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Section A :
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The Cauchy problem for coupled Yang-Mills and scalar fields in the Lorentz gauge

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RÉSUMÉ. — On étudie le problème de Cauchy pour un champ de Yang Mills et un champ scalaire classiques couplés de façon minimale, dans l'espace temps de dimension $n + 1$, en jauge de Lorentz. On démontre l'existence et l'unicité de solutions dans des intervalles de temps petits pour n quelconque, aussi bien dans des espaces locaux que globaux. En dimension deux d'espace temps, les solutions précédentes peuvent être étendues à des temps quelconques par la méthode des estimations *a priori.* En dimension trois d'espace temps, nos estimations ne donnent que des résultats partiels sur le problème d'existence globale.

ABSTRACT. — We study the Cauchy problem for minimally coupled classical Yang-Mills and scalar fields in $n + 1$ dimensional space-time in the Lorentz gauge. We prove the existence and uniqueness of solutions for small time intervals and for any n, both in local and global spaces. In space time dimension two, the previous solutions can be extended to all times by the method of *a priori* estimates. In space time dimension three, our estimates yield only partial results on the global existence problem.

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1. INTRODUCTION

The initial value problem for classical coupled Yang-Mills and scalar fields has been considered recently by several authors [*2*]-[*8*], [*10*]. In particular in a previous paper [*6*], to the introduction of which we refer for more details, we have studied this problem in the so-called temporal gauge. The main results of [*6*] include the existence and uniqueness of solutions in small time intervals for arbitrary space-time dimension and the existence and uniqueness of global solutions in space-time dimension $1 + 1$ and $2 + 1$. All these results hold both in global and local spaces. In the latter case the initial data and the solutions are required to satisfy only local regularity conditions in space, but no restriction on their behaviour at infinity. Remarkably enough all these results can be proved without using the elliptic constraint that generalizes Gauss's law.

In this paper we take up the same problem in the Lorentz gauge $\partial_\mu A^\mu = 0$. In addition to its intrinsic interest, this gauge condition is imposed as a consequence of the Lagrange equations if the Yang-Mills field is massive. In this gauge the situation seems to be less favourable than in the temporal gauge: we are still able to prove existence and uniqueness of solutions of the Cauchy problem in small time intervals for arbitrary space-time dimension; however we are able to prove existence and uniqueness of global solutions only for space-time dimensions $1 + 1$. Furthermore, even in that case, the proof makes explicit use of the elliptic constraint mentioned above. The additional problems posed by the Lorentz gauge as compared with the temporal gauge can best be seen by considering pure Lorentz gauges. If the gauge group is SU(2), the Lorentz condition becomes the equation of motion for the O(4) non linear σ-model. More generally, one can study the Cauchy problem for the O(N) non linear σ-model as well as for the so-called $\mathbb{C}$P(N) and G$\mathbb{C}$(N, p) models. This will be done in a subsequent paper. In all these models, in the same way as for the Yang-Mills field in the Lorentz gauge, our proof of global existence works in dimension $1 + 1$ and breaks down in higher dimensions.

In the same way as in the case of the temporal gauge treated in [*6*], the finite propagation speed and the presence of an elliptic constraint, which produces long range effects in the massless case, lead naturally to study the problem in local spaces, where only local regularity conditions are imposed on the initial data and the solutions.

The methods used here are similar to the ones used in [*6*]. The local problem in global spaces is treated by the method of [*9*]. The local theory in local spaces relies on Section 3 of [*6*]. The global existence problem both in global and local spaces is treated by the standard method of *a priori* estimates. The need to use the elliptic constraint complicates the proof:

it requires a local version of these estimates and an iterative restriction and extension procedure to construct the global solution.

The results of this paper have been announced in [5]. For technical reasons they will be derived here in a slightly different formalism (see Remark 2.1).

The paper is organized as follows. In Section 2 we introduce the notation, choose the dynamical variables, write the equations in suitable form, and treat the local problem in global spaces for arbitrary space-time dimension. In Section 3 we extend the previous treatment to the theory in local spaces. In Section 4 we study the problem of global existence, both in global and local spaces.

2. THE CAUCHY PROBLEM IN GLOBAL SPACES FOR SMALL TIME INTERVALS

In this section we begin our study of the initial value problem for the classical Yang-Mills field minimally coupled to a scalar field, in the Lorentz gauge $\partial_\mu A^\mu = 0$. We first introduce some notation. The Yang-Mills potential $A_\mu(t, x)$ is a function from $n + 1$ dimensional space-time to the Lie algebra $\mathcal{G}$ of a compact Lie group G. The corresponding field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu],$$

where [,] denotes the commutator in $\mathcal{G}$. We assume the existence of a non degenerate positive definite bilinear form in $\mathcal{G}$ denoted by $\langle \cdot, \cdot \rangle$, invariant under the adjoint representation of the group. The scalar field $\phi(t, x)$ belongs to a unitary representation of G in a finite dimensional vector space $\mathcal{F}$. We also denote by $\langle \cdot, \cdot \rangle$ the invariant scalar product in $\mathcal{F}$ and we use the same notation for an element of $\mathcal{G}$ and for its representative in $\mathcal{F}$. We shall write $\langle B, B \rangle = | B |^2$. We use the same notation D_μ for the covariant derivative in $\mathcal{G}$ where $D_\mu = \partial_\mu + e[A_\mu, .]$ and in $\mathcal{F}$ where $D_\mu = \partial_\mu + eA_\mu$. We use the metric $g_{\mu\nu}$ with $g_{00} = 1$, $g_{ii} = -1$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

The field equations are the variational equations associated with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu} \rangle + \frac{1}{2}\kappa^2 \langle A_\mu, A^\mu \rangle + \langle D_\mu\phi, D^\mu\phi \rangle - V(|\phi|^2), \qquad (2.1)$$

where V is a real $\mathcal{C}^1$ function defined in $\mathbb{R}^+$ with $V(0) = 0$. The equations are

$$K^\mu \equiv D_\lambda F^{\lambda\mu} + \kappa^2 A^\mu + J^\mu = 0, \qquad (2.2)$$

$$M \equiv D_\mu D^\mu \phi + \phi V'(|\phi|^2) = 0, \qquad (2.3)$$

where $J^\mu \in \mathcal{G}$ is defined by $\langle J^\mu, C \rangle = 2e \operatorname{Re} \langle D^\mu\phi, C\phi \rangle$ for all $C \in \mathcal{G}$

and V′ is the derivative of V. We shall work in the first order formalism and choose as dynamical variables the quantities A_0, $A = \{ A_j, 1 \leqslant j \leqslant n \}$, $F_0 = \{ F_{0j}, 1 \leqslant j \leqslant n \}$, $F = \{ F_{jk}, 1 \leqslant j < k \leqslant n \}$, ϕ, ψ_0 and $\psi = \{ \psi_j, 1 \leqslant j \leqslant n \}$ (ψ_μ has to be thought of as $D_\mu \phi$). The variational equations and the Lorentz condition can be rewritten as the following system of equations of motion

$$\partial_0 \begin{pmatrix} A_0 \\ A \\ F_0 \\ F \end{pmatrix} = \begin{pmatrix} 0 & R & 0 & 0 \\ -R^* & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & R_A \\ 0 & 0 & -R_A^* & 0 \end{pmatrix} \begin{pmatrix} A_0 \\ A \\ F_0 \\ F \end{pmatrix} + \begin{pmatrix} 0 \\ -e[A_0, A] \\ -e[A^\lambda, F_{\lambda.}] - J - \kappa^2 A \\ -e[A_0, F] + e[A, F_0] + e[F_0, A] \end{pmatrix} \tag{2.4}$$

$$\partial_0 \begin{pmatrix} \phi \\ \psi_0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ 0 & 0 & R \\ 0 & -R^* & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi_0 \\ \psi \end{pmatrix} - \begin{pmatrix} eA_0\phi \\ eA^\mu\psi_\mu + \phi V' \\ eA_0\psi - eA\psi_0 - eF_0\phi \end{pmatrix} \tag{2.5}$$

supplemented by the constraints

$$F_{jk} = \partial_j A_k - \partial_k A_j + e[A_j, A_k], \tag{2.6}$$

$$K_0 = 0, \tag{2.7}$$

$$\psi_j = D_j \phi. \tag{2.8}$$

Here J_0 and $J = \{ J_j, 1 \leqslant j \leqslant n \}$ are supposed to be expressed in terms of ϕ, ψ_0 and ψ_j; R is the $1 \times n$ matrix operator with entries $R_0^j = -\partial^j$ and R_A is the $n \times n(n-1)/2$ matrix operator with entries

$$(R_A)_j^{kl} = -\delta_j^l \partial^k + \delta_j^k \partial^l.$$

The system of equations (2.4) and (2.5) can be written more compactly as

$$\partial_0 u(t) = Tu(t) + f(u(t)) \tag{2.9}$$

where

$$u = \begin{pmatrix} u_A \\ u_\phi \end{pmatrix}, \quad u_A = \begin{pmatrix} A_0 \\ A \\ F_0 \\ F \end{pmatrix}, \quad u_\phi = \begin{pmatrix} \phi \\ \psi_0 \\ \psi \end{pmatrix}, \tag{2.10}$$

$$T = \begin{pmatrix} T_A & 0 \\ 0 & T_\phi \end{pmatrix}, \quad f = \begin{pmatrix} f_A \\ f_\phi \end{pmatrix}$$

and T_A, T_ϕ, f_A and f_ϕ can be read directly from (2.4) and (2.5). The fields u_A and u_ϕ take values in the finite dimensional vector spaces

$$\mathscr{V}_A = \mathscr{G} \otimes \mathbb{R}^{2n+1+n(n-1)/2} \qquad \text{and} \qquad \mathscr{V}_\phi = \mathscr{F} \otimes \mathbb{C}^{2+n}$$

respectively, so that u takes values in $\mathscr{V} = \mathscr{V}_A \oplus \mathscr{V}_\phi$. The Cauchy problem for the equation (2.9) can be transformed into the integral equation

$$u(t) = U(t)u_0 + \int_0^t d\tau U(t-\tau) f(u(\tau)), \tag{2.11}$$

where u_0 is the initial condition,

$$U(t) = \begin{pmatrix} U_A(t) & 0 \\ 0 & U_\phi(t) \end{pmatrix},$$

$$U_A(t) = \begin{pmatrix} \cos \omega t & (\sin \omega t)\omega^{-1}R & (1-\cos \omega t)\omega^{-1}R & 0 \\ -R^*\omega^{-1} \sin \omega t & \cos (R^*R)^{1/2}t & \omega^{-1} \sin \omega t & (1-\cos \omega_\perp t)\omega_\perp^{-2}R_A \\ 0 & 0 & \cos \omega_\perp t & (\sin \omega_\perp t)\omega_\perp^{-1}R_A \\ 0 & 0 & -R_A^*\omega_\perp^{-1} \sin \omega_\perp t & \cos (R_A^*R_A)^{1/2}t \end{pmatrix} \tag{2.12}$$

with $\omega = (RR^*)^{1/2} = (-\Delta)^{1/2}$, $\omega_\perp = (R_A R_A^*)^{1/2}$, and $U_\phi(t)$ is given by a similar formula (see (2.13) of [*6*]).

REMARK 2.1. — In the Lorentz gauge there is a large flexibility in the choice of the dynamical variables. For instance one could take the quantities A_0, $A = \{ A_j, 1 \leqslant j \leqslant n \}$, $B = \{ B_j, 1 \leqslant j \leqslant n \}$ (to be thought of as $B_j = D_0 A_j$), ϕ and ψ_0 (to be thought of as $\psi_0 = D_0\phi$), thereby obtaining the equations

$$\partial_0 \begin{pmatrix} A_0 \\ A \\ B \end{pmatrix} = \begin{pmatrix} 0 & R & 0 \\ 0 & 0 & \mathbb{1} \\ 0 & \Delta & 0 \end{pmatrix} \begin{pmatrix} A_0 \\ A \\ B \end{pmatrix} - \begin{pmatrix} 0 \\ e[A_0, A] \\ e\partial_j [A^j, A] + e[A^\lambda, F_{\lambda .}] + \kappa^2 A + J \end{pmatrix} \tag{2.13}$$

$$\partial_0 \begin{pmatrix} \phi \\ \psi_0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi_0 \end{pmatrix} - \begin{pmatrix} eA_0\phi \\ eA^\mu\psi_\mu + e\partial_j(A^j\phi) + \phi V' \end{pmatrix} \tag{2.14}$$

supplemented by the constraint (2.7). In (2.13), (2.14) and (2.7), $F_{\mu\nu}$, J_μ and $\psi_j \equiv D_j\phi$ are supposed to be expressed in terms of the dynamical variables. In the choice leading to (2.13), (2.14), as well as in the previous choice leading to (2.4), (2.5), the Lorentz condition has been used as the equation of motion for A_0. Another, more standard choice, consists in substituting the Lorentz condition also into the variational equation $K_0 = 0$, thereby converting it into an evolution equation for A_0. One can then choose as dynamical variables A_μ $(0 \leqslant \mu \leqslant n)$, B_μ $(0 \leqslant \mu \leqslant n)$ (to be thought of as $B_\mu = D_0 A_\mu$), ϕ and ψ_0. The equations of motion for A_μ and B_μ become

$$\partial_0 \begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix} - \begin{pmatrix} e[A_0, A_\mu] \\ e\partial_j [A^j, A_\mu] + e[A^\lambda, F_{\lambda\mu}] + \kappa^2 A_\mu + J_\mu \end{pmatrix} \tag{2.15}$$

while the equations for ϕ and ψ_0 are still given by (2.14). These equations must be supplemented by the constraints $\partial_\mu A^\mu = 0$ and $K_0 = 0$ (at $t = 0$). Up to a minor difference in the definition of B_μ, this last formalism is the one used in [5]. It has the well-known drawback that the energy is bounded from below only if the constraints are satisfied.

We now define the spaces in which we shall look for solutions of the integral equation (2.11). For any integer k, we define the space $\mathcal{H}^k$ of those u such that, componentwise, $F_{\mu\nu}$ and ψ_μ belong to the Sobolev space $H^{k-1} \equiv H^{k-1}(\mathbb{R}^n)$ while A_μ and ϕ belong to the Sobolev space $H^k \equiv H^k(\mathbb{R}^n)$. More precisely, for any $u = \{ A_\mu, F_{\mu\nu}, \phi, \psi_\mu \}$ we define the norm $\| u \|$ in $\mathcal{H}^k$ as follows:

$$\| u \|^2 \equiv M_{k-1}(u)^2 + N_{k-1}(u)^2 + P_{k-1}(u)^2 \tag{2.16}$$

where, for $k \geqslant 0$,

$$M_k(u)^2 \equiv \sum_{0 \leqslant |\alpha| \leqslant k} \left\{ \frac{1}{2} \sum_{\mu<\nu} \| \, | \partial^\alpha F_{\mu\nu} | \, \|_2^2 + \sum_\mu \| \, | \partial^\alpha \psi_\mu | \, \|_2^2 \right\} \tag{2.17}$$

$$N_k(u)^2 \equiv \sum_{0 \leqslant |\alpha| \leqslant k} \left\{ \frac{1}{2} \sum_\mu \| \, | \partial^\alpha A_\mu | \, \|_2^2 + \| \, | \partial^\alpha \phi | \, \|_2^2 \right\} \tag{2.18}$$

$$P_k(u)^2 \equiv \sum_{|\alpha| = k} \sum_j \left\{ \frac{1}{2} \sum_\mu \| \, | \partial^\alpha (\partial_j A_\mu + F_{\mu j}) | \, \|_2^2 + \| \, | \partial^\alpha (\partial_j \phi - \psi_j) | \, \|_2^2 \right\}, \tag{2.19}$$

and for $k \leqslant 0$,

$$M_k(u)^2 \equiv \frac{1}{2} \sum_{\mu<\nu} \| \, | (1 - \Delta)^{k/2} F_{\mu\nu} | \, \|_2^2 + \sum_\mu \| \, | (1 - \Delta)^{k/2} \psi_\mu | \, \|_2^2 \tag{2.20}$$

$$N_k(u)^2 \equiv \frac{1}{2} \sum_\mu \| \, | (1 - \Delta)^{k/2} A_\mu | \, \|_2^2 + \| \, | (1 - \Delta)^{k/2} \phi | \, \|_2^2 \tag{2.21}$$

$$P_k(u)^2 \equiv \sum_j \left\{ \frac{1}{2} \sum_\mu \| \, | (1 - \Delta)^{k/2} (\partial_j A_\mu + F_{\mu j}) | \, \|_2^2 + \| \, | (1 - \Delta)^{k/2} (\partial_j \phi - \psi_j) | \, \|_2^2 \right\}. \tag{2.22}$$

Here α is a mutiindex of space derivatives and $\| \cdot \|_q$ denotes the norm in $L^q \equiv L^q(\mathbb{R}^n)$. When equipped with the norm (2.16), $\mathcal{H}^k$ is a Hilbert space which is a direct sum of usual Sobolev spaces. The norm (2.16) is equivalent to the simpler looking norm

$$\{ M_{k-1}(u)^2 + N_k(u)^2 \}^{1/2} . \tag{2.23}$$

However the choice of (2.16) is better adapted to the free evolution $U(t)$ as will be seen in Lemma 2.1 below. For brevity we have not appended an index to the norm $\| u \|$. Furthermore, from now on we shall omit the $| \quad |$ when appearing in an L^p norm.

In order to prove the existence of solutions of (2.11) we need the following properties of $U(t)$.

LEMMA 2.1. — For any integer k, $U(t)$ is a (bounded) strongly continuous one-parameter group in $\mathcal{H}^k$ and, for any $t \in \mathbb{R}$, for any $u \in \mathcal{H}^k$, $U(t)$ satisfies the following estimate

$$\| U(t)u \| \leqslant \mu(t) \| u \| , \tag{2.24}$$

where

$$\mu(t) = \left\{ 1 + \frac{1}{2} | t | (| t | + (t^2 + 4)^{1/2}) \right\}^{1/2} . \tag{2.25}$$

Proof. — The proof runs in the same way as that of Lemma 2.1 of [*6*], after noticing that, for any k, $M_k(U(t)u)$ and $P_k(U(t)u)$ are constant in time, while

$$N_k(U(t)u) \leqslant N_k(u) + | t | M_k(u) . \tag{2.26}$$

Q. E. D.

We can now prove the existence of local solutions of (2.11). For any interval I and any Banach space $\mathcal{B}$ we denote by $\mathcal{C}(I, \mathcal{B})$ the space of continuous functions from I to $\mathcal{B}$, and, for any positive integer l, we denote by $\mathcal{C}^l(I, \mathcal{B})$ the space of l-times continuously differentiable functions from I to $\mathcal{B}$. For compact I, $\mathcal{C}(I, \mathcal{B})$ is a Banach space when equipped with the Sup norm.

PROPOSITION 2.1. — Let $k = [n/2 + 1]$ ($[\lambda]$ is the integral part of λ), $V \in \mathcal{C}^{k+1}(\mathbb{R}^+)$ and $u_0 \in \mathcal{H}^k$. Then, there exists a $T_0 > 0$, depending on $\| u_0 \|$, such that (2.11) has a unique solution $u \in \mathcal{C}([-T_0, T_0], \mathcal{H}^k)$.

Proof. — The proof runs in the same way as that of Proposition 2.1 of [*6*] after noticing that the multiplication by a function of H^k is a bounded operator both in H^k and in H^{k-1}. Q. E. D.

Under additional regularity assumptions on the potential V and on the initial data one can prove additional regularity properties of the solutions.

PROPOSITION 2.2. — Let $k' \geqslant k = [n/2 + 1]$, let $V \in \mathcal{C}^{k'}(\mathbb{R}^+)$ and let $u_0 \in \mathcal{H}^{k'}$. Let I be an interval of $\mathbb{R}$ containing the origin and let $u \in \mathcal{C}(I, \mathcal{H}^k)$ be a solution of (2.11). Then $u \in \mathcal{C}^l(I, \mathcal{H}^{k'-l})$ for any l, $0 \leqslant l \leqslant k'$.

Proof. — The proof is similar to that of Proposition 2.2 of [6], but slightly more complicated because of the choice of the spaces. We shall therefore present the estimates in some detail. First we prove that $u \in \mathscr{C}(\mathrm{I}, \mathscr{H}^{k'})$ by induction. Let therefore $u \in \mathscr{C}(\mathrm{I}, \mathscr{H}^{l-1})$ for some l, $k < l \leqslant k'$. We want to show that $u \in \mathscr{C}(\mathrm{I}, \mathscr{H}^{l})$, or equivalently that, for any multiindex α, with $|\alpha| = l - k$, $v^\alpha \equiv \partial^\alpha u$ belongs to $\mathscr{C}(\mathrm{I}, \mathscr{H}^{k})$. Now $v^\alpha \in \mathscr{C}(\mathrm{I}, \mathscr{H}^{k-1})$ and v^α satisfies the equation

$$v^\alpha(t) = U(t)\partial^\alpha u_0 + \int_0^t d\tau \mathrm{U}(t - \tau) \left\{ \sum_\sigma \frac{\partial f}{\partial u_\sigma}(u(\tau))v^\alpha_\sigma(\tau) + g^\alpha(u(\tau)) \right\} \quad (2.27)$$

where σ labels the various components of u, and g^α is a polynomial in the space derivatives of u of order at most $l - k - 1$ and in the derivatives of f with respect to u of order at most $l - k$. We want to show that (2.27) considered as a linear integral equation for v^α has a unique solution both in $\mathscr{C}(\mathrm{I}, \mathscr{H}^{k})$ and $\mathscr{C}(\mathrm{I}, \mathscr{H}^{k-1})$. This will prove that $\partial^\alpha u \in \mathscr{C}(\mathrm{I}, \mathscr{H}^{k})$ and complete the induction. For this purpose it will be sufficient to show that

$$h^\alpha \equiv \frac{\partial f}{\partial u_\sigma}(u(\tau))v^\alpha_\sigma(\tau)$$

belongs to $\mathscr{C}(\mathrm{I}, \mathscr{H}^{m})$ if v^α belongs to $\mathscr{C}(\mathrm{I}, \mathscr{H}^{m})$ for $m = k, k - 1$, and that $g^\alpha(u(\tau))$ belongs to $\mathscr{C}(\mathrm{I}, \mathscr{H}^{k})$. We consider the terms in h^α and g^α containing only A and F: the terms containing ϕ and ψ, including those coming from the potential V, can be treated in the same way. Now h^α contains terms of the type $\mathrm{A}a^\alpha$, $\mathrm{A}f^\alpha$ and $\mathrm{F}a^\alpha$, where a^α and f^α are the A and F components of v^α. By the induction assumption we know that componentwise $\mathrm{A} \in \mathscr{C}(\mathrm{I}, \mathrm{H}^{l-1})$ and $\mathrm{F} \in \mathscr{C}(\mathrm{I}, \mathrm{H}^{l-2})$. We take $a^\alpha \in \mathscr{C}(\mathrm{I}, \mathrm{H}^{m})$ and $f^\alpha \in \mathscr{C}(\mathrm{I}, \mathrm{H}^{m-1})$ with m being either k or $k - 1$, and we have to show that $\mathrm{A}a^\alpha$ belongs to $\mathscr{C}(\mathrm{I}, \mathrm{H}^{m})$ and that $\mathrm{A}f^\alpha$ and $\mathrm{F}a^\alpha$ belong to $\mathscr{C}(\mathrm{I}, \mathrm{H}^{m-1})$. This is a consequence of the Sobolev inequalities, which imply that multiplication by a function in H^k is a bounded operator in H^r for $-k \leqslant r \leqslant k$ and that the product of two functions in H^{k-1} lies in H^{k-2}, with continuity. Similarly, the terms with A and F in g^α have the form $\partial^{\alpha_1}\mathrm{A}\partial^{\alpha_2}\mathrm{A}$ and $\partial^{\alpha_1}\mathrm{A}\partial^{\alpha_2}\mathrm{F}$ with $|\alpha_j| \geqslant 1$, $|\alpha_1| + |\alpha_2| = l - k$. We have to show that for $\mathrm{A} \in \mathscr{C}(\mathrm{I}, \mathrm{H}^{l-1})$ and $\mathrm{F} \in \mathscr{C}(\mathrm{I}, \mathrm{H}^{l-2})$, these terms lie in $\mathscr{C}(\mathrm{I}, \mathrm{H}^{k})$ and $\mathscr{C}(\mathrm{I}, \mathrm{H}^{k-1})$ respectively. This again follows from the Sobolev inequalities as above.

The end of the proof is the same as that of Proposition 2.2 of [6] with appropriate changes. Q. E. D.

So far we have considered the system (2.4) and (2.5) without taking into account the constraints (2.6), (2.7) and (2.8). We now show that the constraints are preserved in time.

PROPOSITION 2.3. — Let $k = [n/2 + 1]$, let $\mathrm{V} \in \mathscr{C}^k(\mathbb{R}^+)$ and let $u_0 \in \mathscr{H}^k$.

Let I be an interval of $\mathbb{R}$ containing the origin and let $u \in \mathscr{C}(\mathrm{I}, \mathscr{H}^k)$ be a solution of (2.11). Let Ω be an open subset of $\mathbb{R}^n$, possibly $\mathbb{R}^n$ itself. Let u_0 satisfy (2.6) (resp. (2.8)) in Ω at $t = 0$. Then $u(t)$ satisfies (2.6) (resp. (2.8)) in Ω for all $t \in \mathrm{I}$. If in addition u_0 satisfies (2.7) in Ω at $t = 0$, then $u(t)$ satisfies (2.7) in Ω for all $t \in \mathrm{I}$.

Proof. — From Proposition 2.2 it follows that the quantities

$$\lambda_{jk} \equiv \mathrm{F}_{jk} - \partial_j \mathrm{A}_k + \partial_k \mathrm{A}_j - e[\mathrm{A}_j, \mathrm{A}_k]$$

and $\mu_j \equiv \psi_j - \mathrm{D}_j\phi$ belong to $\mathscr{C}(\mathrm{I}, \mathrm{H}^{k-2})$ componentwise. Furthermore, by (2.4) and (2.5), they satisfy

$$\mathrm{D}_0\lambda_{jk} \equiv \partial_0\lambda_{jk} + e[\mathrm{A}_0, \lambda_{jk}] = 0 \tag{2.28}$$

$$\mathrm{D}_0\mu_j \equiv \partial_0\mu_j + e[\mathrm{A}_0, \mu_j] = 0 \tag{2.29}$$

with initial conditions $\lambda_{jk}(0) = 0$ and $\mu_j(0) = 0$ in Ω. From the fact that $\mathrm{A}_0 \in \mathscr{C}(\mathrm{I}, \mathrm{H}^k)$ and that multiplication by a function in H^k is a bounded operator in $\mathrm{H}^{k-2}(\Omega)$ it follows that $\lambda_{jk} = 0$ and $\mu_j = 0$ in Ω for all times. Similarly, from Proposition 2.2 it follows that

$$\mathrm{K}_0 \in \mathscr{C}(\mathrm{I}, \mathrm{H}^{k-2}) \cap \mathscr{C}^1(\mathrm{I}, \mathrm{H}^{k-3})$$

componentwise. From (2.4), (2.5) and the previous constraints it follows that

$$\mathrm{D}_0\mathrm{K}^0 \equiv \partial_0\mathrm{K}^0 + e[\mathrm{A}_0, \mathrm{K}^0] = 0 \tag{2.30}$$

in $\mathrm{H}^{k-3}(\Omega)$. Using again the fact that multiplication by $\mathrm{A}_0 \in \mathrm{H}^k$ is a bounded operator in $\mathrm{H}^{k-2}(\Omega)$, we see that K^0 satisfies (2.30) in $\mathrm{H}^{k-2}(\Omega)$. Together with the initial condition $\mathrm{K}^0 = 0$ in Ω at $t = 0$, (2.30) implies that K^0 vanishes in Ω for all times. Q. E. D.

3. THE CAUCHY PROBLEM IN LOCAL SPACES FOR SMALL TIME INTERVALS

In this section we extend the theory developed in Section 2 to a theory in local spaces. The general framework is described in Section 3 of [*6*] to which we refer for details. For the convenience of the reader we recall a few definitions. We call *dependence domain* any open subset Q of $\mathbb{R}^+ \times \mathbb{R}^n$ such that, for any $(t, x) \in \mathrm{Q}$, Q contains the set

$$\{ (t', x') : t' \geqslant 0 \text{ and } |x - x'| \leqslant t - t' \} .$$

The sections of Q at fixed time t are denoted by

$$\mathrm{Q}(t) = \{ x : x \in \mathbb{R}^n \text{ and } (t, x) \in \mathrm{Q} \} \tag{3.1}$$

and are open. For any open ball $\Omega = B(x, R)$ with center x and radius R and for any $t \in \mathbb{R}$, we define

$$\Omega_{\pm}(t) = B(x, R \pm |t|), \tag{3.2}$$

with the convention that $B(x, R)$ is empty if $R \leqslant 0$. We also define

$$Q(\Omega, t) = \{ (t', x') : 0 \leqslant |t'| < |t|, tt' \geqslant 0 \text{ and } x' \in \Omega_-(t') \} \tag{3.3}$$

which is a dependence domain for $t > 0$.

We next define the local spaces. Let k be an integer, $k \geqslant 1$. In Section 2 we have defined $\mathscr{H}^k$ as a direct sum of usual Sobolev spaces H^m with $m = k$ or $m = k - 1$. For any open (respectively bounded open) set $\Omega \subset \mathbb{R}^n$, we define $\mathscr{H}^k_{\text{loc}}(\Omega)$ (respectively $\mathscr{H}^k(\Omega)$) as the corresponding direct sum of the Sobolev spaces $H^m_{\text{loc}}(\Omega)$ (respectively $H^m(\Omega)$). The space $\mathscr{H}^k(\Omega)$ is a Hilbert space with norm

$$\| u \|^2_{\Omega} = M_{k-1,\Omega}(u)^2 + N_{k-1,\Omega}(u)^2 + P_{k-1,\Omega}(u)^2, \tag{3.4}$$

where $M_{k,\Omega}$, $N_{k,\Omega}$ and $P_{k,\Omega}$ are defined by formulas similar to (2.17), (2.18), (2.19) where however the L^2-norms are now taken in $L^2(\Omega)$. For $\Omega = \mathbb{R}^n$, we shall denote $\mathscr{H}^k_{\text{loc}}(\mathbb{R}^n)$ by $\mathscr{H}^k_{\text{loc}}$. For suitable dependence domains Q, we shall be interested in $\mathscr{H}^k_{\text{loc}}$-valued solutions of equation (2.11) in Q in the sense of Section 3 of [*6*] (see especially (3.17) of [*6*]).

The crucial property of the free evolution is the following local version of Lemma 2.1.

LEMMA 3.1. — For any integer k, $k \geqslant 1$, $U(t)$ is a strongly continuous one-parameter group in $\mathscr{H}^k_{\text{loc}}$ and, for any $t \in \mathbb{R}$ for any open ball $\Omega \subset \mathbb{R}^n$, for any $u \in \mathscr{H}^k_{\text{loc}}$, the following estimate holds

$$\| U(t)u \|_{\Omega_-(t)} \leqslant \mu(t) \| u \|_{\Omega}, \tag{3.5}$$

where $\mu(t)$ is defined by (2.25).

Proof. — The proof runs in the same way as that of Lemma 4.1 of [*6*]. One first proves that, for any smooth u, for any non negative integer k and any open ball Ω,

$$M_{k,\Omega_-(t)}(U(t)u) \leqslant M_{k,\Omega}(u), \tag{3.6}$$

$$N_{k,\Omega_-(t)}(U(t)u) \leqslant N_{k,\Omega}(u) + |t| M_{k,\Omega}(u), \tag{3.7}$$

$$P_{k,\Omega_-(t)}(U(t)u) \leqslant P_{k,\Omega}(u). \tag{3.8}$$

From there on the proof proceeds as that of Lemma 2.1 of [*6*]. Q. E. D.

We can now state the local existence result that follows from the general theory presented in Section 3 of [*6*].

PROPOSITION 3.1. — Let $k = [n/2 + 1]$, $V \in \mathscr{C}^{k+1}(\mathbb{R}^+)$ and $u_0 \in \mathscr{H}^k_{\text{loc}}$. Then there exists a dependence domain Q with $Q(0) = \mathbb{R}^n$ and a (unique)

$\mathcal{H}^k_{\text{loc}}$-valued solution of the equation (2.11) in Q (in the sense of Section 3 of [6]).

The regularity result of Section 2 can also be extended to the theory in local spaces. We denote by r_Ω the operator in $\mathcal{H}^k_{\text{loc}}$ of restriction to Ω, namely the operator of multiplication by the characteristic function of Ω.

PROPOSITION 3.2. — Let $k' \geqslant k = [n/2 + 1]$, let $V \in \mathcal{C}^{k'}(\mathbb{R}^+)$, let Q be a dependence domain, let $u_0 \in \mathcal{H}^{k'}_{\text{loc}}(Q(0))$ and let u be a $\mathcal{H}^k_{\text{loc}}$-valued solution of (2.11) in Q. Then, for any $t > 0$ and any open ball Ω in $\mathbb{R}^n$ with closure contained in $Q(t)$, $r_\Omega u \in \mathcal{C}^l([0, t], \mathcal{H}^{k'-l}(\Omega))$ for any l, $0 \leqslant l \leqslant k'$. In particular u is a $\mathcal{H}^{k'}_{\text{loc}}$-valued solution of (2.11) in Q.

REMARK 3.1. — For $l = k'$, the statement of Proposition 3.2 involves the space $\mathcal{H}^0(\Omega)$, yet undefined. The fact that $r_\Omega u \in \mathcal{C}^{k'}(I, \mathcal{H}^0(\Omega))$ means that each component of $r_\Omega A$ and $r_\Omega \phi$ belongs to $\mathcal{C}^{k'}(I, L^2(\Omega))$ and that each component of $r_\Omega F$ and $r_\Omega \psi$ belongs to $\mathcal{C}^{k'}(I, H^{-1}(\Omega))$, where $H^{-1}(\Omega)$ is defined as in [11] (p. 213).

The proof of Proposition 3.2 is essentially the same as that of Proposition 4.2 of [6] and will be omitted.

Finally the constraints are locally preserved in time also in the local theory.

PROPOSITION 3.3. — Let $k = [n/2 + 1]$, let $V \in \mathcal{C}^k(\mathbb{R}^+)$, let Q be dependence domain, let $u_0 \in \mathcal{H}^k_{\text{loc}}(Q(0))$ and let u be a $\mathcal{H}^k_{\text{loc}}$-valued solution of (2.11) in Q. Let $T > 0$ and let Ω be an open ball in $\mathbb{R}^n$ with closure contained in Q(T). Let u_0 satisfy (2.6) (resp. (2.8)) in Ω. Then $u(t)$ satisfies (2.6) (resp. (2.8)) in Ω for all $t \in [0, T]$. If in addition u_0 satisfies (2.7) in Ω, then $u(t)$ satisfies (2.7) in Ω for all $t \in [0, T]$.

Proof. — Identical with that of Proposition 2.3. Q. E. D.

4. EXISTENCE OF GLOBAL SOLUTIONS

In this section we prove the existence of global solutions of (2.11) for $n = 1$ and make some comments on the case $n = 2$. The proof relies on *a priori* estimates of the solutions in $\mathcal{H}^k$. As mentioned in the introduction, the derivation of some of these estimates requires the use of the elliptic constraint $K^0 = 0$ (see (2.2)). As a consequence the method of proof of global existence in local spaces used in Proposition 5.3 of [6] no longer works since it is based on a cut off procedure which is not compatible with the constraint. We shall therefore use a different method which requires in particular a local version of the basic estimates. These local estimates will be derived in Propositions 4.1 and 4.2 below. Their derivation requires integration by part in truncated cones of the type (3.3).

For this purpose we need smooth approximations of the solutions of (2.11). We therefore begin by introducing a regularization procedure.

Let h be a real positive even function in $\mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\| h \|_1 = 1$ and let $h^{(m)}(x) = m^n h(xm)$. Together with the original problem associated with the Lagrangian density (2.1), we also consider the similar problem obtained by formally replacing $V(|\phi|^2)$ in (2.1) by $V(|h^{(m)} * \phi|^2)$ where $*$ denotes convolution in $\mathbb{R}^n$. The corresponding variational equations are (2.2) and

$$M^{(m)} \equiv D_\mu D^\mu \phi + h^{(m)} * g^{(m)}(\phi) \tag{4.1}$$

where

$$g^{(m)}(\phi) = (h^{(m)} * \phi) V'(|h^{(m)} * \phi|^2). \tag{4.2}$$

They are not compatible in general since the current is no longer conserved. However, if one drops the equation $K^0 = 0$ and chooses the dynamical variables and the gauge condition as in Section 2, the remaining system can be written as

$$\partial_0 u(t) = Tu(t) + f^{(m)}(u(t)) \tag{4.3}$$

supplemented by the constraints (2.6) and (2.8), where $f^{(m)}$ is obtained from f by replacing $\phi V'(|\phi|^2)$ by $h^{(m)} * g^{(m)}(\phi)$. We shall approximate solutions of (2.11) with $u_0 \in \mathscr{H}^k$ for suitable k by solutions of the equation

$$u(t) = U(t)(h^{(m)} * u_0) + \int_0^t d\tau U(t-\tau) f^{(m)}(u(\tau)). \tag{4.4}$$

LEMMA 4.1. — Let $k = [n/2 + 1]$, $V \in \mathscr{C}^{k+1}(\mathbb{R}^+)$ and $u_0 \in \mathscr{H}^k$. Then:

(1) There exists a $T_0 > 0$, depending on $\| u_0 \|$ but independent of m, such that (2.11) has a unique solution $u \in \mathscr{C}([-T_0, T_0], \mathscr{H}^k)$ and (4.4) has a unique solution $u^{(m)} \in \mathscr{C}([-T_0, T_0], \mathscr{H}^k)$.

(2) Let I be a closed interval of $\mathbb{R}$ containing the origin and let $u \in \mathscr{C}(I, \mathscr{H}^k)$ be a solution of (2.11). Then, for m sufficiently large (possibly depending on I and u_0), there exists a unique solution $u^{(m)}$ of (4.4) in $\mathscr{C}(I, \mathscr{H}^k)$ and $u^{(m)}$ tends to u in $\mathscr{C}(I, \mathscr{H}^k)$ as m tends to infinity.

(3) Let I be an interval of $\mathbb{R}$ containing the origin and let $u \in \mathscr{C}(I, \mathscr{H}^k)$ be a solution of (4.4). Then, for any $l \geqslant 1$, $u \in \mathscr{C}^{k+1}(I, \mathscr{H}^l)$.

Proof. — (1) The proof is the same as that of Proposition 2.1. The uniformity in m follows from the fact that the basic estimates can be taken uniform in $h^{(m)}$.

(2) The proof is similar to that of the continuity of the solution of (2.11) with respect to the initial data [9], with the additional complication that the equation itself depends on m. One first proves the statement for small time intervals, as considered in part (1), and then extends it to the whole of I by splitting I into a finite union of such intervals. That the second step

is at all possible follows from the existence of the solution u. We omit the details.

(3) The statement follows by differentiating (4.4) once in $\mathscr{H}^l$, which yields (4.3), and then by successive differentiation of (4.3). The limitation to $\mathscr{C}^{k+1}$ follows from the assumption on V. Q. E. D.

In all this section we shall assume that the potential satisfies the condition

$$V(\rho) \geqslant - a^2 \rho \tag{4.5}$$

for some $a \geqslant 0$. The first estimate we shall use is the local version of the energy conservation. For any open ball Ω in $\mathbb{R}^n$, for any

$$u \equiv (A, F, \phi, \psi) \in \mathscr{H}^k_{\text{loc}},$$

we define the local energy

$$E_\Omega = \frac{1}{2} \sum_{\lambda < \mu} \| F_{\lambda\mu} \|^2_{2,\Omega} + \sum_\mu \| \psi_\mu \|^2_{2,\Omega} + \frac{1}{2} \sum_\mu \kappa^2 \| A_\mu \|^2_{2,\Omega} + \int_\Omega dx V(|\phi|^2) \tag{4.6}$$

where $\|\cdot\|_{q,\Omega}$ denotes the norm in $L^q(\Omega)$, $1 \leqslant q \leqslant \infty$. We also define

$$\bar{E}_\Omega = E_\Omega + a^2 \| \phi \|^2_{2,\Omega}. \tag{4.7}$$

If u depends on t, we denote by $E_\Omega(t)$ and $\bar{E}_\Omega(t)$ the corresponding quantities associated with $u(t)$.

PROPOSITION 4.1. — Let $k = [n/2 + 1]$, let $V \in \mathscr{C}^{k+1}(\mathbb{R}^+)$ satisfy the condition (4.5), let $u_0 \in \mathscr{H}^k$, let I be an interval of $\mathbb{R}$ containing the origin and let $u \in \mathscr{C}(I, \mathscr{H}^k)$ be a solution of (2.11). Then, for any open ball Ω in $\mathbb{R}^n$ and any $t \in I$, u satisfies the estimates

$$\bar{E}_{\Omega - (t)}(t) \leqslant \left(\frac{d\nu_\Omega(t)}{dt} \right)^2, \tag{4.8}$$

$$\| \phi(t) \|_{2,\Omega - (t)} \leqslant \| \phi(0) \|_{2,\Omega} + \nu_\Omega(t), \tag{4.9}$$

$$\left\{ \frac{1}{2} \sum_\mu \| A_\mu(t) \|^2_{2,\Omega - (t)} \right\}^{1/2} \leqslant \left\{ \frac{1}{2} \sum_\mu \| A_\mu(0) \|^2_{2,\Omega} \right\}^{1/2} + \nu_\Omega(t), \tag{4.10}$$

where

$$\nu_\Omega(t) = \| \phi(0) \|_{2,\Omega} (\cosh at - 1) + \bar{E}_\Omega(0)^{1/2} a^{-1} \sinh at. \tag{4.11}$$

Proof. — By Lemma 4.1, for sufficiently large m, (4.4) has a unique solution $u^{(m)} \in \mathscr{C}^{k+1}([0, t], \mathscr{H}^l)$ for any $l \geqslant 1$. In particular

$$u^{(m)} \equiv (A^{(m)}, F^{(m)}, \phi^{(m)}, \psi^{(m)})$$

is $\mathscr{C}^{k+1}$ of space-time componentwise. We now define

$$\overline{\theta}_0^{(m)} = \frac{1}{2}\sum_{\lambda<\mu} | F_{\lambda\mu}^{(m)} |^2 + \sum_{\mu} | \psi_\mu^{(m)} |^2 + \frac{1}{2}\kappa^2 \sum_{\mu} | A_\mu^{(m)} |^2 + V(| h^{(m)} * \phi^{(m)} |^2) + a^2 | \phi^{(m)} |^2 \quad (4.12)$$

and

$$\overline{\theta}_j^{(m)} = \langle F^{(m)0l}, F_{lj}^{(m)} \rangle + \kappa^2 \langle A_0^{(m)}, A_j^{(m)} \rangle + 2 \operatorname{Re} \langle \psi_0^{(m)}, \psi_j^{(m)} \rangle . \quad (4.13)$$

Using the field equations we obtain

$$\partial^\mu \overline{\theta}_\mu^{(m)} = 2a^2 \operatorname{Re} \langle \phi^{(m)}, \psi_0^{(m)} \rangle + \xi^{(m)} \quad (4.14)$$

where

$$\xi^{(m)} = - 2 \operatorname{Re} \langle \psi_0^{(m)}, h^{(m)} * g^{(m)}(\phi^{(m)}) \rangle + 2 \operatorname{Re} \langle D_0^{(m)}(h^{(m)} * \phi^{(m)}), g^{(m)}(\phi^{(m)}) \rangle \quad (4.15)$$

and

$$D_\mu^{(m)} = \partial_\mu + eA_\mu^{(m)} . \quad (4.16)$$

We next integrate (4.14) in the truncated cone $Q(\Omega, t)$ (see (3.3)) and apply Gauss's theorem. Since $\overline{\theta}_\mu^{(m)}$ is outgoing on the side surface of $Q(\Omega, t)$, we obtain

$$\int_{\Omega-(t)} dx \overline{\theta}_0^{(m)}(t, x) \leqslant \int_\Omega dx \overline{\theta}_0^{(m)}(0, x) + \int_0^t d\tau \int_{\Omega-(\tau)} dx \{ 2a^2 \operatorname{Re} \langle \phi^{(m)}, \psi_0^{(m)} \rangle (\tau, x) + \xi^{(m)}(\tau, x) \} . \quad (4.17)$$

Now it follows from Lemma 4.1 that, when m tends to infinity, all terms in (4.17) have well-defined limits (in particular $\int dx \xi^{(m)}(\tau, x)$ tends to zero), so that (4.17) becomes

$$\overline{E}_{\Omega-(t)}(t) \leqslant \overline{E}_\Omega(0) + 2a^2 \int_0^t d\tau \int_{\Omega-(\tau)} dx \operatorname{Re} \langle \phi, \psi_0 \rangle (\tau, x) . \quad (4.18)$$

Similarly, for any $\tau \in [0, t]$, we have

$$\| \phi(\tau) \|_{2,\Omega-(\tau)} \leqslant \| \phi(0) \|_{2,\Omega-(\tau)} + \int_0^\tau d\tau' \| \psi_0(\tau') \|_{2,\Omega-(\tau)} . \quad (4.19)$$

Let

$$y(t) = \overline{E}_{\Omega-(t)}(t) , \qquad z(t) = \| \phi(t) \|_{2,\Omega-(t)} . \quad (4.20)$$

Applying Schwarz's inequality to the last term on the RHS of (4.18) and increasing the integration domain in the RHS of (4.19) we obtain

$$y(t) \leqslant y(0) + 2a^2 \int_0^t d\tau y(\tau)^{1/2} z(\tau) \quad (4.21)$$

and

$$z(t) \leqslant z(0) + \int_0^t d\tau y(\tau)^{1/2} . \quad (4.22)$$

Substituting (4.22) into (4.21) we obtain

$$y(t) \leqslant E_\Omega(0) + a^2 \left\{ \int_0^t d\tau y(\tau)^{1/2} + z(0) \right\}^2, \tag{4.23}$$

from which (4.8) and (4.11) follow by an elementary computation. The estimate (4.9) then follows from (4.22). In order to prove (4.10) we first notice that from the field equations it follows that $A^{(m)}$ satisfies the relation

$$\partial^0 \left\{ \frac{1}{2} \sum_\mu | A_\mu^{(m)} |^2 \right\} + \partial^j \langle A_0^{(m)}, A_j^{(m)} \rangle = - \langle A_j^{(m)}, F^{(m)0j} \rangle . \tag{4.24}$$

Integrating (4.24) over $Q(\Omega, t)$, applying Gauss's theorem, letting m tend to infinity and using (4.8), we finally obtain (4.10). Q. E. D.

The previous proposition holds for any space-time dimension. We now concentrate on the case $n = 1$ and estimate the components of A in H^1_{loc}.

PROPOSITION 4.2. — Let $n = 1$, let $V \in \mathscr{C}^2(\mathbb{R}^+)$ satisfy the condition (4.5), let $\Omega = B(x, R) = (x - R, x + R)$ be an open interval in $\mathbb{R}$ with $R > 1$, let $u_0 \in \mathscr{H}^1$ satisfy the constraints (2.7) and (2.8) in Ω. Let I be an interval of $\mathbb{R}$ containing the origin and let $u \in \mathscr{C}(I, \mathscr{H}^1)$ be a solution of (2.11). Then, for any $t \in I$, $| t | \leqslant R - 1$, A satisfies the estimate

$$\left\{ \sum_\mu \| \partial_1 A_\mu(t) \|^2_{2,\Omega - (t)} \right\}^{1/2} \leqslant \beta_\Omega(\| u_0 \|_\Omega) \tag{4.25}$$

for some estimating function β_Ω independent of t.

Proof. — Let

$$\theta_0(A) = | \partial_1 A_1 |^2 + \frac{1}{2} (| \partial_1 A_0 |^2 + | \partial_0 A_1 |^2) \tag{4.26}$$

and

$$\theta_1(A) = \langle \partial_1 A_1, \partial_1 A_0 \rangle + \langle \partial_1 A_1, \partial_0 A_1 \rangle . \tag{4.27}$$

We shall prove the result by deriving an *a priori* estimate on the quantity

$$\int_{\Omega - (t)} dx(\theta_0(A))(t, x) .$$

This will follow from an integral inequality which will be derived by the use of the cut off procedure of Lemma 4.1. By this lemma, for sufficiently large m, (4.4) has a unique solution $u^{(m)} \in \mathscr{C}^2([0, t], \mathscr{H}^l)$ for any $l \geqslant 1$. In particular, $u^{(m)} \equiv (A^{(m)}, F^{(m)}, \phi^{(m)}, \psi^{(m)})$ is $\mathscr{C}^2$ of space-time componentwise. By a straightforward computation, using the equations of motion, we obtain

$$\partial_0 \theta_0(A^{(m)}) = \partial_1 \theta_1(A^{(m)}) + \eta(u^{(m)}) + \langle \partial_1 A_1^{(m)}, K_0^{(m)} \rangle \tag{4.28}$$

where

$$\eta(u) = - \langle \partial_0 A_1, J_1 + \kappa^2 A_1 \rangle - \langle \partial_1 A_1, J_0 + \kappa^2 A_0 \rangle \\ + 2e \langle F_{01}, [A_1, \partial_1 A_1] \rangle + e \langle F_{01}, [A_0, \partial_0 A_1] \rangle \\ + e^2 \langle [A_1, A_0], [A_1, \partial_1 A_1] \rangle , \tag{4.29}$$

and $K_0^{(m)}$ is the quantity K_0 computed for $u^{(m)}$. We next integrate (4.28) on the truncated cone $Q(\Omega, t)$ (see (3.3)) and apply Gauss's theorem. Since $\theta_\mu(A^{(m)})$ is outgoing on the side surface of $Q(\Omega, t)$, we obtain

$$\int_{\Omega - (t)} dx \theta_0(A^{(m)})(t, x) \leqslant \int_\Omega dx \theta_0(A^{(m)})(0, x) \\ + \int_0^t d\tau \int_{\Omega - (\tau)} dx \{ \eta(u^{(m)}) + \langle \partial_1 A_1^{(m)}, K_0^{(m)} \rangle \} (\tau, x) . \tag{4.30}$$

We now let m tend to infinity. From Lemma 4.1 part (2), it follows that $\theta_0(A^{(m)})$ and $\eta(u^{(m)})$ tend to $\theta_0(A)$ and $\eta(u)$ respectively in $\mathscr{C}([0, t], L^1(\Omega))$. We now show that $K_0^{(m)}$ tends to zero in $\mathscr{C}([0, t], L^2(\Omega))$ so that its contribution to the RHS of (4.30) tends to zero. For this purpose we first consider the constraint (2.8): we note that the quantity $D_1^{(m)}\phi^{(m)} - \psi_1^{(m)}$ satisfies the equation

$$D_0^{(m)}(D_1^{(m)}\phi^{(m)} - \psi_1{}^{(m)}) = 0 \tag{4.31}$$

with the initial condition

$$(D_1^{(m)}\phi^{(m)} - \psi_1^{(m)})(0) = e [h^{(m)} * A_1(0), h^{(m)} * \phi(0)] - eh^{(m)} * [A_1(0), \phi(0)] \tag{4.32}$$

in Ω. From this and from Lemma 4.1, part (2), it follows that $D_1^{(m)}\phi^{(m)} - \psi_1^{(m)}$ tends to zero in $\mathscr{C}([0, t], H^1(\Omega))$ when m tends to infinity. We next consider the quantity $D_\mu^{(m)} J^{(m)\mu}$ where $J^{(m)\mu}$ is the current J^μ computed for $u^{(m)}$. From the equations of motion it follows that, for any $C \in \mathscr{G}$,

$$\langle C, D_\mu^{(m)} J^{(m)\mu} \rangle = - 2e \operatorname{Re} \langle D_1^{(m)}\phi^{(m)} - \psi_1^{(m)}, C\psi_1^{(m)} \rangle \\ - 2e \operatorname{Re} \{ \langle h^{(m)} * g^{(m)}(\phi^{(m)}), C\phi^{(m)} \rangle - \langle g^{(m)}(\phi^{(m)}), C(h^{(m)} * \phi^{(m)}) \rangle \} \tag{4.33}$$

It follows from the previous argument and from Lemma 4.1, part (2), that the first term in the RHS of (4.33) tends to zero in $\mathscr{C}([0, t], L^2(\Omega))$; the second term also tends to zero in $\mathscr{C}([0, t], L^2(\Omega))$ by Lemma 4.1, part (2). We finally turn to $K_0^{(m)}$. From the equations of motion it follows that

$$D_0^{(m)} K_0^{(m)} = D_\mu^{(m)} K^{(m)\mu} = - D_\mu^{(m)} J^{(m)\mu}$$

and therefore

$$K_0^{(m)}(t) = K_0^{(m)}(0) - \int_0^t d\tau (D_\mu^{(m)} J^{(m)\mu} + A_0^{(m)} K_0^{(m)})(\tau) \tag{4.34}$$

with

$$K_0^{(m)}(0) = [h^{(m)} * A_j(0), h^{(m)} * F^{j0}(0)] - h^{(m)} * [A_j(0), F^{j0}(0)] \\ + J_0^{(m)}(0) - h^{(m)} * J_0(0) \tag{4.35}$$

in Ω. In particular $K_0^{(m)}(0)$ tends to zero in $L^2(\Omega)$. From this, from (3.34) and the fact that $D_\mu^{(m)} J^{(m)\mu}$ tends to zero in $\mathscr{C}([0, t], L^2(\Omega))$, it follows that $K_0^{(m)}$ also tends to zero in $\mathscr{C}([0, t], L^2(\Omega))$.

Taking the limit $m \to \infty$ in (4.30), we obtain therefore

$$\int_{\Omega_-(t)} dx \theta_0(A)(t, x) \leqslant \int_\Omega dx \theta_0(A)(0, x) + \int_0^t d\tau \int_{\Omega_-(\tau)} dx \eta(u)(\tau, x) \quad (4.36)$$

The quantity $\eta(u)$ contains terms of various types and we now estimate their contributions to the space integral at time τ in (4.36). For brevity we omit the time τ and the spacetime indices; all space integrals and L^p norms are taken in $\Omega_-(\tau)$. We obtain

$$\left| \int dx \langle \partial A, J \rangle \right| \leqslant C \| \partial A \|_2 \| \psi \|_2 \| \phi \|_\infty$$

$$\left| \int dx \langle \partial A, A \rangle \right| \leqslant C \| \partial A \|_2 \| A \|_2$$

$$\left| \int dx \langle \partial A, [A, F] \rangle \right| \leqslant C \| \partial A \|_2 \| F \|_2 \| A \|_\infty$$

$$\left| \int dx \langle [A, A], [A, \partial A] \rangle \right| \leqslant C \| \partial A \|_2 \| A \|_2 \| A \|_\infty^2 .$$

We estimate the L^∞ norms through the local one dimensional Sobolev inequality (covariant or non covariant)

$$\| v \|_\infty^2 \leqslant L^{-1} \| v \|_2^2 + 2 \| v \|_2 \| Dv \|_2 , \quad (4.37)$$

where L is the length of the interval under consideration, in the present case $L = 2(R - \tau) \geqslant 2(R - t) \geqslant 2$. We then estimate the norms $\| \psi \|_2$, $\| F \|_2$, $\| \phi \|_2$ and $\| A \|_2$ by Proposition 4.1 and obtain from (4.36) a sublinear inequality for the function of t defined by the L. H. S. The result finally follows by a straightforward application of Gronwall's inequality.

Q. E. D.

For $n = 1$, Propositions 4.1 and 4.2 provide an *a priori* control of F and ψ in L^2 (componentwise) and of A in H^1. In order to complete the *a priori* control of u in $\mathscr{K}^1$, which is the relevant space for $n = 1$, it suffices in addition to estimate ϕ in H^1. This is done easily by using the equations of motion and the fact that $\phi \in L^2$ and $A \in H^1$, so that the norms defined with ordinary and covariant derivatives are equivalent (see Lemma 5.6 of [*6*]).

REMARK 4.1. — In Propositions (4.1) and (4.2) we have given a local version of the estimates needed to prove the existence of global solutions. As mentioned at the beginning of this section, this local form of the estimates is necessary to establish global existence in local spaces. If one is interested in solutions in global spaces, one needs these estimates only in global form, namely with Ω replaced by $\mathbb{R}^n$. In this case the proof simpler (compare with Lemma 5.1 of [*6*]).

We are now in a position to prove the global existence result in the case $n = 1$. We first state the result in global spaces.

PROPOSITION 4.3. — Let $n = 1$. let $V \in \mathscr{C}^2(\mathbb{R}^+)$ satisfy the condition (4.5), let $u_0 \in \mathscr{K}^1$ satisfy the constraints (2.7) and (2.8). Then the equation (2.11) has a unique solution u in $\mathscr{C}(\mathbb{R}, \mathscr{K}^1)$ and u satisfies the constraints (2.7) and (2.8) for all times.

Proof. — The result follows by standard arguments from Proposition 2.1 and from the estimates of Propositions 4.1 and 4.2 in the global form described in Remark 4.1. Q. E. D.

The global theory in local spaces requires a more careful treatment. In the proof we shall need the fact that, for any open ball $\Omega = B(x, R)$, there exists an extension j_Ω which is a bounded map from $\mathscr{K}^k(\Omega)$ to $\mathscr{K}^k$ such that $r_\Omega j_\Omega u = u$ for all $u \in \mathscr{K}^k(\Omega)$ and that

$$|| j_\Omega r_\Omega u || \leqslant C_1(\Omega) || r_\Omega u ||_\Omega \tag{4.38}$$

for all $u \in \mathscr{K}^k_{loc}$. Furthermore, one can choose $C_1(\Omega)$ independent of x and uniformly bounded for $R \geqslant 1$ (see Sect. 3 of [*6*] and [*1*]): $C_1(\Omega) \leqslant C_1$ for $R \geqslant 1$. In what follows we shall make such a choice. We can now state the global existence result.

PROPOSITION 4.4. — Let $n = 1$, let $V \in \mathscr{C}^2(\mathbb{R}^+)$ satisfy the condition (4.5), let $u_0 \in \mathscr{K}^1_{loc}$ satisfy the constraints (2.7) and (2.8). Then the equation (2.11) has a unique solution u in $\mathscr{C}(\mathbb{R}, \mathscr{K}^1_{loc})$ and u satisfies the constraints (2.7) and (2.8) for all times.

Proof. — We first prove that for any open interval $\Omega = B(0, R) = (-R, R)$, there exists a (unique) $\mathscr{K}^1_{loc}$-valued solution u^R of (2.11) in the truncated double cone $\Delta_1(R) \equiv Q(\Omega, R-1) \cup Q(\Omega, -R+1)$ (see (3.3)), in the sense that the restriction of u^R to $Q(\Omega, R-1)$ is a $\mathscr{K}^1_{loc}$-valued solution of (2.11) in $Q(\Omega, R-1)$ and that a similar property, defined by an obvious symmetry, holds in $Q(\Omega, -R+1)$. For this purpose, we first note that if u is a $\mathscr{K}^1_{loc}$-valued solution of (2.11) in $\Delta_1(R)$, then by Propositions 4.1 and 4.2, u satisfies an estimate of the type

$$|| u(t) ||_{\Omega_-(t)} \leqslant \gamma_R(|| u_0 ||_\Omega) \tag{4.39}$$

for some estimating function γ_R, uniformly in t for $0 \leqslant |t| \leqslant R-1$. As a consequence, for any t with $0 \leqslant |t| \leqslant R-1$,

$$\begin{aligned} || j_{\Omega_-(t)} r_{\Omega_-(t)} u(t) || &\leqslant C_1(\Omega_-(t)) \gamma_R(|| u_0 ||_\Omega) \\ &\leqslant C_1 \gamma_R(|| u_0 ||_\Omega) \end{aligned} \tag{4.40}$$

It follows now from Proposition 2.1 and its proof that, for any s, $0 \leqslant |s| \leqslant R - 1$, the equation

$$v(t) = U(t - s) j_{\Omega - (s)} r_{\Omega - (s)} u(s) + \int_0^t d\tau U(t - \tau) f(v(\tau)) \tag{4.41}$$

has a unique solution in $\mathscr{C}([s - T_0, s + T_0], \mathscr{K}^1)$, where T_0 can be taken independent of s because of (4.40). We choose such a T_0 and construct now a solution of (2.11) in $\Delta_1(R)$. For brevity we consider only the case of positive time. Let $t_j = jT_0$, $0 \leqslant j \leqslant [(R - 1)/T_0]$ and $I_j = [t_j, t_{j+1}]$. Applying Proposition 2.1, we define $v_j \in \mathscr{C}(I_j, \mathscr{K}^1)$ as the solution of (4.41) with $s = t_j$ and with $u(s)$ replaced by u_0 for $j = 0$ and by $v_{j-1}(t_j)$ for $j \geqslant 1$. Clearly, if we define $u(t) = v_j(t)$ for $t \in I_j$, then the restriction u^R of u to $Q(\Omega, R - 1)$ is a $\mathscr{K}^1_{loc}$-valued solution of (2.11) in $Q(\Omega, R - 1)$. Combining this construction with the similar one in $Q(\Omega, -R + 1)$, we obtain the announced solution u^R in $\Delta_1(R)$.

The end of the proof consists in taking an increasing sequence $\{ R_n \}$ tending to infinity and gluing together the solutions u^{R_n} in $\Delta_1(R_n)$ constructed as above. The argument is the same as in the proof of Proposition 5.3 of [6] to which we refer for more details. Q. E. D.

We conclude this section with some comments on the case of dimension $n = 2$. In this case (as well as for $n = 3$), the relevant space is $\mathscr{K}^2$ so that one needs to control F and ψ in H^1 (componentwise) and A and ϕ in H^2. In order to estimate F and ψ in H^1, it is natural to consider the quantity E_1 as in [6] (see (5.2) of [6]). For $n = 2$ and if the Yang-Mills field is massless ($\kappa = 0$), E_1 which is gauge invariant, can be controlled exactly in the same way as in the temporal gauge, both globally (see Lemma 5.2 of [6]) and locally. However, even in the massless case, the proof of Proposition 4.2 breaks down for $n = 2$ and we are unable to control A in H^1. If the Yang-Mills field is massive, the difficulty occurs already at the level of E_1, and we are unable to control E_1 in that case. In all cases, the last step, namely the control of A in H^2, knowing that F, ψ, ϕ and A are in H^1, can be done easily.

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REFERENCES

[1] R. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
[2] D. CHRISTODOULOU, *C. R. Acad. Sci. Paris*, t. **293**, 1981, p. 139-141.
[3] Y. CHOQUET-BRUHAT, D. CHRISTODOULOU, Existence of global solutions of the Yang-Mills-Higgs and spinor field equations in 3 + 1 dimensions, *C. R. Acad. Sci. Paris*, t. **293**, 1981, p. 195-200, and *Ann. Scient. E. N. S.* (in press).

[4] D. EARDLEY, V. MONCRIEF, *The global existence of Yang-Mills-Higgs field in 4 dimensional Minkowski space. I. Local existence and smoothness properties, II. Completion of the proof*, *Commun. Math. Phys.* (in press).
[5] J. GINIBRE, G. VELO, *Phys. Lett.*, t. **B99**, 1981, p. 405-410.
[6] J. GINIBRE, G. VELO, *The Cauchy problem for coupled Yang-Mills and scalar fields in the temporal gauge*, *Commun. Math. Phys.* (in press).
[7] R. T. GLASSEY, W. A. STRAUSS, *Commun. Math. Phys.*, t. **81**, 1981, p. 171-188.
[8] R. KERNER, *Ann. IHP*, t. **20**, 1974, p. 279-284.
[9] I. E. SEGAL, *Ann. Math.*, t. **78**, 1963, p. 339-364.
[10] I. E. SEGAL, *J. Funct. Anal.*, t. **33**, 1979, p. 175-194.
[11] F. TRÈVES, *Basic linear partial differential equations*, Academic press, New York, 1975.