Annales de l'I. H. P., section A

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Annales de l'I. H. P., section A, tome 36, n° 3 (1982), p. 225-237

<http://www.numdam.org/item?id=AIHPA_1982__36_3_225_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXVI, n° 3, 1982, p. 225-237. *Physique théorique.*

Matrix elements and highest weight Wigner coefficients of GL(n, $\mathbb{C}$)

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ABSTRACT. — A simple non inductive method for computing matrix elements of GL(n, $\mathbb{C}$) (U(n)) is exhibited. Matrix elements of arbitrary irreducible representations are given in terms of the highest weight Wigner coefficients and the matrix elements of the fundamental representations of GL(n, $\mathbb{C}$).

RÉSUMÉ. — Nous donnons une méthode non itérative d'évaluation des coefficients matriciels des représentations irréductibles du groupe GL(n, $\mathbb{C}$). Les coefficients matriciels d'une représentation irréductible arbitraire s'obtiennent à partir de ceux des représentations fondamentales et des coefficients de Wigner du sous GL(n, $\mathbb{C}$)-module de poids dominant des produits tensoriels de représentations fondamentales.

I. INTRODUCTION

The matrix elements of GL(n, $\mathbb{C}$) (U(n)) were first computed by Gelfand and Graev [*1*]. Their results were obtained from a complicated induction procedure, which made it extremely difficult to explicitly write out a matrix element for a given irreducible representation. Since the work of Gelfand and Graev, other methods [2] have been used to compute matrix elements, but they all are complicated for use in explicit computations of matrix elements. In this paper we will present a simple non inductive procedure for computing the matrix elements of $GL(n, \mathbb{C})$, in which the matrix elements of a given irreducible representation are given as products of matrix elements of the fundamental representations (i. e., representations with signatures $(\underbrace{1, \ldots, 1}_{s}, 0, \ldots, 0)$, $1 \leq s \leq n$), which themselves are simply certain minors of matrix variables. Our results hinge on the fact that all of the finite dimensional irreducible representations of $GL(n, \mathbb{C})$-modules can be realized as spaces of polynomials of a matrix variable. More precisely one has irreducible $GL(n, \mathbb{C})$-modules characterized by

$$V^{(m)} = \{ f : \mathbb{C}^{n \times n} \to \mathbb{C} \mid f \text{ polynomial and } f(by) = \pi^{(m)}(b) f(y), \quad \forall (b, y) \in B \times \mathbb{C}^{n \times n} \} \tag{1}$$

where $\pi^{(m)}$ is a holomorphic character of B, the subgroup of lower triangular matrices of $GL(n, \mathbb{C})$, and $(m) = (m_1, \ldots, m_n)$ is an n-tuple belonging to the integral lattice Z^n with $m_1 \geq m_2 \geq \ldots \geq m_n$. We denote an orthogonal basis of $V^{(m)}$ by $\{ h^{(m)}_{[k]} \}$, where $[k]$ is a Gelfand pattern, and equip $V^{(m)}$ with the differentiation inner product $(.,.)$ [3]. Then it can be shown [3] that the representation $R^{(m)}$ defined on $V^{(m)}$ by $(R^{(m)}(g)f)(y) = f(yg)$, $f \in V^{(m)}$, $g \in GL(n, \mathbb{C})$ is irreducible, and when restricted to the unitary subgroup $U(n)$, the representation $R^{(m)} \mid U(n)$ is unitary with respect to $(.,.)$.

It is known that in general the decomposition of the tensor product space $V^{(m')} \otimes V^{(m'')}$ of two irreducible representations labeled by (m') and (m'') into direct sum irreducible subspaces involves multiplicity. However, for the highest weight irreducible representation with signature $(m) = (m') + (m'')$, there is no multiplicity. Further, $V^{(m)}$ can be obtained from $V^{(m')} \otimes V^{(m'')}$ via the map

$$(\Phi_e F)(y) = F(y, y), \qquad \forall F \in V^{(m')} \otimes V^{(m'')} . \tag{2}$$

The two-fold tensor products can be generalized to r-fold tensor products; that is, if $V^{(m^{(1)})} \otimes \ldots \otimes V^{(m^{(r)})}$ is an r-fold tensor product, then under the map

$$(\Phi_e F)(y) = F(y, \ldots, y), \qquad F \in V^{(m^{(1)})} \otimes \ldots \otimes V^{(m^{(r)})}$$

an irreducible representation with signature $(m) = (m^{(1)}) + \ldots + (m^{(r)})$ is obtained. Furthermore, as we shall see in Section III, basis elements for $V^{(m)}$ can be obtained from basis elements in $V^{(m^{(1)})} \otimes \ldots \otimes V^{(m^{(r)})}$. Now if the $(m^{(1)}) \ldots (m^{(r)})$ are chosen to be fundamental representations, it is seen that an arbitrary representation with signature (m) is obtained through the above Φ_e map from fundamental representations by

$$(m) = (m^{(1)}) + \ldots + (m^{(r)}).$$

II. MATRIX ELEMENTS

Thus, we first discuss the matrix elements of the fundamental representations. If $(m) = (\underbrace{1, \ldots, 1}_{s}, 0, \ldots, 0)$, $1 \leq s \leq n$, is the signature of a fundamental representation of GL(n, $\mathbb{C}$), and if $\Delta^{1 \ldots s}_{k_1 \ldots k_s}(y)$ denotes the minor of the matrix $y \in \mathbb{C}^{n \times n}$ formed from the rows $1, \ldots, s$, and the columns $k_1, \ldots, k_s$ where $(k_1, \ldots, k_s)$ is an s-shuffle; i. e., $1 \leq k_1 < \ldots < k_s \leq n$, then $h^{(m(}_{[k]}(y) = \Delta^{1 \ldots s}_{k_1 \ldots k_s}(y)$. The $k_1 \ldots k_s$ are related to the Gelfand pattern $[k]$ in the following way:

$$\begin{array}{lccccccccccc}
 & \overbrace{1 \ldots 1}^{s} & 0 & & & \ldots & & & & 0 & & \\
k_s \rightarrow & & \overbrace{1 \ldots 1}^{s} & & 0 & & \ldots & & 0 & & & \\
k_2 \rightarrow & & & 1 & 1 & 0 & \ldots & 0 & & & & \equiv \binom{(m)}{[k]}. \quad (3) \\
k_1 \rightarrow & & & & 1 & 0 & \ldots & 0 & & & & \\
 & & & & & 0 & \ldots & 0 & & & & \\
 & & & & & & 0 & & & & &
\end{array}$$

By an elementary fact concerning minors of the product of two matrices. we have

$$\begin{aligned}
\mathrm{R}^{(m)}(g)h^{(m)}_{[k]}(y) &= \Delta^{1 \ldots s}_{k_1 \ldots k_s}(yg) \\
&= \sum_{j_1 < \ldots < j_s} \Delta^{1 \ldots s}_{j_1 \ldots j_s}(y)\Delta^{1 \ldots j_s}_{k_1 \ldots k_s}(g) \\
&= \sum_{j_1 < \ldots < j_s} \Delta^{j_1 \ldots j_s}_{k_1 \ldots k_s}(g)h^{(m)}_{[j]}(y).
\end{aligned}$$

It follows that matrix elements associated with the fundamental representations $\mathrm{R}^{(m)}$ are given by

$$\mathrm{D}^{(m)}_{[j][k]}(g) = \Delta^{j_1 \ldots j_s}_{k1 \ldots k_s}(g), \qquad \forall g \in \mathrm{GL}(n, \mathbb{C}). \tag{4}$$

To obtain the matrix elements for the non fundamental representations, it is necessary to have explicit polynomial realizations of orthogonal bases for all the irreducible representations of GL(n, $\mathbb{C}$). In Ref. [*4*] we have shown that such polynomial orthogonal bases are given by so-called pattern addition. In this reference coefficients appearing in this pattern addition were incorrectly identified as being highest weight Clebsch-Gordan coefficients. We wish to emphasize here that these coefficients are not highest weight Clebsch-Gordan coefficients; such coefficients will in fact be needed and computed in Section III of this paper. In the appendix we give an explicit proof that pattern addition indeed gives polynomial orthogonal bases for all irreducible representations of $GL(n, \mathbb{C})$.

Before turning to the computation of the matrix elements of $GL(n, \mathbb{C})$, it is useful to review the relationship between orthogonal bases, Clebsch-Gordan coefficients and matrix elements, using the abstract ket notation of Dirac. Let $|(m)[k]\rangle$ denote an orthonormal basis, with (m) a signature and $[k]$ a Gelfand labelling as before. Then the highest weight in the tensor product $(m^{(1)} \otimes \ldots \otimes (m^{(r)})$ is given by

$$|(m)[k]\rangle = \sum_{[k^{(1)}]\ldots[k^{(r)}]} \times \langle (m^{(1)})[k^{(1)}] \ldots (m^{(r)})[k^{(r)}] \,|\, (m)[k]\rangle \times |(m^{(1)})[k^{(1)}]\rangle \ldots |(m^{(r)}[k^{(r)}]\rangle \tag{5}$$

where $\langle (m^{(1)})[k^{(1)}] \ldots (m^{(r)}[k^{(r)}] \,|\, (m)[k]\rangle$ denotes a highest weight Wigner coefficient. A distinction is being made between Wigner and Clebsch-Gordan coefficients in that Wigner coefficients are defined as those coefficients that reduce an ortho*normal* basis in an r-fold tensor product space to an ortho*normal* direct sum basis, whereas Clebsch-Gordan coefficients merely reduce an orthogonal basis to an orthogonal direct sum basis. The reason for this distinction is that the various Φ maps do not in general preserve norms. Thus, it is convenient not to fix the norms of various bases until the very end of a calculation.

If Eq. (5) is right translated by $g \in GL(n, \mathbb{C})$, it follows that

$$\begin{aligned} R^{(m)}(g)\,|(m)[k]\rangle &= \sum_{[k^{(1)}]\ldots[k^{(r)}]} \times \langle (m^{(1)})[k^{(1)}] \ldots (m^{(r)})[k^{(r)}] \,|\, (m)[k]\rangle \\ &\qquad \times R^{(m)}(g)(\,|(m^{(1)})[k^{(1)}]\rangle \ldots |(m^{(r)})[k^{(r)}]\rangle\,), \\ &= \sum_{[k']} D^{(m)}_{[k'][k]}(g)\,|(m)[k']\rangle \\ &= \sum_{\substack{[k^{(1)}]\ldots[k^{(r)}] \\ [k'^{(1)}]\ldots[k'^{(r)}]}} \langle (m^{(1)})[k^{(1)}] \ldots (m^{(r)})[k^{(r)}] \,|\, (m)[k]\rangle \\ &\qquad \times D^{(m^{(1)})}_{[k'^{(1)}][k^{(1)}]}(g) \ldots D^{(m^{(r)})}_{[k'^{(r)}][k^{(r)}]}(g) \\ &\qquad \times |(m^{(1)})[k'^{(1)}]\rangle \ldots (m^{(r)})[k'^{(r)}]\rangle\,. \end{aligned} \tag{6}$$

Then

$$D^{(m)}_{[k'][k]}(g) = \sum_{\substack{[k^{(1)}]\dots[k^{(r)}] \\ [k'^{(1)}]\dots[k'^{(r)}]}} \langle (m)[k'] \mid (m^{(1)})[k'^{(1)}] \dots (m^{(r)}[k'^{(r)}] \rangle$$
$$\times D^{(m^{(1)})}_{[k'^{(1)}][k^{(1)}]}(g) \dots D^{(m^{(r)})}_{[k'^{(r)}][k^{(r)}]}(g)$$
$$\times \langle (m^{(1)})[k^{(1)}] \dots (m^{(r)})[k^{(r)}] \mid (m)[k] \rangle . \tag{7}$$

Thus, to compute the matrix elements $D^{(m)}_{[k'][k]}(g)$, it is necessary to know the matrix elements of the fundamental representations $D^{(m^{(i)})}_{[k'^{(i)}][k^{(i)}]}(g)$ and the highest weight Wigner coefficients relating $(m) = \Sigma_{i=1}^{r}(m^{(i)})$ to the fundamental representations. But the matrix elements of the fundamental representations are given by Eq. (4). Thus, it remains to compute the highest weight « Wigner » coefficients [5].

III. THE HIGHEST WEIGHT CLEBSCH-GORDAN COEFFICIENTS OF GL(n, $\mathbb{C}$)

For any signature (m) define a map from $V^{(m)}$ to $V^{(m^{(1)})} \otimes \dots \otimes V^{(m^{(r)})}$ by

$$\Lambda^{(m)}_{\eta} h^{(m)}_{[k]} = \sum_{[k^{(1)}]\dots[k^{(r)}]} C\,\eta^{(m)(m^{(1)})\dots(m^{(r)})}_{[k][k^{(1)}]\dots[k^{(r)}]} \, h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]} \tag{8}$$

where η is a multiplicity parameter, needed to distinguish equivalent representations of signature (m) in the tensor product space. If we denote the inverses of the Clebsch-Gordan coefficients $C\,\eta^{(m)(m^{(1)})\dots(m^{(r)})}_{[k][k^{(1)}]\dots[k^{(r)}]}$ by $K\,\eta^{(m)(m^{(1)})\dots(m^{(r)})}_{[k][k^{(1)}]\dots[k^{(r)}]}$, then

$$h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]} = \sum_{(m),\eta[k]} K\,\eta^{(m)(m^{(1)})\dots(m^{(r)})}_{[k][k^{(1)}]\dots[k^{(r)}]} \, \Lambda^{(m)}_{\eta} h^{(m)}_{[k]} . \tag{9}$$

If Φ_e is applied to both sides of Eq. (9), only the highest weight representation, $(m) = (m^{(1)}) + \dots + (m^{(r)})$, survives, for which there is no multiplicity. Therefore,

$$\Phi_e\big(h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]}(g)\big) = \sum_{[k]} K^{(m)(m^{(1)})\dots(m^{(r)})}_{[k][k^{(1)}]\dots[k^{(r)}]} h^{(m)}_{[k]}(g) . \tag{10}$$

Using the orthogonality of $h^{(m)}_{[k]}$ then gives

$$K^{(m)(m^{(1)})\dots(m^{(r)})}_{[k][k^{(1)}]\dots[k^{(r)}]} = \frac{1}{\| h^{(m)}_{[k]} \|^2} \left(h^{(m)}_{[k]}, \Phi_e \left(h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]} \right) \right) \tag{11}$$

which gives an explicit formula for the computation of the coefficients $K^{-\,-\,-}_{-\,-\,-}$, since $\{ h^{(m^{(i)})}_{[k^{(i)}]} \}$ are bases for the fundamental representations, given in Eq. (3), $\{ h^{(m)}_{[k]} \}$ is given by Eq. (A.1), and the norms $\| h^{(m)}_{[k]} \|^2$ were computed in Ref. 6.

To connect the coefficients $K^{-\,-\,-}_{-\,-\,-}$ with the highest weight Wigner coefficients needed to compute a general matrix element, we note that Eq. (9) can be rewritten as

$$\frac{h^{(m^{(1)})}_{[k^{(1)}]}}{\| h^{(m^{(1)})}_{[k^{(1)}]} \|} \otimes \dots \otimes \frac{h^{(m^{(r)})}_{[k^{(r)}]}}{\| h^{(m^{(r)})}_{[k^{(r)}]} \|} = \sum_{(m)\eta[k]} \times \langle (m)\eta [k] \,|\, (m^1)[k^1] \dots (m^r)[k^r] \rangle \frac{\Lambda^{(m)}_{\eta} h^{(m)}_{[k]}}{\| \Lambda^{(m)}_{\eta} h^{(m)}_{[k]} \|}. \tag{12}$$

Comparing this expression with Eq. (9) and Eq. (11), and restricting (m) to be the highest weight, $(m) = (m^1) + \dots + (m^r)$, gives

$$\langle (m)[k] \,|\, (m^{(1)})[k^{(1)}] \dots (m^{(r)})[k^{(r)}] \rangle = \frac{\left(h^{(m)}_{[k]}, \Phi_e h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]} \right)}{\sum_{i=1}^{r} \| h^{(m)(i)}_{[k^{(i)}]} \|} \frac{\| \Lambda^{(m)} h^{(m)}_{[k]} \|}{\| h^{(m)}_{[k]} \|^2}. \tag{13}$$

Now $\| \Lambda^{(m)} h^{(m)}_{[k]} \|$ is in general rather difficult to compute. But if the highest weight Wigner coefficients are viewed as an orthogonal matrix in the variables $(m)\eta[k]$ and $[k^1], \dots, [k^r]$, then if (m) is the highest weight and $[k]$ is held fixed, it follows that

$$\sum_{[k^{(1)}]\dots[k^{(r)}]} | \langle (m)[k] \,|\, (m^{(1)})[h^{(1)}] \dots (m^{(r)})[k^{(r)}] \rangle |^2 = 1$$

$$= \sum_{[k^{(1)}]\dots[k^{(r)}]} \left[\frac{\left(h^{(m)}_{[k]}, \Phi_e h^{(m^{(1)})}_{[k^1]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^r]} \right)}{\prod_{i=1}^{r} \| h^{(m^i)}_{[k^i]} \|} \right]^2 \times \left[\frac{\| \Lambda^{(m)} h^{(m)}_{[k]} \|}{\| h^{(m)}_{[k]} \|^2} \right] \tag{14}$$

It follows that

$$\frac{\| \Lambda^{(m)} h^{(m)}_{[k]} \|}{\| h^{(m)}_{[k]} \|} = \left\{ \sum_{[k^{(1)}]\dots[k^{(r)}]} \left[\frac{\left(h^{(m)}_{[k]}, \Phi_e h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]}\right)}{\prod_{i=1}^{r} \| h^{(m^{(i)})}_{[k^{(i)}]} \|} \right]^2 \right\}^{-\frac{1}{2}} . \quad (15)$$

If this equation is combined with Eq. (13), the following explicit formula for the highest Wigner coefficients is obtained :

$$\langle (m)[k] \,|\, (m^{(1)})[k^{(1)}] \dots (m^{(r)})[k^{(r)}] \rangle = \frac{\left(h^{(m)}_{[k]}, \Phi_e h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]}\right)}{\prod_{i=1}^{r} \| h^{(m^{(i)})}_{[k^{(i)}]} \|}$$

$$\left\{ \sum_{[\tilde{k}^{(1)}]\dots[\tilde{k}^{(r)}]} \frac{\left[\left(h^{(m)}_{[k]}, \Phi_e h^{(m^{(1)})}_{[\tilde{k}^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[\tilde{k}^{(r)}]}\right)\right]^2}{\prod_{i=1}^{r} \| h^{(m^{(i)})}_{[\tilde{k}^{(i)}]} \|} \right\}^{-\frac{1}{2}} . \quad (16)$$

Now in Eq. (16) the norms of the basis elements of the fundamental representations are very simple expressions; for example, if $(m^{(i)}) = (\underbrace{1, \dots, 1}_{s}, 0, \dots, 0)$ then $\| h^{(m^{(i)})}_{[k^{(i)}]} \|^2 = s\,!$ for all $[k^{(i)}]$. And since

$$h^{(m)}_{[k]} = C \sum_{[k^{(1)}] + \dots + [k^{(r)}] = [k]} \Phi_e\left(h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]}\right)$$

where C is a constant (cf. Ref. [*14*] Eq. (11)), a quantity such as $\left(h^{(m)}_{[k]}, \Phi_e h^{(m^{(1)})}_{[k^{(1)}]} \otimes \dots \otimes h^{(m^{(r)})}_{[k^{(r)}]}\right)$ can be easily obtained as a sum of simple differentiations of polynomials. Thus, all the elements of Eq. (7) needed to compute a general matrix element are known. It should be noted that Eq. (14) can also be used for computing $\| \Lambda^{(m)} h^{(m)}_{[k]} \|$ since $\| h^{(m)}_{[k]} \|$ is known [*6*].

IV. AN EXAMPLE

To illustrate our method for computing matrix elements of GL(n, $\mathbb{C}$), we consider a very simple representation, namely the eight-dimensional representation (210) of SU(3). We wish to compute

$$\begin{aligned} D^{(210)}_{[k'][k]}(g) = \sum_{\substack{[k^1][k^2] \\ [k^{1'}][k^{2'}]}} & \langle (210)[k'] \,|\, (100)[k^{1'}](110)[k^{2'}] \rangle \\ & \times D^{(100)}_{[k^{1'}][k^1]}(g) D^{(110)}_{[k^{2'}][k^2]} \\ & \times \langle (100)[k^1](110)[k^2] \,|\, (210)[k] \rangle . \end{aligned}$$

To keep the computation as specific as possible we will look only at $[k] = [k'] = \begin{bmatrix} 20 \\ 1 \end{bmatrix}$. Thus, it is necessary to compute the inner product $(h^{(210)}_{\left[\begin{smallmatrix}20\\1\end{smallmatrix}\right]}, \Phi_e h^{(100)}_{[k^1]} \otimes h^{(110)}_{[k^2]})$. Now the Appendix shows that

$$\begin{aligned} h^{(210)}_{\left[\begin{smallmatrix}20\\1\end{smallmatrix}\right]}(g) &= h^{(100)}_{\left[\begin{smallmatrix}10\\1\end{smallmatrix}\right]} h^{(110)}_{\left[\begin{smallmatrix}10\\0\end{smallmatrix}\right]} + h^{(100)}_{\left[\begin{smallmatrix}10\\0\end{smallmatrix}\right]} h^{(100)}_{\left[\begin{smallmatrix}10\\1\end{smallmatrix}\right]} \\ &= g_{11}\Delta^{12}_{23}(g) + g_{12}\Delta^{12}_{13}(g) \\ &= g_{11}(g_{12}g_{23} - g_{13}g_{22}) + g_{12}(g_{11}g_{23} - g_{13}g_{21}) ; \end{aligned}$$

performing the simple polynomial differentiations of Eqs. (13) and (14) results in Table I.

TABLE I. — *Highest Weight Wigner Coefficients of* $\begin{bmatrix} 210 \\ 20 \\ 1 \end{bmatrix}$.

$[k^1]$	$[k^2]$	$\Phi_e(h^{(100)}_{[k^1]} \otimes h^{(110)}_{[k^2]})$	$(h^{(210)}_{\left[\begin{smallmatrix}20\\1\end{smallmatrix}\right]}, \Phi_e h^{(100)}_{[k^1]} \otimes h^{(110)}_{[k^2]})$	$\langle (210) \begin{bmatrix} 20 \\ 1 \end{bmatrix} \mid (100)[k^1](110)[k^2] \rangle$
$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 11 \\ 1 \end{bmatrix}$	$g_{11}\Delta^{12}_{12} = g_{11}(g_{11}g_{22} - g_{12}g_{21})$	0	0
$\begin{bmatrix} 10 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 11 \\ 1 \end{bmatrix}$	$g_{12}\Delta^{12}_{12} = g_{12}(g_{11}g_{22} - g_{12}g_{21})$	0	0
$\begin{bmatrix} 00 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 11 \\ 1 \end{bmatrix}$	$g_{13}\Delta^{12}_{12} = g_{13}(g_{11}g_{22} - g_{12}g_{21})$	0	0
$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$	$g_{11}\Delta^{12}_{13} = g_{11}(g_{11}g_{23} - g_{13}g_{21})$	0	0
$\begin{bmatrix} 10 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$	$g_{12}\Delta^{12}_{13} = g_{12}(g_{11}g_{23} - g_{13}g_{21})$	3	$1/\sqrt{2}$
$\begin{bmatrix} 00 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$	$g_{13}\Delta^{12}_{13} = g_{13}(g_{11}g_{23} - g_{13}g_{21})$	0	0
$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 0 \end{bmatrix}$	$g_{11}\Delta^{12}_{23} = g_{11}(g_{12}g_{23} - g_{13}g_{22})$	3	$1/\sqrt{2}$
$\begin{bmatrix} 11 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 0 \end{bmatrix}$	$g_{12}\Delta^{12}_{23} = g_{12}(g_{12}g_{23} - g_{13}g_{22})$	0	0
$\begin{bmatrix} 00 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 0 \end{bmatrix}$	$g_{13}\Delta^{12}_{23} = g_{13}(g_{12}g_{23} - g_{13}g_{22})$	0	0

Thus,

$$
\begin{aligned}
D^{(210)}_{\left[\begin{smallmatrix}2\,0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}2\,0\\1\end{smallmatrix}\right]}(g) &= \frac{1}{\sqrt{2}}\Big[D^{(100)}_{\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]}(g)D^{(110)}_{\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]}(g) \\
&+ D^{(100)}_{\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]}(g)D^{(110)}_{\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]}(g) + D^{(100)}_{\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]}D^{(110)}_{\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]}(g) \\
&+ D^{(100)}_{\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right]}(g)D^{(110)}_{\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right]}(g) \\
&= \frac{1}{\sqrt{2}}\left[\Delta^{2}_{2}(g)\Delta^{13}_{13}(g) + \Delta^{2}_{1}(g)\Delta^{13}_{23}(g) + \Delta^{1}_{2}(g)\Delta^{23}_{13}(g) + \Delta^{1}_{1}(g)\Delta^{23}_{23}(g)\right] \\
&= \frac{1}{\sqrt{2}}\left[g_{22}(g_{11}g_{33} - g_{13}g_{31}) + g_{21}(g_{12}g_{33} - g_{13}g_{32})\right. \\
&\left. + g_{12}(g_{21}g_{33} - g_{31}g_{23}) + g_{11}(g_{22}g_{33} - g_{23}g_{32})\right].
\end{aligned}
$$

APPENDIX

Proof that pattern addition gives orthogonal polynomial bases for GL(n, $\mathbb{C}$)

In this appendix we prove that the basis given by pattern addition, namely

$$h^{(m)}_{[k]}(y) = \sum_{[k^{(1)}]+\ldots+[k^{(r)}]=[k]} (\Phi_e h^{(m^{(1)})}_{[k^{(1)}]} \otimes \ldots \otimes h^{(m^{(r)})}_{[k^{(r)}]})(y) \tag{A.1}$$

indeed is orthogonal, as claimed in ref. 4. This result is used in Section III to compute the highest weight Wigner coefficients.

If $\begin{matrix}(k)\\ [l]\end{matrix}$ is the Gelfand label obtained from tableau (3) by removing the top row of $\begin{matrix}(m)\\ [k]\end{matrix}$, and if γ belongs to the subgroup GL($n-1$, $\mathbb{C}$), then an easy computation shows that

$$R^{(m)}(\gamma)h^{(m)}_{\substack{(k)\\ [l]}} = \sum_{[\bar{l}]} D^{(k)}_{[\bar{l}][l]}(\gamma)h^{(m)}_{\substack{(k)\\ [\bar{l}]}} \tag{A.2}$$

where in Eq. (A.2) $\begin{matrix}(k)\\ [l]\end{matrix} \equiv [k]$, and $D^{(k)}_{[\bar{l}][l]}$ are matrix elements corresponding to the representation of GL($n-1$, $\mathbb{C}$) with signature (k). Now let $(m) = (m_1, \ldots, m_n)$ be an arbitrary signature and let us show by induction that an orthogonal polynomial basis corresponding to this signature can be given by pattern addition (cf. ref. 4). The case $n = 2$ being obviously true, let us assume that it holds for all integers $\leq n-1$.

Let $\begin{matrix}(m)\\ [k]\end{matrix}$ of the Gelfand pattern given by

$$\begin{bmatrix} m_{1n} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & m_{nn} \\ & m_{1,n-1} & & \cdot & \cdot & \cdot & \cdot & & m_{n-1,n-1} & \\ & & & \cdot & & & & \cdot & & \\ & & & & & m_{11} & & & & \end{bmatrix} \tag{A.3}$$

where $m_{in} = m_i$, $1 \leq i \leq n$, and each integer m_{ij} in row j is subject to the constraint $m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1}$. If the notations $\begin{matrix}(k)\\ [l]\end{matrix}$ and $D^{(k)}_{[\bar{l}][l]}$ introduced before are preserved, then by assumption

$$h^{(k)}_{[l]}(y) = \sum_{[l^{(1)}]+\ldots+[l^{(r)}]=[l]} \times\, h^{(k^{(1)})}_{[l^{(1)}]}(y) \ldots h^{(k^{(r)})}_{[l^{(r)}]}(y), \qquad y \in \mathbb{C}^{(n-1)\times(n-1)} \tag{A.4}$$

is an orthogonal polynomial basis for $V^{(k)}$ (cf. ref. 4). For $\gamma \in$ GL($n-1$, $\mathbb{C}$), we have

$$R^{(k)}(\gamma)h^{(k)}_{[l]}(y) = h^{(k)}_{[l]}(y\gamma) = \sum_{[l^{(1)}]+\ldots+[l^{(r)}]=[l]} h^{(k^{(1)})}_{[l^{(1)}]}(y\gamma) \ldots h^{(k^{(r)})}_{[l^{(r)}]}(y\gamma)$$

$$= \sum_{[l^{(1)}]+\ldots+[l^{(r)}]=[l]} \left(\sum_{[\bar{l}^{(1)}]} D^{(k^{(1)})}_{[\bar{l}^{(1)}][l^{(1)}]}(\gamma)h^{(k^{(1)})}_{[\bar{l}^{(1)}]}(y)\right) \times \left(\sum_{\bar{l}^{(r)}} D^{(k^{(r)})}_{[\bar{l}^{(r)}][l^{(r)}]}(\gamma)h^{(k^{(r)})}_{[\bar{l}^{(r)}]}(y)\right)$$

$$= \sum_{[\bar{l}^{(1)}],\ldots,[\bar{l}^{(r)}]} \left(\sum_{[l^{(1)}]+\ldots+[l^{(r)}]=[l]} \times\, D^{(k^{(1)})}_{[\bar{l}^{(1)}][l^{(1)}]}(\gamma) \ldots D^{(k^{(r)})}_{[\bar{l}^{(r)}][l^{(r)}]}(\gamma)\right) \times h^{(k^{(1)})}_{[\bar{l}^{(1)}]}(y) \ldots h^{(k^{(r)})}_{[\bar{l}^{(r)}]}(y). \tag{A.5}$$

Since $\{ h_l^k \}$ is a basis in $V^{(k)}$, we can write

$$h^{(k^{(1)})}_{[l^{(1)}]}(y) \dots h^{(k^{(r)})}_{[l^{(r)}]}(y) = \sum_{\bar{l}} \alpha^{k,k^{(1)}\dots k^{(r)}}_{\bar{l},\bar{l}^{(1)}\dots\bar{l}^{(r)}} h^{(k)}_{[\bar{l}]}(y) \tag{A.6}$$

where the coefficients $\alpha^{k,k^{(1)}\dots k^{(r)}}_{\bar{l},\bar{l}^{(1)}\dots\bar{l}^{(r)}}$ are uniquely determined. Equations (A.5) and (A.6) imply that

$$R^{(k)}_{(\gamma)} h^{(k)}_{[l]}(y) = \sum_{[\bar{l}]} \Bigg(\sum_{[l^{(1)}],\dots,[\bar{l}^{(r)}]} \Bigg(\sum_{[\bar{l}^{(1)}]+\dots+[l^{(r)}]=[l]} D^{(k^{(1)})}_{[\bar{l}^{(1)}][l^{(1)}]}(\gamma) \dots D^{(k^{(r)})}_{[\bar{l}^{(r)}][l^{(r)}]}(\gamma) \times \alpha^{k,k^{(1)}\dots k^{(r)}}_{\bar{l},\bar{l}^{(1)}\dots\bar{l}^{(r)}} \Bigg) h^{(k)}_{[\bar{l}]}(y) \Bigg). \tag{A.7}$$

But by induction we also have

$$R^{(k)}_{(\gamma)} h^{(k)}_{[l]}(y) = \sum_{[\bar{l}]} D^{(k)}_{[\bar{l}][l]}(\gamma) h^{(k)}_{[\bar{l}]}(y). \tag{A.8}$$

Thus, it follows from Eqs. (A.7) and (A.8) that

$$D^{(k)}_{[\bar{l}][l]}(\gamma) = \sum_{[\bar{l}^{(1)}]\dots[\bar{l}^{(r)}]} \Bigg(\sum_{[l^{(1)}]+\dots+[l^{(r)}]=[l]} D^{(k^{(1)})}_{[\bar{l}^{(1)}][l^{(1)}]}(\gamma) \dots D^{(k^{(r)})}_{[\bar{l}^{(r)}][l^{(r)}]}(\gamma) \alpha^{k,k^{(1)}\dots k^{(r)}}_{\bar{l},\bar{l}^{(1)}\dots\bar{l}^{(r)}} \Bigg). \tag{A.9}$$

Now let $\{ h^{(m)}_{[k]} \}$ be elements of $V^{(m)}$ defined by pattern addition, *i. e.*,

$$h^{(m)}_{\substack{(k)\\ [l]}} = \sum_{\substack{(k^{(1)})+\dots+(k^{(r)})=(k)\\ [l^{(1)}]+\dots+[l^{(r)}]=[l]}} h^{(m^{(1)})}_{\substack{(k^{(1)})\\ [l^{(1)}]}} \dots h^{(m^{(r)})}_{\substack{(k^{(r)})\\ [l^{(r)}]}} \tag{A.10}$$

where the $h^{(m^{(i)})}_{\substack{(k^{(i)})\\ [l^{(i)}]}}$'s are basis elements of fundamental representations for all i, $1 \le i \le r$.

Let Φ_D and $\Omega^{(k)}$ denote the maps defined in Ref. 4, then for

$$(y, \gamma) \in \mathbb{C}^{(n-1)\times(n-1)} \times GL(n-1, \mathbb{C}),$$

we have

$$R^{(m)}(\gamma)(\Phi_D h^{(m)}_{\substack{(k)\\ [l]}})(y) = h^{(m)}_{\substack{(k)\\ [l]}}(y_D y\gamma) = \sum_{\substack{(k^{(1)})+\dots+(k^{(r)})=(k)\\ [l^{(1)}]+\dots+[l^{(r)}]=[l]}} h^{(m^{(1)})}_{\substack{(k^{(1)})\\ [l^{(1)}]}}(y_D y\gamma) \dots h^{(m^{(r)})}_{\substack{(k^{(r)})\\ [l^{(r)}]}}(y_D y\gamma). \tag{A.11}$$

Set $k_i = m_{i,n-1}$ for $1 \le i \le n-1$ and

$$C = \binom{m_1 - m_2}{k_1 - m_2}\binom{m_1 - m_2}{m_2 - k_2} \dots \binom{m_{n-1} - m_n}{k_{n-1} - m_n}$$

then from Eq. (A.11) it is easy to verify that

$$R^{(m)}(\gamma)(\Phi_D h^{(m)}_{\substack{(k)\\ [l]}})(y) = C \sum_{[l^{(1)}]+\dots+[l^{(r)}]=[l]} \times h^{(m^{(1)})}_{\substack{(k^{(1)})\\ [l^{(1)}]}}(y_D y\gamma) \dots h^{(m^{(r)})}_{\substack{(k^{(r)})\\ [l^{(r)}]}}(y_D y\gamma). \tag{A.12}$$

Now Eq. (A.2) applied to each fundamental representation in the expression (A.12) shows that

$$R^{(m)}(\gamma)(\Phi_D h^{(m)}_{\substack{(k)\\ [l]}})(y) = C \sum_{[l^{(1)}]+\dots+[l^{(r)}]=[l]} \times \Bigg(\sum_{[\bar{l}^{(1)}],\dots,[\bar{l}^{(r)}]} D^{(k^{(1)})}_{[\bar{l}^{(1)}][l^{(1)}]}(\gamma) \dots D^{(k^{(r)})}_{[\bar{l}^{(r)}][l^{(r)}]}(\gamma) \times h^{(m^{(1)})}_{\substack{(k^{(1)})\\ [\bar{l}^{(1)}]}}(y_D y) \dots h^{(m^{(r)})}_{\substack{(k^{(r)})\\ [\bar{l}^{(r)}]}}(y_D y) \Bigg). \tag{A.13}$$

Since the functions $h^{(m^{(1)})}_{(k^{(1)})}(y_D y) \dots h^{(m^{(r)})}_{(k^{(r)})}(y_D y)$ belong to $\Omega^{(k)}(V^{(k)})$, it is easy to find some subset $\{ f^{(k)}_{[l]} \}$ of $\Omega^{(k)}(V^{(k)})$ (not necessarily a spanning set and not necessarily unique) so that

$$h^{(m^{(1)})}_{\substack{(k^{(1)})\\ [\bar{l}^{(1)}]}}(y_D y) \ldots h^{(m^{(r)})}_{\substack{(k^{(r)})\\ [\bar{l}^{(r)}]}}(y_D y) = \sum_{[\bar{l}]} \alpha^{k,k^{(1)}\ldots k^{(r)}}_{[\bar{l}],[\bar{l}^{(1)}]\ldots[\bar{l}^{(r)}]} f^{(k)}_{[\bar{l}]}(y) \tag{A.14}$$

where the α^{---}_{---} are the same as in Eq. (A.6). It follows from Eqs. (A.13) and (A.14) that

$$R^{(m)}(\gamma)(\Phi_D h^{(m)}_{\substack{(k)\\ [l]}})(y) = C \sum_{[\bar{l}]} \Bigg(\sum_{[\bar{l}^{(1)}],\ldots,[\bar{l}^{(r)}]} \times \Bigg(\sum_{[l^{(1)}]+\ldots+[l^{(r)}]=[l]} D^{(k^{(1)})}_{[\bar{l}^{(1)}][l^{(1)}]}(\gamma) \ldots D^{(k^{(r)})}_{[\bar{l}^{(r)}][l^{(r)}]}(\gamma) \alpha^{k,k^{(1)}\ldots k^{(r)}}_{[\bar{l}],[l^{(1)}][l^{(r)}]} f^{(k)}_{[\bar{l}]}(y) \Bigg)\Bigg). \tag{A.15}$$

Finally, Eqs. (A.9) and (A.15) imply that

$$R^{(m)}(\gamma)(\Phi_D h^{(m)}_{\substack{(k)\\ [l]}})(y) = C \sum_{[\bar{l}]} D^{(k)}_{[\bar{l}][l]}(\gamma) f^{(k)}_{[\bar{l}]}(y). \tag{A.16}$$

By applying the map ψ_D (cf. Ref. 5) to Eq. (A.16), we get

$$R^{(m)}(\gamma) h^{(m)}_{\substack{(k)\\ [l]}} = \sum_{[\bar{l}]} D^{(k)}_{[\bar{l}][l]}(\gamma) \psi_D(f^{(k)}_{[\bar{l}]}). \tag{A.17}$$

Now consider any two elements $h^{(m)}_{\substack{(k)\\ [l]}}$ and $h^{(m)}_{\substack{(k')\\ [l']}}$ and let γ belong to U$(n-1)$. Using the fact that the representation $R^{(m)}(\gamma)$ is unitary with respect to the inner product (., .), and the Haar measure on U$(n-1)$ is normalized, we get

$$\int_{U(n-1)} (R^{(m)}(\gamma) h^{(m)}_{\substack{(k)\\ [l]}}, R^{(m)}(\gamma) h^{(m)}_{\substack{(k')\\ [l']}}) d\gamma = (h^{(m)}_{\substack{(k)\\ [l]}}, h^{(m)}_{\substack{(k')\\ [l']}}). \tag{A.18}$$

Thus, Eq. (A.17) and (A.18) imply that

$$(h^{(m)}_{\substack{(k)\\ [l]}}, h^{(m)}_{\substack{(k')\\ [l']}}) = \sum_{[\bar{l}],[\bar{l}']} (\psi_D(f^{(k)}_{[\bar{l}]}), \psi_D(f^{(k')}_{[\bar{l}']}) \times \int_{U_{n-1}} D^{(k)}_{[\bar{l}][l]}(\gamma) D^{*(k')}_{[\bar{l}'][l]}(\gamma) d\gamma. \tag{A.19}$$

where the * means complex conjugation. Schur's orthogonality relations applied to the right-hand side of Eq. (A.19) imply that

$$(h^{(m)}_{\substack{(k)\\ [l]}}, h^{(m)}_{\substack{(k')\\ [l']}}) = 0 \qquad \text{if} \qquad (k) \neq (k') \tag{A.20}$$

and

$$(h^{(m)}_{\substack{(k)\\ [l]}}, h^{(m)}_{\substack{(k)\\ [l']}}) = \sum_{[\bar{l}]} \| \psi_D(f^{(k)}_{[\bar{l}]}) \|^2 \delta_{[l],[l']} / \text{degree } (R^{(k)}), \tag{A.21}$$

where $\delta_{[l],[l']}$ is the Kronecker delta, set equal to one when $[l] = [l']$; 0 elsewhere.

Equations (A.20) and (A.21) show that our assertion that the functions $h^{(m)}_{[k]}$ given by pattern addition do indeed form an orthogonal polynomial basis for $V^{(m)}$. Our induction is thus completed. Note that if we want our basis elements to have the same norms as those given abstractly by Gelfand and Graev, we must multiply $h^{(m)}_{[k]}$ by the factor

$$\left[\prod_{j=2}^{n} \left(\prod_{i=1}^{j-1} \begin{pmatrix} m_{ij} - m_{i+1,j} \\ m_{i,j-1} - m_{i+1,j} \end{pmatrix} \right) \right]^{-1}$$

where $m_{i,j}$ are entries of the tableau $\begin{matrix}(m)\\ [k]\end{matrix}$ and $\begin{pmatrix} p \\ q \end{pmatrix}$ denotes the binomial coefficients $\dfrac{p!}{p!(p-q)!}$.

ACKNOWLEDGMENTS

We wish to thank Dr. James Louck for several helpful discussions.

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(Manuscrit reçu le 7 septembre 1981)