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## **C\*-algebraic generalization of relative entropy and entropy (\*)**

by

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**SUMMARY.** — The concept of differentiability of a state with respect to a weight (state) on C\*-algebra recently introduced by authors generalizes the notion of almost majorising introduced by Naudts in a von Neumann algebra context. It enables us to introduce the notions of entropy and relative entropy in the case of C\*-algebraic description of a physical system. Our generalization of relative entropy leads to some modification of this notion concerning in the quantum case the effect of possible noncommutativity of the states.

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### **1. INTRODUCTION**

Let  $\mathcal{H}$  denote the Hilbert space corresponding to a quantum system and  $B(\mathcal{H})$  denote the algebra of bounded operators on  $\mathcal{H}$ . Observables are represented by selfadjoint elements of  $B(\mathcal{H})$ . In the most cases the statistical states of the system are described by normal states on  $B(\mathcal{H})$ . To each normal state  $\sigma$  on  $B(\mathcal{H})$  corresponds a unique density operator  $\Sigma$  (semi-positive trace-class operator satisfying the condition  $\text{Tr } \Sigma = 1$ ) and  $\sigma(A) = \text{Tr } (\Sigma A)$ ,  $\forall A \in B(\mathcal{H})$ . The entropy of the normal state  $\sigma$  on  $B(\mathcal{H})$  (called von Neumann entropy) is defined by the formula

$$\mathcal{L}^\sigma = - \text{Tr } (\Sigma \ln \Sigma). \quad (1.1)$$

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In the classical case the phase space of the system is a measure space  $(\Omega, \mathcal{B}, \varphi)$ . The macroscopic state of the system is described by a probability measure  $\sigma$  (positive normalized measure) absolutely continuous with respect to  $\varphi$ . Then there exists a positive integrable function  $f$  (Radon-Nikodym derivative) which satisfies  $\int f d\varphi = 1$ ,  $d\sigma = f d\varphi$  and the entropy of the state described by  $\sigma$  (called generalized Boltzmann-Gibbs-Shannon entropy) is given by the formula

$$\mathcal{S}^\sigma = - \int f \ln f d\varphi = - \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi. \quad (1.2)$$

Let us remind another notion of entropy, the so-called relative entropy, cf. for instance [5]. In the classical case consider two states described in terms of the probability measures  $\sigma$  and  $\varphi$  and assume  $\sigma$  to be absolutely continuous with respect to  $\varphi$ . Denote  $g = \frac{d\sigma}{d\varphi}$ . The relative entropy of the state  $\sigma$  with respect to  $\varphi$  is defined by the formula

$$\mathcal{S}^{\sigma/\varphi} = \int g \ln g d\varphi = \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi. \quad (1.3)$$

(This entropy is frequently called Kullback information or information gain).

The quantum analogue of (1.3) is usually written in the form [5]

$$\mathcal{S}^{\sigma/\varphi} = \text{Tr} \{ \Sigma (\ln \Sigma - \ln \Phi) \}, \quad (1.4)$$

where  $\sigma(A) = \text{Tr} (\Sigma A)$ ,  $\varphi(A) = \text{Tr} (\Phi A)$ ,  $\forall A \in \mathcal{B}(\mathcal{H})$ .

The aim of this paper is to generalize the notion of entropy and relative entropy in the case of a physical system described in terms of  $C^*$ -algebra. For this purpose we use the notion of differentiability of the state  $\sigma$  with respect to a weight  $\varphi$  on  $C^*$ -algebra  $\mathcal{A}$  recently introduced by authors [2]. In the case of  $\mathcal{A}$  being a von Neumann algebra,  $\varphi$  — a faithful normal semi-finite weight on  $\mathcal{A}$  and  $\sigma$  — a normal state on  $\mathcal{A}$ ,  $\sigma$  differentiable with respect to  $\varphi$  means that  $\sigma$  is almost majorised by  $\varphi$  in the sense of Naudts, [4]. Next, following Naudts, the  $C^*$ -algebraic generalization of entropy is defined (Sec. 3). It is verified that this expression for entropy in the cases  $\mathcal{A} = \mathcal{B}(\mathcal{H})$  and  $\mathcal{A} = \mathcal{F}(\Omega, \mathcal{B})$  takes the form (1.1) and (1.2), respectively. In Section 2 we consider the case of  $\varphi$  being a state on  $C^*$ -algebra  $\mathcal{A}$  and generalize the notion of relative entropy via the density operator of a state  $\sigma$  differentiable with respect to the state  $\varphi$ . Our generalization of the relative entropy leads to the expression (1.3) in the classical case but in the quantum case  $\mathcal{A} = \mathcal{B}(\mathcal{H})$  we obtain some modification of (1.4) concerning the effect of possible noncommutativity of the states  $\sigma$  and  $\varphi$ .

Namely, if  $\sigma(A) = \text{Tr}(\Sigma A)$ ,  $\varphi(A) = \text{Tr}(\Phi A)$ , assuming  $\Phi$  to be strictly positive we obtain

$$\mathcal{S}^{\sigma|\varphi} = \text{Tr} \{ \Sigma \ln(\Phi^{-1}\Sigma) \} = \text{Tr} \{ \Phi(\Phi^{-1/2}\Sigma\Phi^{-1/2}) \ln(\Phi^{-1/2}\Sigma\Phi^{-1/2}) \} \tag{1.5}$$

which differs from (1.4) except for the case of commuting  $\Sigma$  and  $\Phi$ .

## 2. RELATIVE ENTROPY

Let  $\varphi$  be a state on a C\*-algebra  $\mathcal{A}$  and let  $\pi_\varphi : \mathcal{A} \rightarrow B(\mathcal{H}_\varphi)$  denote the cyclic representation of  $\mathcal{A}$  with respect to  $\varphi$ . Let moreover  $\langle . | . \rangle$  denote the inner product in  $\mathcal{H}_\varphi$ .

A state  $\sigma$  on  $\mathcal{A}$  will be called differentiable with respect to  $\varphi$  ([2]) if it has the form

$$\sigma(a) = \langle \xi | \pi_\varphi(a)\xi \rangle, \tag{2.1}$$

where  $\xi \in \mathcal{H}_\varphi$  is the vector for which there exists a closable operator  $\rho(\xi)$ , densely defined in  $\mathcal{H}_\varphi$  by the formula

$$\rho(\xi) | a \rangle = \pi_\varphi(a)\xi, \quad \forall a \in \mathcal{A} \text{ (*)}. \tag{2.2}$$

It is easy to verify that  $\rho(\xi)$  is affiliated with  $\pi_\varphi(\mathcal{A})'$ . In this case there exists a unique vector  $\xi$  for which  $\overline{\rho(\xi)}$  is positive and selfadjoint [4]. Such vector will be called the positive vector. An operator  $P = \rho(\xi) + \overline{\rho(\xi)}$  is called the density operator of the state  $\sigma$  with respect to  $\varphi$  and  $\overline{\rho(\xi)} = P^{1/2}$  for positive  $\xi$ , which we denote  $\xi = (d\sigma/d\varphi)^{1/2}$ . Following [4] we define the entropy  $\mathcal{S}^{\sigma|\varphi}$  of the state  $\sigma$  differentiable with respect to  $\varphi$  in the following way

$$\mathcal{S}^{\sigma|\varphi} = \lim_{\delta \downarrow 0} \langle \xi | \ln(PE_\delta)\xi \rangle, \tag{2.3}$$

whenever this limit exists,  $E_\delta = E([\delta, \delta^{-1}])$ , where  $E(d\lambda)$  stands for the spectral measure of  $P$ .

a) Let us first consider the case  $\mathcal{A} = B(\mathcal{H})$ . Let  $\varphi(A) = \text{Tr}(\Phi A)$ ,  $A \in \mathcal{A}$ , be a fixed state described by a density operator  $\Phi$ . Let  $\mathcal{H}_\varphi$  denote the Hilbert space of cyclic representation of  $\mathcal{A} = B(\mathcal{H})$  with respect to  $\varphi$ . The inner product in  $\mathcal{H}_\varphi$  has the form

$$\langle A | B \rangle = \varphi(A^+B) = \text{Tr}(\Phi A^+B). \tag{2.4}$$

Let

$$\sigma(A) = \text{Tr}(\Sigma A) \tag{2.5}$$

and let moreover  $\sigma$  fulfil the conditions

$$\sigma(A) = \langle D | \pi_\varphi(A)D \rangle = \langle D | AD \rangle = \text{Tr}(\Phi D^+AD) \tag{2.6}$$

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(\*) We employ the following notation:  $| a \rangle$  stand for vectors belonging to the pre-Hilbert space obtained via the G. N. S. construction while  $\xi$  can be an element of its  $\langle . | . \rangle$ -completion  $\mathcal{H}_\varphi$ .

and there exists an operator  $\rho(D)$  such that

$$\rho(D)|A\rangle = |AD\rangle \quad (2.7)$$

$$\rho(D)^+|A\rangle = |AD^*\rangle. \quad (2.8)$$

(The involution  $^*$ , conjugated to the involution  $^+$ , is defined in the following manner: if  $\rho(D) = R(|D\rangle)$  then  $\rho(D)^+ = R(|D\rangle^*)$  and  $|D\rangle^* = |D^*\rangle$ ). Hence, according to (2.1) and (2.2) the state  $\sigma$  is differentiable with respect to the state  $\varphi$ . From (2.5) and (2.6) one can easily obtain

$$\Sigma = D\Phi D^+. \quad (2.9)$$

Taking into account (2.7) and (2.8) we have

$$\langle B|AD\rangle = \langle B|\rho(D)A\rangle = \langle \rho(D)^+B|A\rangle = \langle BD^*|A\rangle, \quad \forall A, B \in \mathcal{A}. \quad (2.10)$$

From this condition and (2.4) assuming  $\Phi$  to be strictly positive one can find

$$D^* = \Phi D^+ \Phi^{-1}. \quad (2.11)$$

As mentioned above there is exactly one vector  $|D\rangle$  for which  $\rho(D)$  is selfadjoint and positive. Deriving the appropriate conditions from (2.10) we obtain with the help of (2.9) and (2.11) that  $\rho(D)$  is selfadjoint and positive for

$$D^* = D = (\Sigma\Phi^{-1})^{1/2} \equiv \Phi^{1/2}(\Phi^{-1/2}\Sigma\Phi^{-1/2})\Phi^{-1/2}. \quad (2.12)$$

The operator  $\Phi^{-1/2}\Sigma\Phi^{-1/2}$  is obviously selfadjoint and positive with respect to the initial inner product  $(x, y)$  in  $\mathcal{H}$ . Let

$$\Phi^{-1/2}\Sigma\Phi^{-1/2} = \sum_n \lambda_n E_n \quad (2.13)$$

stand for its spectral decomposition. The operators  $\Sigma\Phi^{-1}$  and  $\Phi^{-1}\Sigma$  are not selfadjoint with respect to  $(x, y)$  but they are selfadjoint and positive with respect to  $(x, y)_{\Phi^{-1}} = (\Phi^{-1}x, y)$  and  $(x, y)_{\Phi} = (\Phi x, y)$ , respectively. Denote by  $\{E_n^{\Phi^{-1}}\}_{n \in \mathbb{N}}$  and  $\{E_n^{\Phi}\}_{n \in \mathbb{N}}$  spectral families of the operators  $\Sigma\Phi^{-1}$  and  $\Phi^{-1}\Sigma$  (with respect to the inner products  $(x, y)_{\Phi^{-1}}$  and  $(x, y)_{\Phi}$ , resp.). Then

$$\Sigma\Phi^{-1} = \sum_n \lambda_n E_n^{\Phi^{-1}}, \quad (2.14)$$

$$\Phi^{-1}\Sigma = \sum_n \lambda_n E_n^{\Phi} \quad (2.15)$$

where

$$E_n^{\Phi^{-1}} = \Phi^{1/2} E_n \Phi^{-1/2}, \quad (2.16)$$

$$E_n^{\Phi} = \Phi^{-1/2} E_n \Phi^{1/2}. \quad (2.17)$$

From (2.14)-(2.17) we obtain

$$f(\Sigma\Phi^{-1}) = \Phi^{1/2} f(\Phi^{-1/2}\Sigma\Phi^{-1/2})\Phi^{-1/2}, \quad (2.18)$$

$$f(\Phi^{-1}\Sigma) = \Phi^{-1/2} f(\Phi^{-1/2}\Sigma\Phi^{-1/2})\Phi^{1/2}. \quad (2.19)$$

Moreover one can easily verify the identity

$$f(\Sigma\Phi^{-1})\Phi = \Phi f(\Phi^{-1}\Sigma). \tag{2.20}$$

Because

$$P = [\rho \{ (\Sigma\Phi^{-1})^{1/2} \}]^2 \tag{2.21}$$

we easily obtain

$$P | A \rangle = | \Lambda \Sigma \Phi^{-1} \rangle. \tag{2.22}$$

Taking into account (2.14) define bounded operators  $G_n^{\Phi^{-1}}$  by the formula

$$G_n^{\Phi^{-1}} | A \rangle = | \Lambda E_n^{\Phi^{-1}} \rangle, \quad \forall A \in \mathcal{A}. \tag{2.23}$$

The operators  $G_n^{\Phi^{-1}}$  are mutually orthogonal projectors in  $\pi_\varphi(\mathcal{A})'$  with sum I. Then by (2.21) and (2.22) obviously  $\rho = \sum_n \lambda_n^{1/2} G_n^{\Phi^{-1}}$ ,

$$P = \sum_n \lambda_n G_n^{\Phi^{-1}}. \tag{2.24}$$

Now let us calculate from (2.3) the entropy of the state  $\sigma$  differentiable with respect to the state  $\varphi$ . Using (2.12), (2.24), (2.4) and (2.20) we obtain

$$\begin{aligned} \mathcal{H}^{\sigma/\varphi} &= \lim_{n \rightarrow \infty} \left\langle (\Sigma\Phi^{-1})^{1/2} \left| \ln \left( P \sum_{\substack{p=1 \\ \lambda_p \neq 0}}^n G_p^{\Phi^{-1}} \right) (\Sigma\phi^{-1})^{1/2} \right\rangle \\ &= \lim_{n \rightarrow \infty} \sum_{p=1}^n \ln \lambda_p \langle (\Sigma\Phi^{-1})^{1/2} | G_p^{\Phi^{-1}} (\Sigma\phi^{-1})^{1/2} \rangle \\ &= \lim_{n \rightarrow \infty} \sum_{p=1}^n \ln \lambda_p \langle (\Sigma\phi^{-1})^{1/2} | (\Sigma\phi^{-1})^{1/2} E_p^{\Phi^{-1}} \rangle \\ &= \lim_{n \rightarrow \infty} \sum_{p=1}^n \ln \lambda_p \text{Tr} \{ \Phi(\Phi^{-1}\Sigma)^{1/2} (\Sigma\phi^{-1})^{1/2} E_p^{\Phi^{-1}} \} \\ &= \lim_{n \rightarrow \infty} \sum_{p=1}^n \ln \lambda_p \text{Tr} \{ (\Sigma\Phi^{-1})\Phi(\Sigma\Phi^{-1})^{1/2} E_p^{\Phi^{-1}} \} \\ &= \lim_{n \rightarrow \infty} \sum_{p=1}^n \ln \lambda_p \text{Tr} \{ \Phi(\Sigma\Phi^{-1})G_p^{\Phi^{-1}} \} \\ &= \lim_{n \rightarrow \infty} \sum_{p=1}^n \lambda_p \ln \lambda_p \text{Tr} \{ \Phi G_p^{\Phi^{-1}} \} \\ &= \text{Tr} \{ \Phi(\Sigma\phi^{-1}) \ln (\Sigma\phi^{-1}) \} \\ &= \text{Tr} \{ \Sigma \ln (\Sigma\Phi^{-1}) \} \\ &= \text{Tr} \{ \Sigma \ln (\phi^{-1}\Sigma) \}. \end{aligned} \tag{2.25}$$

Let us generalize the expression (2.25) in the case of noninvertible  $\Phi$ . In this case our relative entropy is well defined by the formula

$$\mathcal{S}^{\sigma|\varphi} = \text{Tr}(\Sigma \ln X) \quad (2.26)$$

for every differentiable  $\sigma$  having the density operator  $\Sigma = \Phi X$ , where  $X$  is an essentially selfadjoint operator with respect to the inner product  $(x, y)_{\Phi} = (\Phi x, y)$ , which for invertible  $\Phi$  takes the form  $X = \Phi^{-1}\Sigma$ . It is obvious that  $X$  is positive with respect to the inner product  $(x, y)_{\Phi}$ ;  $(x, Xx)_{\Phi} = (\Phi x, Xx) = (x, \Phi Xx) = (x, \Sigma x) \geq 0$  for all  $x \in \mathcal{D}(X) \subseteq \mathcal{H}$  due to the positivity of  $\Sigma$ .

Note that  $\ln X \neq \ln \Sigma - \ln \Phi$ , except for the case of commuting  $\Sigma$  and  $\Phi$ . Hence our relative entropy differs from the Araki's relative entropy [1], which is well defined only in the case of faithful states and in this example takes the form (1.4).

b) In the classical case consider  $\mathcal{A} = \mathcal{F}(\Omega, \mathcal{B})$  the  $C^*$ -algebra of bounded measurable functions with the norm  $\|g\|_{\infty} = \sup \{|g(\omega)| : \omega \in \Omega\}$ . Let  $\varphi$  and  $\sigma$  be probability measures on  $(\Omega, \mathcal{B})$  which define the states on  $\mathcal{A}$

$$\varphi(g) = \int g d\varphi, \quad (2.27)$$

$$\sigma(g) = \int g d\sigma. \quad (2.28)$$

Let  $\mathcal{H}_{\varphi} = L^2(\Omega, \mathcal{B}, \varphi)$  and  $\pi_{\varphi} : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_{\varphi})$  be the cyclic  $*$ -representation with domain  $\mathcal{D}(\pi_{\varphi}) = \mathcal{H}_{\varphi}$  defined by

$$[\pi(g)h](\omega) = g(\omega)h(\omega). \quad (2.29)$$

The inner product in  $\mathcal{H}_{\varphi}$  has the form

$$\langle g | h \rangle = \int \bar{g} h d\varphi. \quad (2.30)$$

Assume  $\sigma$  to be differentiable with respect to  $\varphi$ . It is easy to verify that in this case  $\rho(\xi) = \left(\frac{d\sigma}{d\varphi}\right)^{1/2}$ ,  $P = \frac{d\sigma}{d\varphi} \equiv f$ . Let  $P = \int_0^{\infty} \lambda dE_{\lambda}$ , where  $E_{\lambda} = \chi(N_{\lambda})$  denotes the characteristic function of  $N_{\lambda} = \{\omega \in \Omega : f(\omega) \leq \lambda\}$ . From (2.3) we obtain

$$\begin{aligned} \mathcal{S}^{\sigma|\varphi} &= \lim_{\delta \downarrow 0} \langle \xi | \ln (PE_{\delta})\xi \rangle = \lim_{\lambda \rightarrow \infty} \langle \xi | \ln P(E_{\lambda} - E_{1/\lambda})\xi \rangle \\ &= \lim_{\lambda \rightarrow \infty} \int f \ln f(E_{\lambda} - E_{1/\lambda}) d\varphi = \int f \ln f d\varphi \\ &= \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi. \end{aligned} \quad (2.31)$$

### 3. ENTROPY

We will briefly sketch some elements of the theory of weights on C\*-algebra according to [3] and [4].

A weight on C\*-algebra is a function  $\varphi : \mathcal{A}_+ \rightarrow \mathbb{R}^+ + \{ + \infty \}$  satisfying the conditions:

$$\begin{aligned} \varphi(a + b) &= \varphi(a) + \varphi(b), & \forall a, b \in \mathcal{A}_+ \\ \varphi(\alpha a) &= \alpha\varphi(a), & \forall \alpha \in \mathbb{R}^+, \forall a \in \mathcal{A}_+ \end{aligned}$$

(with the convention  $0.(+ \infty) = 0$ ).

A trace on  $\mathcal{A}$  is a weight  $\varphi$  for which

$$\varphi(a^*a) = \varphi(aa^*), \quad \forall a \in \mathcal{A}.$$

Define

$$\mathcal{L}_\varphi := \{ a \in \mathcal{A} : \varphi(a^*a) < + \infty \},$$

then  $\mathcal{L}_\varphi$  is a left ideal in  $\mathcal{A}$ . Let  $\mathcal{A}_\varphi = \mathcal{L}_\varphi^* \mathcal{L}_\varphi$ , that is the set of all complex linear combinations of elements  $a^*b, a, b \in \mathcal{L}_\varphi$ . Then  $\mathcal{A}_\varphi$  is a \*-subalgebra of  $\mathcal{A}$  and  $\mathcal{A}_{\varphi+} = \mathcal{A}_\varphi \cap \mathcal{A}_+$  is exactly the set  $\{ a \in \mathcal{A}_+ : \varphi(a) < + \infty \}$  and  $\mathcal{A}_\varphi$  is the complex linear span of  $\mathcal{A}_{\varphi+}$ . Moreover  $\mathcal{A}_{\varphi+}$  is a cone in  $\mathcal{A}_+$  which is hereditary, i. e.,

$$0 \leq a \leq b \in \mathcal{A}_{\varphi+} \Rightarrow a \in \mathcal{A}_{\varphi+},$$

hence  $\mathcal{A}_\varphi$  has the property that  $b^*ac \in \mathcal{A}_\varphi$  if  $b, c \in \mathcal{A}_\varphi$ . The weight  $\varphi$  can be extended uniquely to a linear positive functional on  $\mathcal{A}_\varphi$  (again denoted by  $\varphi$ ).

**THEOREM** (cf. [3], [4]). — For each weight  $\varphi$  on  $\mathcal{A}$  there exists a Hilbert space  $\mathcal{H}_\varphi$  and two mappings:  $\Lambda_\varphi : \mathcal{L}_\varphi \rightarrow \mathcal{H}_\varphi$  and  $\pi_\varphi : \mathcal{A} \rightarrow \mathbf{B}(\mathcal{H}_\varphi)$  such that  $\Lambda_\varphi$  is linear with the range dense in  $\mathcal{H}_\varphi$ ,  $\pi_\varphi$  is a representation of  $\mathcal{A}$ , and

$$\langle \Lambda_\varphi b \mid \pi_\varphi(a) \Lambda_\varphi c \rangle = \varphi(c^*ab)$$

for all  $a \in \mathcal{A}$  and  $b, c \in \mathcal{L}_\varphi$ .

Let  $\varphi$  be a weight on C\*-algebra  $\mathcal{A}$  and let  $\pi_\varphi : \mathcal{A} \rightarrow \mathbf{B}(\mathcal{H}_\varphi)$  denote the representation of  $\mathcal{A}$  corresponding to  $\varphi$ . A state  $\sigma$  on  $\mathcal{A}$  will be called differentiable with respect to  $\varphi$  if it has the form

$$\sigma(a) = \langle \xi \mid \pi_\varphi(a) \xi \rangle, \tag{3.1}$$

where  $\xi \in \mathcal{H}_\varphi$  is the vector for which there exists a closable operator  $\rho(\xi)$ , densely defined in  $\mathcal{H}_\varphi$  by the formula

$$\rho(\xi) \mid \Lambda_\varphi a \rangle = \pi_\varphi(a) \xi, \quad \forall a \in \mathcal{A}. \tag{3.2}$$

Again, there exists a unique vector  $\xi$  for which  $\overline{\rho(\xi)}$  is positive and selfadjoint and an operator  $\mathbf{P} = \rho(\xi)^+ \overline{\rho(\xi)}$  is called the density operator of the state  $\sigma$



with respect to  $\varphi$ . In the case of  $\mathcal{A}$  being a von Neumann algebra,  $\varphi$  — a faithful normal semi-finite weight on  $\mathcal{A}$  and  $\sigma$  — a normal state on  $\mathcal{A}$ ,  $\sigma$  differentiable with respect to  $\varphi$  means that  $\sigma$  is almost majorised by  $\varphi$  ([4]).

Analogously to [4] we define the entropy  $\mathcal{S}^{\sigma/\varphi}$  of a state  $\sigma$  differentiable with respect to a weight  $\varphi$  by the formula

$$\mathcal{S}^{\sigma/\varphi} = - \lim_{\delta \downarrow 0} \langle \xi | \ln (PE_{\delta})\xi \rangle \quad (3.3)$$

whenever this limit exists,  $E_{\delta} = E([\delta, \delta^{-1}])$ , where  $E(d\lambda)$  stands for the spectral measure of  $P$ .

As in the previous section one can verify that (3.3) is a generalization of entropy.

a) Let  $\varphi(A^+A) = \text{Tr}(A^+A)$  for all  $A \in \mathcal{B}(\mathcal{H})$ . Assuming that a state  $\sigma$  is differentiable with respect to  $\text{Tr}(\cdot)$  we can find that  $\rho(D)$  is positive and selfadjoint for  $D^* = D = \Sigma^{1/2}$ ,  $P = [\rho(\Sigma^{1/2})]^2$  and consequently

$$\mathcal{S}^{\sigma/\varphi} = - \text{Tr}(\Sigma \ln \Sigma).$$

b) Let  $\varphi(\bar{g}g) = \int \bar{g}gd\varphi$ ,  $\forall g \in \mathcal{F}(\Omega, \mathcal{B})$ , where  $\varphi$  is positive measure on  $(\Omega, \mathcal{B})$ . Assuming  $\sigma$  differentiable with respect to  $\varphi$  we obtain  $\rho(\xi) = \left(\frac{d\sigma}{d\varphi}\right)^{1/2}$ ,  $P = \frac{d\sigma}{d\varphi} \equiv f$ . Then from (3.3) we obtain

$$\mathcal{S}^{\sigma/\varphi} = - \int \frac{d\sigma}{d\varphi} \ln \frac{d\sigma}{d\varphi} d\varphi.$$

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