(t^2) \quad \text{for} \quad t < 0$$

and

$$\nu_{2m} = \mu_m, \quad \nu_{2m+1} = 0 .$$

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(Manuscrit reçu le 3 avril 1981)