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On the supergroup SU(m | n) and its superfield representations

by

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ABSTRACT. — The representations of the supergroup SU(m | n) are studied on the basis of the generalized Gell-Mann's matrices and identitites involving d- and f-tensors occuring in their commutation relations. The superfield representation in the space of functions of an anticommuting $m.n$-component parameter is also considered.

1. INTRODUCTION

There have been recently several attempts to consider the unification model of strong, weak and electromagnetic interactions in the context of the gauge theory of the supergroup SU(m | n) [*1*]-[*7*]. This formulation has many attractive features—it allows to obtain naturally the reasonable bare Weinberg angle, it also appears to lead naturally to a spectrum of particles in the theory, in which the Higgs mesons are various components of the gauge fields along with the usual vector mesons.

The aim of this paper is to discuss the properties of the representations of the supergroup SU(m | n). In section 2 we study some general properties of the graded SU(m | n) algebra, especially those of the generalized Gell-Mann's matrices realizing its basic matrix representation. Section 3 is devoted to spinor and tensor representations. Here too, some rules are derived for constructing scalar and vector representations from the product of two representations, and the eigenvalues of second-order Casimir operators are found for some simple representations. In section 4 we consider the superfield representation which is realized in the space of functions of an anticommuting $m.n$-component parameter.

2. GENERALIZED GELL-MANN'S MATRICES, f- AND d-TENSORS

The graded algebra $SU(m\,|\,n)$ consists of the generators T^B_A,

$$A, B = 1, 2, \ldots, m + n$$

satisfying the commutation and anticommutation relations [7]:

$$[T^B_A, T^D_C]_{-(AB,CD)} = \delta^D_A T^B_C - (AB, CD)\delta^B_C T^D_A \tag{2.1}$$

where the following notations are used:

$$(AB\ldots, CD\ldots) \equiv (-1)^{((A)+(B)+\ldots)((C)+(D)+\ldots)}$$
$$(A), (B), \ldots = \begin{cases} 0 \text{ for even indices } i = 1, 2, \ldots, m \\ 1 \text{ for odd indices } m+\alpha,\ \alpha = 1, 2, \ldots, n \end{cases} \tag{2.2}$$

Due to the relation

$$\sum_{A=1}^{m+n} T^A_A = 0 \tag{2.3}$$

the number of independent generators is $(m + n)^2 - 1$.

It is more convenient to use instead of T^B_A the following generators:

$$F_a \equiv \frac{1}{2}\sum_{i,j=1}^{m} (\lambda^{(m)}_a)^j_i T^i_j$$
$$G_p \equiv \frac{1}{2}\sum_{\alpha,\beta=1}^{n} (\lambda^{(n)}_p)^\beta_\alpha T^{m+\alpha}_{m+\beta}$$
$$H \equiv \frac{1}{2}\sum_{i=1}^{m} T^i_i = -\frac{1}{2}\sum_{\alpha=1}^{n} T^{m+\alpha}_{m+\alpha} \tag{2.4}$$
$$S^\alpha_i \equiv \frac{1}{2} T^{m+\alpha}_i$$
$$R^i_\alpha \equiv \frac{1}{2} T^i_{m+\alpha}$$

where $\lambda_a^{(m)}$ and $\lambda_p^{(n)}$ are Gell-Mann's matrices for SU(m) and SU(n) groups, $a = 1, 2, \ldots, m^2 - 1$; $p = 1, 2, \ldots, n^2 - 1$.

In terms of these new generators the relations (2.1) read:

$$\begin{aligned}
&[F_a, F_b] = if_{abc}F_c, \qquad a, b, c = 1, 2, \ldots, m^2 - 1\\
&[G_p, G_q] = if_{pqr}G_r, \qquad p, q, r = 1, 2, \ldots, n^2 - 1\\
&[F_a, G_p] = 0, \qquad [H, F_a] = 0, \qquad [H, G_p] = 0\\
&[F_a, S_i^\alpha] = -\frac{1}{2}(\lambda_a^{(m)})_i^j S_j^\alpha\\
&[F_a, R_\alpha^i] = \frac{1}{2} R_\alpha^K (\lambda_a^{(m)})_K^i\\
&[G_p, S_i^\alpha] = \frac{1}{2} S_i^\beta (\lambda_p^{(n)})_\beta^\alpha\\
&[G_p, R_\alpha^i] = -\frac{1}{2}(\lambda_p^{(n)})_\alpha^\beta R_\beta^i\\
&[H, S_i^\alpha] = -\frac{1}{2} S_i^\alpha, \qquad [H, R_\alpha^i] = \frac{1}{2} R_\alpha^i\\
&\{ S_i^\alpha, S_j^\beta \} = 0, \qquad \{ R_\alpha^i, R_\beta^j \} = 0\\
&\{ S_i^\alpha, R_\beta^j \} = \frac{1}{4}\delta_\beta^\alpha(\lambda_a^{(m)})_i^j F_a + \frac{1}{4}\delta_i^j(\lambda_p^{(n)})_\beta^\alpha G_p - \frac{1}{2}\delta_\beta^\alpha\delta_i^j \frac{m - n}{mn} H
\end{aligned} \tag{2.5}$$

Here f_{abc} and f_{pqr} are the structure constants of SU(m) and SU(n). We see that the graded group SU($m \mid n$) contains SU(m) × SU(n) × U(1) as its subgroup.

It is easy to find $(m + n)^2 - 1$ matrices $\frac{\beta_1}{2}$ of rank $m + n$ satisfying the same commutation relations as those for F_a, G_p, H, S_i^α, R_α^i and therefore realizing the basic matrix representation of the algebra. They are:

$$\begin{aligned}
&\beta_{a(m)} \equiv M_a = \left(\begin{array}{c|c} \lambda_a^{(m)} & 0 \\ \hline 0 & 0 \end{array}\right)\\
&\beta_{p(n)} \equiv N_p = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \lambda_p^{(n)} \end{array}\right)\\
&\beta_h \equiv \varphi = \frac{1}{n - m}\left(\begin{array}{c|c} \begin{matrix} n & & \\ & \ddots & \\ & & n \end{matrix} & 0 \\ \hline 0 & \begin{matrix} m & & \\ & \ddots & \\ & & m \end{matrix} \end{array}\right)\\
&\beta_{\binom{\alpha}{i}} \equiv \xi_i^\alpha, \qquad (\xi_i^\alpha)_A^B = \delta_A^{m+\alpha}\delta_i^B\\
&\beta_{\binom{i}{\alpha}} \equiv \eta_\alpha^i, \qquad (\eta_\alpha^i)_A^B = \delta_A^i\delta_{m+\alpha}^B
\end{aligned} \tag{2.6}$$

The matrices β_I will be refered to as generalized SU($m \mid n$) Gell-Mann's matrices.

Let us denote the generator associated to the matrix β_I by $\mathcal{F}_I$, so that

$$\mathcal{F}_{a^{(m)}} \equiv F_a, \qquad \mathcal{F}_{p^{(n)}} \equiv G_p, \qquad \mathcal{F}_h \equiv H, \qquad \mathcal{F}_{\binom{\alpha}{i}} \equiv S^\alpha_i, \qquad \mathcal{F}_{\binom{i}{\alpha}} \equiv R^i_\alpha$$

We then have:

$$\begin{gathered} [\mathcal{F}_I, \mathcal{F}_J]_{-(I,J)} = i f_{IJK} \mathcal{F}_K \\ \left[\frac{\beta_I}{2}, \frac{\beta_J}{2}\right]_{-(I,J)} = i f_{IJK} \frac{\beta_K}{2} \end{gathered} \tag{2.7}$$

Here (I, J) is the same notation as in (2.2) with (I) = 0 for I = $a^{(m)}$, $p^{(n)}$, h and (I) = 1 for I = $\binom{\alpha}{i}, \binom{i}{\alpha}$; f_{IJK} – the structure constants having the symmetry property followed from definition

$$f_{IJK} = -(I, J) f_{JIK} \tag{2.8}$$

their non-vanishing values are:

$$\begin{aligned}
&f_{a^{(m)}b^{(m)}c^{(m)}} = f_{abc}, && f_{p^{(n)}q^{(n)}r^{(n)}} = f_{pqr} \\
&f_{a^{(m)}\binom{\alpha}{i}\binom{\alpha}{j}} = \frac{i}{2}(\lambda_a^{(m)})^j_i, && f_{a^{(m)}\binom{i}{\alpha}\binom{j}{\alpha}} = -\frac{i}{2}(\lambda_a^{(m)})^i_j \\
&f_{p^{(n)}\binom{\alpha}{i}\binom{\beta}{i}} = -\frac{i}{2}(\lambda_p^{(n)})^\alpha_\beta, && f_{p^{(n)}\binom{i}{\alpha}\binom{i}{\beta}} = \frac{i}{2}(\lambda_p^{(n)})^\beta_\alpha \\
&f_{\binom{\alpha}{i}\binom{j}{\alpha}a^{(m)}} = -\frac{i}{4}(\lambda_a^{(m)})^j_i, && f_{\binom{\alpha}{i}\binom{i}{\beta}p^{(n)}} = -\frac{i}{4}(\lambda_p^{(n)})^\alpha_\beta \\
&f_{\binom{\alpha}{i}\binom{i}{\alpha}h} = \frac{i}{2}\frac{m-n}{mn} && \\
&f_{\binom{\alpha}{i}h\binom{\alpha}{i}} = -\frac{i}{2}, && f_{\binom{i}{\alpha}h\binom{i}{\alpha}} = \frac{i}{2}
\end{aligned} \tag{2.9}$$

In terms of β_I and $\mathcal{F}_I$ the equations (2.4) can be rewritten in a compact form:

$$\mathcal{F}_I = \frac{1}{2}\operatorname{Tr}\beta_I T \equiv \frac{1}{2}\sum_{A,B}(\beta_I)^B_A T^A_B \tag{2.10}$$

Note some simple properties of the generalized Gell-Mann's matrices.

a) They are graded-traceless:

$$S\operatorname{Tr}\beta_I \equiv \sum_A [A](\beta_I)^A_A = 0, \qquad [A] \equiv (-1)^{(A)} \tag{2.11}$$

b) They satisfy the orthogonal relations:

$$\mathrm{S\,Tr}\,\beta_I\beta_J = \eta(I)\delta_{I\tilde{J}} \tag{2.12 a}$$

$$\mathrm{Tr}\,\beta_I\beta_J = \sigma(I)\delta_{I\tilde{J}} \tag{2.12 b}$$

where $\tilde{J} \equiv J$ for even indices

$$\widetilde{\begin{pmatrix} i \\ \alpha \end{pmatrix}} \equiv \begin{pmatrix} \alpha \\ i \end{pmatrix}, \quad \widetilde{\begin{pmatrix} \alpha \\ i \end{pmatrix}} \equiv \begin{pmatrix} i \\ \alpha \end{pmatrix}$$

$$\eta(a^{(m)})=2, \quad \eta(p^{(n)})=-2, \quad \eta\begin{pmatrix} \alpha \\ i \end{pmatrix}=-1, \quad \eta\begin{pmatrix} i \\ \alpha \end{pmatrix}=1, \quad \eta(h)=\frac{mn}{n-m} \tag{2.13}$$

$$\sigma(I) = \begin{cases} |\eta(I)|, & I \neq h \\ \dfrac{mn(m+n)}{(m-n)^2}, & I = h \end{cases}$$

As a consequence of (2.12) and (2.13) we have

$$\begin{aligned} \eta(I) &= [I]\eta(\tilde{I}) = \mathrm{S\,Tr}\,\beta_I\beta_{\tilde{I}} = [I]\mathrm{S\,Tr}\,\beta_{\tilde{I}}\beta_I \\ \sigma(I) &= \mathrm{Tr}\,\beta_I\beta_{\tilde{I}} \end{aligned} \tag{2.14}$$

From (2.6) and (2.13) it is evident that

$$\beta_J^+ = \beta_{\tilde{J}} \tag{2.15}$$

c) They satisfy the completeness identity:

$$\sum_I \frac{1}{\eta(I)} (\beta_{\tilde{I}})^B_A(\beta_I)^D_C = [B]\delta^D_A\delta^B_C + \frac{1}{n-m}\delta^B_A\delta^D_C \tag{2.16}$$

d) They obey the multiplication law:

$$\beta_I\beta_J = (if_{IJK} + d_{IJK})\beta_K + \frac{\eta(I)}{m-n}\delta_{IJ} \tag{2.17}$$

where d_{IJK} are the constants appearing in the commutation relations

$$[\beta_I, \beta_J]_{+(I,J)} = 2d_{IJK}\beta_K + \frac{2\eta(I)}{m-n}\delta_{I\tilde{J}} \tag{2.18}$$

They have the symmetry property

$$d_{IJK} = (I, J)d_{JIK}, \tag{2.19}$$

their non-vanishing values are:

$$
\begin{aligned}
&d_{a^{(m)}b^{(m)}c^{(m)}} = d_{abc}, && d_{p^{(n)}q^{(n)}r^{(n)}} = d_{pqr} \\
&d_{a^{(m)}a^{(m)}h} = \frac{2}{m}, && d_{p^{(n)}p^{(n)}h} = -\frac{2}{n} \\
&d_{a^{(m)}\binom{\alpha}{i}\binom{\alpha}{j}} = \frac{1}{2}(\lambda_a^{(m)})_i^j, && d_{a^{(m)}\binom{i}{\alpha}\binom{j}{\alpha}} = \frac{1}{2}(\lambda_a^{(m)})_j^i \\
&d_{a^{(m)}ha^{(m)}} = \frac{n}{n-m}, && d_{p^{(n)}hp^{(n)}} = \frac{m}{n-m} \\
&d_{\binom{\alpha}{i}h\binom{\alpha}{i}} = \frac{1}{2}\frac{n+m}{n-m}, && d_{\binom{i}{\alpha}h\binom{i}{\alpha}} = \frac{1}{2}\frac{n+m}{n-m} \\
&d_{hhh} = \frac{n+m}{n-m} && \\
&d_{p^{(n)}\binom{\alpha}{i}\binom{\beta}{i}} = \frac{1}{2}(\lambda_p^{(n)})_\beta^\alpha, && d_{p^{(n)}\binom{i}{\alpha}\binom{i}{\beta}} = \frac{1}{2}(\lambda_p^{(n)})_\alpha^\beta \\
&d_{\binom{\alpha}{i}\binom{j}{\alpha}a^{(m)}} = -\frac{1}{4}(\lambda_a^{(m)})_i^j, && d_{\binom{\alpha}{i}\binom{i}{\beta}p^{(n)}} = \frac{1}{4}(\lambda_p^{(n)})_\beta^\alpha \\
&d_{\binom{\alpha}{i}\binom{i}{\alpha}h} = -\frac{1}{2}\frac{m+n}{mn} &&
\end{aligned}
\tag{2.20}
$$

where d_{abc} and d_{pqr} are the usual d-tensors of SU(m) and SU(n).

It is useful for further purposes to quote here some simple identities involving f_{IJK} and d_{IJK}:

$$f_{\mathrm{IJK}} = -(\mathrm{I},\mathrm{J})\frac{\eta(\mathrm{J})}{\eta(\mathrm{K})}f_{\mathrm{I}\tilde{\mathrm{K}}\tilde{\mathrm{J}}} = -(\mathrm{J},\mathrm{K})\frac{\eta(\mathrm{I})}{\eta(\mathrm{K})}f_{\tilde{\mathrm{K}}\mathrm{J}\tilde{\mathrm{I}}} \tag{2.21}$$

$$d_{\mathrm{IJK}} = (\mathrm{I},\mathrm{J})\frac{\eta(\mathrm{J})}{\eta(\mathrm{K})}d_{\mathrm{I}\tilde{\mathrm{K}}\tilde{\mathrm{J}}} = (\mathrm{J},\mathrm{K})\frac{\eta(\mathrm{I})}{\eta(\mathrm{K})}d_{\tilde{\mathrm{K}}\mathrm{J}\tilde{\mathrm{I}}} \tag{2.22}$$

$$(\mathrm{I},\mathrm{K})f_{\mathrm{ILM}}f_{\mathrm{JKL}} + (\mathrm{J},\mathrm{I})f_{\mathrm{JLM}}f_{\mathrm{KIL}} + (\mathrm{K},\mathrm{J})f_{\mathrm{KLM}}f_{\mathrm{IJL}} = 0 \tag{2.23}$$

$$(\mathrm{I},\mathrm{K})f_{\mathrm{ILM}}d_{\mathrm{JKL}} + (\mathrm{J},\mathrm{I})f_{\mathrm{JLM}}d_{\mathrm{KIL}} + (\mathrm{K},\mathrm{J})f_{\mathrm{KLM}}d_{\mathrm{IJL}} = 0 \tag{2.24}$$

The validity of (2.21) and (2.22) can be checked immediately from the explicit expressions (2.9) and (2.20) for f and d. The equations (2.23) and (2.24) are consequences of the following identity for any three graded operators $\mathscr{M}_{\mathrm{A}}$, $\mathscr{M}_{\mathrm{B}}$, $\mathscr{M}_{\mathrm{C}}$:

$$
\begin{aligned}
(\mathrm{A},\mathrm{C})[\mathscr{M}_{\mathrm{A}}, [\mathscr{M}_{\mathrm{B}}, \mathscr{M}_{\mathrm{C}}]_{\pm(\mathrm{B},\mathrm{C})}]_{-(\mathrm{A},\mathrm{BC})} + (\mathrm{B},\mathrm{A})[\mathscr{M}_{\mathrm{B}}, [\mathscr{M}_{\mathrm{C}}, \mathscr{M}_{\mathrm{A}}]_{\pm(\mathrm{C},\mathrm{A})}]_{-(\mathrm{B},\mathrm{CA})} \\
+ (\mathrm{C},\mathrm{B})[\mathscr{M}_{\mathrm{C}}, [\mathscr{M}_{\mathrm{A}}, \mathscr{M}_{\mathrm{B}}]_{\pm(\mathrm{A},\mathrm{B})}]_{-(\mathrm{B},\mathrm{CA})} = 0
\end{aligned}
\tag{2.25}
$$

when applied to the matrices β_{I}, β_{J}, β_{K}.

3. SPINOR AND TENSOR REPRESENTATIONS

The spinor representation of SU($m \mid n$) consists of $m + n$ operators ψ_A transforming in the following way:

$$[\mathscr{F}_I, \psi_A]_{-(I,A)} = -\frac{1}{2}(I, A)(\beta_I\psi)_A \qquad (3.1)$$

Its conjugate representation $\bar{\psi}^A$ is defined by

$$[\mathscr{F}_I, \bar{\psi}^A]_{-(I,A)} = \frac{1}{2}(\bar{\psi}\beta_I)^A \qquad (3.2)$$

The formulae (3.1) and (3.2) can be easily generalized for the spinors of arbitrary rank $\psi_{A_1\ldots A_r}$, $\bar{\psi}^{A_1\ldots A_r}$ and mixed spinors $\phi^{B_1\ldots B_S}_{A_1\ldots A_r}$. We have:

$$[\mathscr{F}_I, \psi_{A_1\ldots A_r}]_{-(I,A_1\ldots A_r)} = -\frac{1}{2}\sum_{K=1}^{r}(I, A_1 \ldots A_K)(\beta_I)^B_{A_K}\psi_{A_1\ldots B\ldots A_r} \qquad (3.3)$$

$$[\mathscr{F}_I, \bar{\psi}^{A_1\ldots A_r}]_{-(I,A_1\ldots A_r)} = \frac{1}{2}\sum_{K=1}^{r}(I, A_1 \ldots A_{K-1})(\beta_I)^{A_K}_B\bar{\psi}^{A_1\ldots B\ldots A_r} \qquad (3.4)$$

$$[\mathscr{F}_I, \phi^{B_1\ldots B_S}_{A_1\ldots A_r}]_{-(I,A_1\ldots A_r B_1\ldots B_S)} = \frac{1}{2}\sum_{K=1}^{S}(I, B_1 \ldots B_{K-1})(\beta_I)^{B_K}_C\phi^{B_1\ldots C\ldots B_S}_{A_1\ldots A_r} - \frac{1}{2}\sum_{K=1}^{r}(I, A_1 \ldots A_K B_1 \ldots B_S)(\beta_I)^C_{A_K}\phi^{B_1\ldots B_S}_{A_1\ldots C\ldots A_r} \qquad (3.5)$$

The « vector » representation consists of $(m + n)^2 - 1$ operators φ_I transforming in a similar way as $\mathscr{F}_I$, namely:

$$[\mathscr{F}_I, \varphi_J]_{-(I,J)} = if_{IJK}\varphi_K \qquad (3.6)$$

In some cases it is more convenient to use instead of φ_I the traceless mixed spinor φ^B_A defined by:

$$\varphi^B_A \equiv \sum_I \frac{\sqrt{2}}{\eta(\tilde{I})}[A](\beta_{\tilde{I}})^B_A\varphi_I = \sum_I \frac{\sqrt{2}}{\eta(I)}[B](\beta_{\tilde{I}})^B_A\varphi_I \qquad (3.7)$$

The reciprocal formula of (3.7) is:

$$\varphi_I = \frac{1}{\sqrt{2}}\operatorname{Tr}\beta_I\varphi \qquad (3.8)$$

The generalization of the formula (3.6) to higher tensor representation is straightforward. We have:

$$[\mathcal{F}_{\mathrm{J}}, \varphi_{\mathrm{I}_1\mathrm{I}_2\ldots\mathrm{I}_p}]_{-(\mathrm{J},\mathrm{I}_1\mathrm{I}_2\ldots\mathrm{I}_p)} = i\sum_{l=1}^{p}(\mathrm{I}, \mathrm{J}_1 \ldots \mathrm{J}_{l-1})f_{\mathrm{IJ}_l\mathrm{K}}\varphi_{\mathrm{J}_1\ldots\mathrm{K}\ldots\mathrm{J}_p} \quad (3.9)$$

From (3.1)-(3.4) it follows that if ψ_{A} and χ_{A} are spinor operators then

$$\omega \equiv \bar{\psi}^{\mathrm{A}}\chi_{\mathrm{A}} \quad (3.10\,a)$$

is SU($m \mid n$) invariant, and

$$\varphi_{\mathrm{I}} \equiv \bar{\psi}^{\mathrm{A}}(\beta_{\mathrm{I}})_{\mathrm{A}}^{\mathrm{B}}\chi_{\mathrm{B}} \quad (3.11\,a)$$

is a « vector » operator transforming according to (3.6). Similarly, if ψ_{AB} and χ_{CD} are second-rank spinor operators, then

$$\begin{gathered}\bar{\psi}^{\mathrm{AB}}\chi_{\mathrm{BA}}\\ (\mathrm{A}, \mathrm{B})\bar{\psi}^{\mathrm{AB}}\chi_{\mathrm{AB}}\end{gathered} \quad (3.10\,b)$$

are invariant, and

$$\begin{gathered}\bar{\psi}^{\mathrm{AB}}(\beta_{\mathrm{I}})_{\mathrm{A}}^{\mathrm{C}}\chi_{\mathrm{BC}}\\ (\mathrm{A}, \mathrm{B})\bar{\psi}^{\mathrm{BA}}(\beta_{\mathrm{I}})_{\mathrm{A}}^{\mathrm{C}}\chi_{\mathrm{BC}}\\ (\mathrm{B}, \mathrm{C})\bar{\psi}^{\mathrm{AB}}(\beta_{\mathrm{I}})_{\mathrm{A}}^{\mathrm{C}}\chi_{\mathrm{CB}}\\ (\mathrm{AC}, \mathrm{B})\bar{\psi}^{\mathrm{BA}}(\beta_{\mathrm{I}})_{\mathrm{A}}^{\mathrm{C}}\chi_{\mathrm{CB}}\end{gathered} \quad (3.11\,b)$$

are vector operators.

These rules can be easily generalized for higher-rank spinors.

From (3.6), using the identities (2.21)-(2.24) we can prove that if φ_{I} and ϕ_{I} are vector operators, then

$$\varphi \equiv \sum_{\mathrm{I}} \frac{2}{\eta(\mathrm{I})}\phi_{\tilde{\mathrm{I}}}\varphi_{\mathrm{I}} \quad (3.12)$$

is SU($m \mid n$) invariant, and

$$\psi_{\mathrm{K}}^{(\mathrm{F})} \equiv \sum_{\mathrm{I,J}}(\mathrm{I}, \mathrm{J})\frac{2\eta(\tilde{\mathrm{K}})}{\eta(\mathrm{I})\eta(\mathrm{J})} f_{\mathrm{IJ}\tilde{\mathrm{K}}}\phi_{\tilde{\mathrm{I}}}\varphi_{\tilde{\mathrm{J}}} \quad (3.13)$$

$$\psi_{\mathrm{K}}^{(\mathrm{D})} \equiv \sum_{\mathrm{I,J}}(\mathrm{I}, \mathrm{J})\frac{2\eta(\tilde{\mathrm{K}})}{\eta(\mathrm{I})\eta(\mathrm{J})} d_{\mathrm{IJ}\tilde{\mathrm{K}}}\phi_{\tilde{\mathrm{I}}}\varphi_{\tilde{\mathrm{J}}} \quad (3.14)$$

are vector operators.

The expression (3.12) can be rewritten in terms of matrices $\phi_{\mathrm{A}}^{\mathrm{B}}$ and $\varphi_{\mathrm{A}}^{\mathrm{B}}$ defined by (3.7) as follows:

$$\varphi = \sum_{\mathrm{I}} \frac{2}{\eta(\mathrm{I})}\phi_{\tilde{\mathrm{I}}}\varphi_{\mathrm{I}} = \mathrm{S\,Tr}\,\phi\varphi = \mathrm{S\,Tr}\,\varphi\phi \quad (3.15)$$

In writing the last equation we have used the fact that the matrices ϕ and φ have graded elements satisfying the commutation relation:

$$[\phi_A^B, \varphi_C^D]_{-(AB,CD)} = 0 \tag{3.16}$$

More general, it is easy to prove that the supertrace of any product of the matrices with graded elements has the cyclic property:

$$\mathrm{S\,Tr}\, \phi\varphi \ldots \chi = \mathrm{S\,Tr}\, \varphi \ldots \chi\phi = \ldots \tag{3.17}$$

In a similar way, with the help of (2.7), (2.16), (2.18), (2.21) and (2.22) the expressions (3.13) and (3.14) can be rewritten as follows:

$$\psi_K^{(F)} = \frac{i}{2}\, \mathrm{S\,Tr}\, (\phi\beta_K\varphi - \varphi\beta_K\phi) \tag{3.18}$$

$$\psi_K^{(D)} = \frac{1}{2}\, \mathrm{S\,Tr}\, (\phi\beta_K\varphi + \varphi\beta_K\phi) \tag{3.19}$$

According to (3.12) the second order Casimir operator is of the form:

$$\begin{aligned} C \equiv & \sum_I \frac{2}{\eta(I)} \mathscr{F}_{\bar{I}} \mathscr{F}_I \\ & = \sum_{a=1}^{m} F_a^2 - \sum_{p=1}^{n} G_p^2 + 2S_i^\alpha R_\alpha^i - 2R_\alpha^i S_i^\alpha - \frac{2(m-n)}{mn} H^2 \end{aligned} \tag{3.20}$$

From the transformation laws (3.1)-(3.6), using the above quoted properties of the matrices β_I we can find the eigenvalues of C for each irreducible representation. Thus, we have

$$C = \frac{(m-n)^2 - 1}{2(m-n)}$$

for the spinor representation ψ_A,

$$C = m - n$$

for the vector representation ϕ_I,

$$C = \frac{r(r + m - n)(m - n - 1)}{2(m - n)}$$

for the graded totally symmetrized r-rank spinor

$$\psi^{(+)}_{C_1C_2\ldots C_r} = (C_1, C_2)\psi^{(+)}_{C_2C_1\ldots C_r} = \ldots,$$

etc.

4. SUPERFIELD REPRESENTATION

Consider the space of functions of an anticommuting $m.n$-component parameter θ_α^i, $\alpha = 1, 2, \ldots, m$; $i = 1, 2, \ldots, n$. In this space the generators of the $SU(m \mid n)$ algebra can be realized as follows:

$$
\begin{aligned}
F_a &= \frac{1}{2}(\lambda_a^{(m)})_i^j \theta_\alpha^i \frac{\partial}{\partial \theta_\alpha^j} \\
G_p &= -\frac{1}{2}(\lambda_p^{(n)})_\alpha^\beta \theta_\beta^i \frac{\partial}{\partial \theta_\alpha^i} \\
H &= \frac{1}{2}\theta_\alpha^i \frac{\partial}{\partial \theta_\alpha^i} \\
S_i^\alpha &= -\frac{i}{2}\frac{\partial}{\partial \theta_\alpha^i} \\
R_\alpha^i &= -\frac{i}{2}\theta_\beta^i \theta_\alpha^j \frac{\partial}{\partial \theta_\beta^j}
\end{aligned}
\tag{4.1}
$$

Consider now the transformation laws of the superfield operators $\phi(\theta)$ defined in the space of the parameters θ_α^i. From (4.1) we note that the point $\theta_\alpha^i = 0$ remains unchanged under F_a, G_p, H and R_α^i transformations. So, these transformations form the little group of the $SU(m \mid n)$ group. According to any given representation of this little group we can define the entire action of the generators of the $SU(m \mid n)$ group on the field operators. This is done by the method of the theory of induced representations [*8*] [*10*] in the following manner.

Let

$$
\begin{aligned}
[F_a, \phi_{\mathscr{A}}(0)]_- &= -(f_a^{(\phi)}\phi(0))_{\mathscr{A}} \\
[G_p, \phi_{\mathscr{A}}(0)]_- &= -(g_p^{(\phi)}\phi(0))_{\mathscr{A}} \\
[H, \phi_{\mathscr{A}}(0)]_- &= -(h^{(\phi)}\phi(0))_{\mathscr{A}} \\
[R_\alpha^i, \phi_{\mathscr{A}}(0)]_{-[\mathscr{A}]} &= -(r^{(\phi)i}{}_\alpha \phi(0))_{\mathscr{A}}
\end{aligned}
\tag{4.2}
$$

where f_a, g_p, h, r_α^i are some matrices obeying the analogous commutation relations as those for F_a, G_p, H, R_α^i. We are to find the commutation rule $[\mathscr{F}_I, \phi_{\mathscr{A}}(\theta)]_{-(I,\mathscr{A})}$. Choose the basis in index space $\mathscr{A}$ in such a way that the operators S_i^α do not act on the indices, *i. e.*

$$
[S_i^\alpha, \phi_{\mathscr{A}}(\theta)]_{-[\mathscr{A}]} = \frac{i}{2}\frac{\partial}{\partial \theta_\alpha^i}\phi_{\mathscr{A}}(\theta) \tag{4.3}
$$

and, therefore,

$$
\phi_{\mathscr{A}}\left(\theta_\gamma^K + \frac{1}{2}\eta_\gamma^K\right) = e^{-i\eta_\alpha^i S_i^\alpha}\phi_{\mathscr{A}}(\theta)e^{i\eta_\alpha^i S_i^\alpha} \tag{4.4}
$$

With the help of (4.4) we can write

$$[\mathscr{F}_{\mathrm{I}}, \phi_{\mathscr{A}}(\theta)]_{-(\mathrm{I},\mathscr{A})} = e^{-2i\theta^i_\alpha S^\alpha_i}[\mathscr{F}'_{\mathrm{I}}, \phi_{\mathscr{A}}(0)]_{-(\mathrm{I},\mathscr{A})}e^{2i\theta^i_\alpha S^\alpha_i} \quad (4.5)$$

where we denote

$$\mathscr{F}'_{\mathrm{I}} \equiv e^{2i\theta^i_\alpha S^\alpha_i}\mathscr{F}_{\mathrm{I}}e^{-2i\theta^i_\alpha S^\alpha_i} \quad (4.6)$$

Using the commutation relations (2.5) we find:

$$\begin{aligned}
\mathrm{F}'_a &= \mathrm{F}_a + i\theta^i_\alpha(\lambda^{(m)}_a)^j_i\mathrm{S}^\alpha_j \\
\mathrm{G}'_p &= \mathrm{G}_p + i\mathrm{S}^\alpha_i(\lambda^{(n)}_p)^\beta_\alpha\theta^i_\beta \\
\mathrm{H}' &= \mathrm{H} + i\theta^i_\alpha\mathrm{S}^\alpha_i \\
\mathrm{R}^{i\prime}_\alpha &= \mathrm{R}^i_\alpha + \frac{i}{2}\left[\theta^j_\alpha(\lambda^{(m)}_a)^i_j\mathrm{F}_a + \theta^i_\beta(\lambda^{(n)}_p)^\beta_\alpha\mathrm{G}_p - 2\theta^i_\alpha\frac{m-n}{mn}\mathrm{H}\right] - \theta^j_\alpha\theta^i_\beta\mathrm{S}^\beta_j
\end{aligned} \quad (4.7)$$

By inserting (4.7) into (4.5) and taking into account (4.2) we get, after some manipulations:

$$\begin{aligned}
[\mathrm{F}_a, \phi_{\mathscr{A}}(\theta)]_- &= -\left\{(f^{(\phi)}_a\phi(\theta))_{\mathscr{A}} + \frac{1}{2}(\lambda^{(m)}_a)^j_i\theta^i_\alpha\frac{\partial\phi_{\mathscr{A}}(\theta)}{\partial\theta^j_\alpha}\right\} \\
[\mathrm{G}_p, \phi_{\mathscr{A}}(\theta)]_- &= -\left\{(g^{(\phi)}_p\phi(\theta))_{\mathscr{A}} - \frac{1}{2}(\lambda^{(n)}_p)^\beta_\alpha\theta^i_\beta\frac{\partial\phi_{\mathscr{A}}(\theta)}{\partial\theta^i_\alpha}\right\} \\
[\mathrm{H}, \phi_{\mathscr{A}}(\theta)]_- &= -\left\{(h^{(\phi)}\phi(\theta))_{\mathscr{A}} + \frac{1}{2}\theta^i_\alpha\frac{\partial\phi_{\mathscr{A}}(\theta)}{\partial\theta^i_\alpha}\right\} \\
[\mathrm{R}^i_\alpha, \phi_{\mathscr{A}}(\theta)]_{-[\mathscr{A}]} &= -\left\{(r^{(\phi)i}_\alpha\phi(\theta))_{\mathscr{A}} + i\theta^j_\alpha\left(\frac{\lambda^{(m)}_a}{2}\right)^i_j(f^{(\phi)}_a\phi(\theta))_{\mathscr{A}}\right. \\
&\left. + i\theta^i_\beta\left(\frac{\lambda^{(n)}_p}{2}\right)^\beta_\alpha(g^{(\phi)}_p\phi(\theta))_{\mathscr{A}} - i\theta^i_\alpha\frac{m-n}{mn}(h^{(\phi)}\phi(\theta))_{\mathscr{A}} + \frac{i}{2}\theta^j_\alpha\theta^i_\beta\frac{\partial\phi_{\mathscr{A}}(\theta)}{\partial\theta^j_\beta}\right\}
\end{aligned} \quad (4.8)$$

Let us consider the simplest case when

$$f_a = 0, \qquad g_p = 0, \qquad r^i_\alpha = 0$$

and h is a number. Then the formulae (4.8) become:

$$\begin{aligned}
[\mathrm{F}_a, \phi(\theta)] &= -\frac{1}{2}(\lambda^{(m)}_a)^j_i\theta^i_\alpha\frac{\partial\phi(\theta)}{\partial\theta^j_\alpha} \\
[\mathrm{G}_p, \phi(\theta)] &= \frac{1}{2}(\lambda^{(n)}_p)^\beta_\alpha\theta^i_\beta\frac{\partial\phi(\theta)}{\partial\theta^i_\alpha} \\
[\mathrm{H}, \phi(\theta)] &= -h^{(\phi)}\phi(\theta) - \frac{1}{2}\theta^i_\alpha\frac{\partial\phi(\theta)}{\partial\theta^i_\alpha} \\
[\mathrm{R}^i_\alpha, \phi(\theta)] &= i\frac{m-n}{mn}h\theta^i_\alpha\phi(\theta) - \frac{i}{2}\theta^j_\alpha\theta^i_\beta\frac{\partial\phi(\theta)}{\partial\theta^j_\beta}
\end{aligned} \quad (4.9)$$

The superfield $\phi(\theta)$ can be expanded in a polynomial series of order 2^{mn}:

$$\phi(\theta) = \varphi + \theta^i_\alpha\varphi^\alpha_i + \theta^{i_1}_{\alpha_1}\theta^{i_2}_{\alpha_2}\varphi^{\alpha_1\alpha_2}_{i_1i_2} + \ldots + \theta^{i_1}_{\alpha_1}\ldots\theta^{i_{mn}}_{\alpha_{mn}}\varphi^{\alpha_1\ldots\alpha_{mn}}_{i_1\ldots i_{mn}} \quad (4.10)$$

Here the tensors $\varphi^{\alpha_1\alpha_2\ldots\alpha_K}_{i_1i_2\ldots i_K}$ are totally antisymmetric in the pairs of indices $\begin{pmatrix}\alpha_l\\ i_l\end{pmatrix}$. Their infinitesimal change can be easily found from (4.3) and (4.9):

$$\delta^{(S)}\varphi^{\alpha_1\alpha_2\ldots\alpha K}_{i_1i_2\ldots iK} = \frac{K+1}{2}\eta^i_\alpha\varphi^{\alpha_1\ldots\alpha K\alpha}_{i_1\ldots iKi} \tag{4.11}$$

$$\delta^{(R)}\varphi^{\alpha_1\alpha_2\ldots\alpha_K}_{i_1i_2\ldots i_K} = (-1)^K\left\{\frac{m-n}{mn}h.\left[\sigma^{\alpha_1}_{i_1}\varphi^{\alpha_2\ldots\alpha_K}_{i_2\ldots i_K}\right] - \frac{K-1}{2}\left[\sigma^{\alpha_1}_{i_2}\varphi^{\alpha_2\alpha_3\ldots\alpha_K}_{i_1i_3\ldots i_K}\right]\right\} \tag{4.12}$$

$$\delta^{(H)}\varphi^{\alpha_1\alpha_2\ldots\alpha_K}_{i_1t\ldots i_K} = i\varepsilon\left(h + \frac{K}{2}\right)\varphi^{\alpha_1\alpha_2\ldots\alpha_K}_{i_1i_2\ldots i_K} \tag{4.13}$$

$$\delta^{(F)}\varphi^{\alpha_1\alpha_2\ldots\alpha_K}_{i_1i_2\ldots i_K} = \frac{i\omega_a}{2}\sum_{l=1}^{K}(\lambda^{(m)}_a)^j_{i_l}\varphi^{\alpha_1\ldots\alpha_l\ldots\alpha_K}_{i_1\ldots j\ldots i_K} \tag{4.14}$$

$$\delta^{(G)}\varphi^{\alpha_1\alpha_2\ldots\alpha_K}_{i_1i_2\ldots i_K} = -\frac{i\omega_p}{2}\sum_{l=1}^{K}(\lambda^{(n)}_p)^{\alpha_l}_\beta\varphi^{\alpha_1\ldots\beta\ldots\alpha_K}_{i_1\ldots i_l\ldots i_K} \tag{4.15}$$

Where η, σ, ε, ω are infinitesimal parameters, the symbol $[\ldots]$ in the r. h. s. of (4.12) means the antisymmetrization over all the pairs of indices $\begin{pmatrix}\alpha_l\\ i_l\end{pmatrix}$.

The equations (4.11), (4.14), (4.15) show in particular that the highest field component $\varphi^{\alpha_1\alpha_2\ldots\alpha_{mn}}_{i_1i_2\ldots i_{mn}}$ is invariant under S, F and G transformations. In order to see in what condition it is R-invariant we write (see (4.10)):

$$\varphi^{\alpha_1\alpha_2\ldots\alpha_{mn}}_{i_1i_2\ldots i_{mn}} = \frac{1}{(mn)!}\frac{\partial^{(mn)}}{\partial\theta^{i_{mn}}_{\alpha_{mn}}\ldots\partial\theta^{i_2}_{\alpha_2}\partial\theta^{i_1}_{\alpha_1}}\phi(\theta)$$

and therefore (using the last equation of (4.9):

$$\delta^{(R)}\varphi^{\alpha_1\alpha_2\ldots\alpha_{mn}}_{i_1i_2\ldots i_{mn}} = \frac{1}{(mn)}\cdot\frac{m-n}{mn}\cdot\left(h + \frac{mn}{2}\right)\cdot\frac{\partial^{(mn)}}{\partial\theta^{i_{mn}}_{\alpha_{mn}}\ldots\partial\theta^{i_2}_{\alpha_2}\partial\theta^{i_1}_{\alpha_1}}(\overline{\eta}^\alpha_i\theta^i_\alpha\phi(\theta))$$

From here we see that $\varphi^{\alpha_1\alpha_2\ldots\alpha_{mn}}_{i_1i_2\ldots i_{mn}}$ is R-invariant if $h = -\frac{mn}{2}$. It is obvious from (4.13) that with this value of h this component is also H-invariant.

Finally, we note that if $\phi_1(\theta)$ and $\phi_2(\theta)$ are superfields transforming according to (4.9) with h_1 and h_2 then their product $\psi \equiv \phi_1(\theta)\phi_2(\theta)$ is also a superfield transforming in the same way with

$$h = h_1 + h_2 .$$

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