ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 38, nº 1 (1983), p. 59-68

<http://www.numdam.org/item?id=AIHPA_1983__38_1_59_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXVIII, n° 1, 1983, p. 59-68. *Physique théorique.*

An analog of the RAGE Theorem for the impact parameter approximation to three particle scattering

by

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ABSTRACT. — The impact parameter model is governed by the time dependent Schrödinger equation $i\frac{\partial\psi}{\partial t} = (-\Delta/2m + V_1(x - v_1 t) + V_2(x - v_2 t))\psi$, with $v_1 \neq v_2$. In $n \geq 3$ dimensions we show that if ψ is orthogonal to the states which are asymptotically bound to $V_1(x - v_1 t)$ or $V_2(x - v_2 t)$ as $t \to \infty$, then there are arbitrarily large positive times τ such that the probability of finding the particle described by $\psi(\tau)$ near either of the scatterers $V_1(x - v_1\tau)$ or $V_2(x - v_2\tau)$ is arbitrarily small. The analogous result holds for $t \to -\infty$.

RÉSUMÉ. — Le modèle du paramètre d'impact est décrit par l'équation de Schrödinger dépendant du temps $i\frac{d\psi}{\partial t} = (-\Delta/2m + V_1(x - v_1 t) + V_2(x - v_2 t))\psi$, avec $v_1 \neq v_2$. En dimension $n \geq 3$, on montre que, si ψ est orthogonal aux états asymptotiquement liés dans les potentiels $V_1(x - v_1 t)$ ou $V_2(x - v_2 t)$ quand $t \to \infty$, alors il existe des temps τ positifs arbitrairement grands, tels que la probabilité de trouver la particule décrite par $\psi(\tau)$ près des potentiels diffuseurs $V_1(x - v_1 t)$ ou $V_2(x - v_2 t)$ est arbitrairement petite. Un résultat analogue vaut pour $t \to -\infty$.

(*) Supported in part by the National Science Foundation under Grant MCS-8100738.

In two body quantum scattering there is a result which distinguishes bound states from scattering states in a geometric way. Roughly speaking, this result states that the bound states are the only ones which do not from time to time leave the region of space where the potential is large. This was first proved by Ruelle [*11*], and then by Amrein and Georgescu [*1*]. Its importance was recognized by Enss [*3*]. Consequently, the result has been dubbed the « RAGE Theorem » [*9*].

The RAGE Theorem played an important role in Enss's original geometric proof of asymptotic completeness [*9*]. Subsequently several authors [[*2*] [*3*] [*4*] [*5*] [*9*] [*13*] [*14*] [*17*], and references therein] have refined the Enss method. The RAGE Theorem plays an important role in some versions, but it is not explicitly used in others.

We prove a result analogous to the RAGE Theorem for the impact parameter approximation to three body scattering in three or more dimensions. Rather than distinguish bound states from scattering states, our result distinguishes between states in the different scattering channels [*9*] for this model. The time dependent Hamiltonian for this model is

$$H(t) = H_0 + V_1(x - v_1 t) + V_2(x - v_2 t),$$

where $H_0 = -\dfrac{1}{2m}\Delta$ and $v_1 \neq v_2$. For convenience we will assume the mass m is 1. We assume that the potentials $V_j(x)$ satisfy the following condition:

$$\begin{gathered} V_j(x) = (1 + x^2)^{-\delta}[W_{j,1}(x) + W_{j,2}(x)], \\ \text{where } 1 < \delta \text{ and } W_{j,1} \text{ and } W_{j,2} \text{ are real valued functions on } \mathbb{R}^n \\ \text{for some } n \geq 3, \text{ such that } W_{j,1} \in W^{1,s}(\mathbb{R}^n) \text{ for some } s > n/2 \\ \text{and } W_{j,2} \in W^{1,\infty}(\mathbb{R}^n). \end{gathered} \tag{1}$$

Here $W^{1,q}(\mathbb{R}^n)$ denotes the Sobolev space [*16*] of $L^q(\mathbb{R}^n)$ functions whose weak first partial derivatives belong to $L^q(\mathbb{R}^n)$.

Under this hypothesis $H_1(t)=H_0+V_1(x-v_1t)$, $H_2(t)=H_0+V_2(x-v_2t)$, and $(H(t)$ are self-adjoint on the domain of H_0. Furthermore ([*12*], Theorem II.27; [*15*]), condition (1) guarantees that $H_1(t)$, $H_2(t)$, and $H(t)$ generate strongly continuous unitary propagators $U_1(t, s)$, $U_2(t, s)$, and $U(t, s)$, respectively. For $H(t)$ this means

(1) $U(t, s)$ is unitary on $L^2(\mathbb{R}^n)$ and is strongly continuous in (t, s);

(2) $U(t, s) = U(t, r)U(r, s)$ for $-\infty < t, r, s < \infty$;

(3) if $\psi \in D(H_0^{1/2})$, then $U(t, s)\psi \in D(H_0^{1/2})$ and $i(\partial/\partial t)U(t, s)\psi = H(t)U(t, s)\psi$, where the derivative is understood as the strong derivative in the space $\mathscr{H}_{-1} = \mathscr{H}_{+1}^* =$ dual space of $W^{1,2}(\mathbb{R}^n)$.

The analogous statements hold for $H_1(t)$ and $H_2(t)$.

Condition (1) also implies that $H_1(t)$ and $H_2(t)$ have only finitely many eigenvalues, and the orthogonal projection $P_j^{(0)}(t)$ onto the span of the eigenfunctions of $H_j(t)$ has finite rank [*9*]. We define $P_j(t) = e^{imv_jx}P_j^{(0)}(t)e^{-imv_jx}$.

It is not hard to prove [15] that condition (1) implies the existence of the wave operators

$$\Omega_j^{\pm}(s) = \underset{t \to \mp \infty}{\text{strong-lim}}\ \mathrm{U}(s, t)\mathrm{U}_j(t, s)\mathrm{P}_j(s) .$$

The ranges of $\Omega_j^{\pm}(s)$ are of the same dimension as the ranges of $\mathrm{P}_j(s)$ (whose dimensions are independent of s). Ran $\Omega_1^{+}(s)$ is orthogonal to Ran $\Omega_2^{+}(s)$, and Ran $\Omega_1^{-}(s)$ is orthogonal to Ran $\Omega_2^{-}(s)$ [15]. The states in Ran $\Omega_j^{+}(s)$ or Ran $\Omega_j^{-}(s)$ are those states at time s which are asymptotically bound to $\mathrm{V}_j(x - v_j t)$ in the remote past or remote future, respectively. Our main result is the following:

THEOREM 1. — Suppose V_1 and V_2 satisfy condition (1), and let χ_1 and χ_2 be characteristic functions of any two bounded subsets of $\mathbb{R}^n$. If $\phi(s)$ is orthogonal to Ran $\Omega_1^{\pm}(s) \oplus$ Ran $\Omega_2^{\pm}(s)$, then

$$\liminf_{t \to \mp \infty} \{ \| \chi_1(. - v_1 t)\mathrm{U}(t, s)\phi(s) \| + \| \chi_2(. - v_2 t)\mathrm{U}(t, s)\phi(s) \| \} = 0 .$$

Heuristically this result shows that a particle whose motion is governed by $\mathrm{H}(t)$ and which is asymptotically bound to neither $\mathrm{V}_1(x - v_1 t)$ nor $\mathrm{V}_2(x - v_2 t)$ must move away from both potentials at some large times. In contrast, it is easy to prove that modulo arbitrarily small errors, the asymptotically bound states stay near one potential or the other at large times. Thus, we have characterizations of the different scattering channels for $\mathrm{H}(t)$.

REMARKS 1. — Due to the lack of energy conservation in the impact parameter model we have not been able to prove a theorem of this sort for 1 or 2 dimensions. In three or more dimensions we have been able to make use of the diffusion effects of the free Schrödinger propagator to overcome this difficulty.

2. We hope that Theorem 1 will help provide an « Enss method » type proof of asymptotic completeness for the impact parameter model. The lack of energy conservation has so far prevented us from finding such a proof.

3. The presently available asymptotic completeness theorem [6] [15] for the impact parameter model has a shortcoming. One must assume there are no zero energy resonances or zero energy bound states for H_1 or H_2 in order to apply the theorem. Yajima has pointed out to us privately that if H_1 and H_2 both have zero energy resonances, then $\mathrm{H}(t)$ should have a bound state for each t which is asymptotically absorbed into the continuum as $t \to \pm \infty$. One might conjecture that asymptotic completeness fails in this case due to the presence of this extra bound state, but Theorem 1 strongly suggests that asymptotic completeness still holds.

4. The RAGE Theorem is a stronger type of result than Theorem 1 in the sense that it states that if ϕ is orthogonal to the bound states of $H = H_0 + V$, then $\frac{1}{2T}\int_{-T}^{T} \| \chi_e^{-itH}\phi \| \, dt$ tends to zero as $T \to \infty$. It follows that $\liminf_{t\to\pm\infty} \| \chi_e^{-itH}\phi \| = 0$. This second result is the useful one [3], and is the one to which Theorem 1 is an analog.

5. The variable s which appears in Theorem 1 plays a non-essential role, since $U(s', s)$ maps $\operatorname{Ran} \Omega_j^{\pm}(s)$ isomorphically onto $\operatorname{Ran} \Omega_j^{\pm}(s')$ for any s and s'. In our proofs we will take $s = 0$.

Since $V_1(t)$ has no t dependence in one frame of reference and $V_2(t)$ has no t dependence in a different frame, we will find it useful to work in both frames. To minimize the resulting confusion which might arise we will use superscripts (1) and (2) to indicate that we are working in frames in which $V_1(t)$ and $V_2(t)$, respectively, are time dependent. For example $U_1(t, s)$ is generated by $H_1(t)$. In the frame in which $V_1(t)$ is time independent we denote the corresponding propagator by $U_1^{(1)}(t, s) = e^{-i(t-s)H_1}$, where $H_1 = H_0 + V_1(x)$. For $\phi \in \mathscr{H}$, the corresponding $\phi^{(1)} \in \mathscr{H}^{(1)}$ and $\phi^{(2)} \in \mathscr{H}^{(2)}$ are related by simple time dependent unitary transformations.

To prove Theorem 1 we need two preliminary results. The first one contains most of the hard work. The second is required because of the lack of energy conservation.

Proposition 2. — Assume V_1 and V_2 satisfy condition (1). Suppose $A(t)$ and $B(t)$ are compact operator valued functions such that $A^{(1)}(t)$ and $B^{(2)}(t)$ are both independent of t. If $\phi \in \mathscr{H}$ is orthogonal to $\operatorname{Ran} \Omega_1^{\pm}(0) \oplus \operatorname{Ran} \Omega_2^{\pm}(0)$, then

$$\liminf_{t\to\pm\infty} \{ \| A(t)U(t, 0)\phi \| + \| B(t)U(t, 0)\phi \| \} = 0 .$$

Proof. — Consider only the case in which $t \to +\infty$ and ϕ is orthogonal to $\operatorname{Ran} \Omega_1^{-}(0) \oplus \operatorname{Ran} \Omega_2^{-}(0)$. The other case is similar. We may assume without loss of generality that $\| \phi \| = 1$, and for convenience we set $\phi^{(1)}(t) = U^{(1)}(t, 0)\phi^{(1)}$ and $\phi^{(2)}(t) = U^{(2)}(t, 0)\phi^{(2)}$.

We may rewrite the conclusion to the proposition as

$$\liminf_{t\to\infty} \{ \| A^{(1)}\phi^{(1)}(t) \| + \| B^{(2)}\phi^{(2)}(t) \| \} = 0 .$$

Let $E_1^{(1)}(t)$ denote the orthogonal projection onto the orthogonal complement of $\operatorname{Ran} \Omega_1^{-(1)}(t)$ in $\mathscr{H}^{(1)}$, and let $E_2^{(2)}(t)$ denote the orthogonal projection onto the orthogonal complement of $\operatorname{Ran} \Omega_2^{-(2)}(t)$ in $\mathscr{H}^{(2)}$. Since $U(t, 0)\phi$ is orthogonal to both $\operatorname{Ran} \Omega_1^{-}(t)$ and $\operatorname{Ran} \Omega_2^{-}(t)$ we can rewrite the conclusion again as

$$\liminf_{t\to\infty} \{ \| A^{(1)}E_1^{(1)}(t)\phi^{(1)}(t) \| + \| B^{(2)}E_2^{(2)}(t)\phi^{(2)}(t) \| \} = 0 .$$

Assume this is false. Then the quantity inside the brackets must be greater than some $\varepsilon > 0$ for all t greater than some $S_0 > 0$. Using the canonical form for compact operators [7, Theorem VI.17], we see that there exist four finite orthonormal sets $\{\alpha_n^{(1)}\}$, $\{\beta_n^{(1)}\}$, $\{\zeta_n^{(2)}\}$, $\{\eta_n^{(2)}\}$, and two finite sequences of non-negative numbers λ_n, μ_n $(1 \leq n \leq N)$, such that

$$\left\| A^{(1)} - \sum_{n=1}^{N} \lambda_n \alpha_n^{(1)} \langle \beta_n^{(1)}, . \rangle \right\| \quad \text{and} \quad \left\| B^{(2)} - \sum_{n=1}^{N} \mu_n \zeta_n^{(2)} \langle \eta_n^{(2)}, . \rangle \right\|$$

are both less than $\varepsilon/4$. Since $t > S_0$ implies

$$\begin{aligned}
\varepsilon &< \| A^{(1)} E_1^{(1)}(t)\phi^{(1)}(t) \| + \| B^{(2)} E_2^{(2)}(t)\phi^{(2)}(t) \| \\
&\leq \left\| A^{(1)} E_1^{(1)}(t)\phi^{(1)}(t) - \sum_{n=1}^{N} \lambda_n \alpha_n^{(1)} \langle \beta_n^{(1)}, E_1^{(1)}\phi^{(1)}(t) \rangle \right\| \\
&+ \left\| \sum_{n=1}^{N} \lambda_n \alpha_n^{(1)} \langle \beta_n^{(1)}, E_1^{(1)}\phi^{(1)}(t) \rangle \right\| \\
&+ \left\| B^{(2)} E_2^{(2)}(t)\phi^{(2)}(t) - \sum_{n=1}^{N} \mu_n \zeta_n^{(2)} \langle \eta_n^{(2)}, E_2^{(2)}\phi^{(2)}(t) \rangle \right\| \\
&+ \left\| \sum_{n=1}^{N} \mu_n \zeta_n^{(2)} \langle \eta_n^{(2)}, E_2^{(2)}(t)\phi^{(2)}(t) \rangle \right\|
\end{aligned}$$

we consequently have

$$\left\| \sum_{n=1}^{N} \lambda_n \alpha_n^{(1)} \langle \beta_n^{(1)}, E_1^{(1)}(t)\phi^{(1)}(t) \rangle \right\| + \left\| \sum_{n=1}^{N} \mu_n \zeta_n^{(2)} \langle \eta_n^{(2)}, E_2^{(2)}(t)\phi^{(2)}(t) \rangle \right\| > \varepsilon/2$$

for all $t > S_0$.

From this it follows that

$$\left\| \sum_{n=1}^{N} \lambda_n \alpha_n^{(1)} \langle \beta_n^{(1)}, E_1^{(1)}(t)\phi^{(1)}(t) \rangle \right\|^2 + \left\| \sum_{n=1}^{N} \mu_n \zeta_n^{(2)} \langle \eta_n^{(2)}, E_2^{(2)}(t)\phi^{(2)}(t) \rangle \right\|^2 > \varepsilon^2/8$$

for all $t > S_0$, or equivalently

$$\sum_{n=1}^{N} \{ \lambda_n^2 | \langle E_1^{(1)}(t)\beta_n^{(1)}, \phi^{(1)}(t) \rangle |^2 + \mu_n^2 | \langle E_2^{(2)}(t)\eta_n^{(2)}, \phi^{(2)}(t) \rangle |^2 \} > \varepsilon^2/8$$

for all $t > S_0$.

Since the λ_n and μ_n are bounded, we therefore have

$$\sum_{n=1}^{N} \{ | \langle E_1^{(1)}(t)\beta_n^{(1)}, \phi^{(1)}(t) \rangle |^2 + | \langle E_2^{(2)}(t)\eta_n^{(2)}, \phi^{(2)}(t) \rangle |^2 \} > \varepsilon'$$

for all $t > S_0$ and some $\varepsilon' > 0$.

By examining the proof of existence of the wave operators [15] it is easy to see that $E_1^{(1)}(t)$ and $E_2^{(2)}(t)$ converge strongly to $E_1^{(1)}$ and $E_2^{(2)}$ as $t \to \infty$, respectively, where $E_j^{(j)}$ is the projection onto the orthogonal complement of the bound states of $H_j = H_0 + V_j(x)$. As a consequence there exists $S_1 \geq S_0$ such that $t > S_1$ implies

$$\sum_{n=1}^{N} \{ | \langle E_1^{(1)}\beta_n^{(1)}, \phi^{(1)}(t) \rangle |^2 + | \langle E_2^{(2)}\eta_n^{(2)}, \phi^{(2)}(t) \rangle |^2 \} > \varepsilon'/2. \qquad (2)$$

For each n consider $e^{itH_1}E_1^{(1)}\beta_n^{(1)}$ and $e^{itH_2}E_2^{(2)}\eta_n^{(2)}$. By asymptotic completeness for H_1 and H_2 [3] we can find $t_{1,n}$ such that

$$| \langle E_1^{(1)}\beta_n^{(1)}, e^{itH_1}E_1^{(1)}\beta_n^{(1)} \rangle | < \delta/3 \quad \text{and} \quad | \langle E_2^{(2)}\eta_n^{(2)}, e^{itH_2}E_2^{(2)}\eta_n^{(2)} \rangle | < \delta/3$$

for all $t > t_{1,n}$ (where δ is arbitrarily small). Let $T_1 = \text{Max}\,\{ t_{1,n} \}$. Then for all $n = 1, 2, \ldots, N$ and all non-negative integers $j > l \geq 0$ we have

$$| \langle e^{ilT_1H_1}E_1^{(1)}\beta_n^{(1)}, e^{ijT_1H_1}E_1^{(1)}\beta_n^{(1)} \rangle | = | \langle E_1^{(1)}\beta_n^{(1)}, e^{i(j-l)T_1H_1}E_1^{(1)}\beta_n^{(1)} \rangle | < \delta/3,$$

and

$$| \langle e^{ilT_1H_2}E_2^{(2)}\eta_n^{(2)}, e^{ijT_1H_2}E_2^{(2)}\eta_n^{(2)} \rangle | < \delta/3 .$$

If $\psi \in D(H_1) = D(H_0)$, then $U_1^{(1)}(t, s)\psi = e^{-i(t-s)H_1}\psi$ is continuous in the Hilbert space $D(H_1)$ with norm $\| \psi \|_{D(H_1)} = \| (H_1 + c)\psi \|_{L^2}$, where $c \geq -\inf \sigma(H_1)$. Condition (1) guarantees that $V_2^{(1)}(t)$ is bounded from $D(H_1)$ to $\mathscr{H}^{(1)} = L^2(\mathbb{R}^n)$ [8], and $V_2^{(1)}(t): D(H_1) \to \mathscr{H}^{(1)}$ clearly converges strongly to zero as $t \to \infty$. With this information and the formula

$$\| U^{(1)}(t, s)\psi - U_1^{(1)}(t, s)\psi \| = \left\| \int_s^t U^{(1)}(t, r)V_2^{(1)}(r)U_1^{(1)}(r, s)\psi dr \right\|,$$

one easily sees that for each integer j, $U^{(1)}(T_0, T_0 + jT_1)$ converges strongly to $e^{ijT_1H_1}$ as $T_0 \to \infty$. Similarly, $U^{(2)}(T_0, T_0 + jT_1)$ converges strongly to $e^{ijT_1H_2}$ as $T_0 \to \infty$. As a consequence, we can choose $T_0 > S_1$ so large that

$$\begin{aligned}
&| \langle U^{(1)}(T_0, T_0 + lT_1)E_1^{(1)}\beta_n^{(1)}, U^{(1)}(T_0, T_0 + jT_1)E_1^{(1)}\beta_n^{(1)} \rangle | \\
&\quad \leq | \langle [U^{(1)}(T_0, T_0 + lT_1) - e^{ilT_1H_1}]E_1^{(1)}\beta_n^{(1)}, U^{(1)}(T_0, T_0 + jT_1)E_1^{(1)}\beta_n^{(1)} \rangle | \\
&\quad + | \langle e^{ilT_1H_1}E_1^{(1)}\beta_n^{(1)}, [U^{(1)}(T_0, T_0 + jT_1) - e^{ijT_1H_1}]E_1^{(1)}\beta_n^{(1)} \rangle | \\
&\quad + | \langle e^{ilT_1H_1}E_1^{(1)}\beta_n^{(1)}, e^{ijT_1H_1}E_1^{(1)}\beta_n^{(1)} \rangle | \\
&\quad < \delta \quad \text{for} \quad 0 \leq l < j \leq 2KN \quad \text{and} \quad 1 \leq n \leq N .
\end{aligned}$$

(Here K is fixed, but arbitrarily large; l and j are integers). Similarly by increasing T_0 if necessary, we can conclude that

$$|\langle U^{(2)}(T_0, T_0 + lT_1)E_2^{(2)}\eta_n^{(2)}, U^{(2)}(T_0, T_0 + jT_1)E_2^{(2)}\eta_n^{(2)} \rangle| < \delta$$

for $0 \leq l < j \leq 2NK$ and $1 \leq n \leq N$.

Let

$$A_n = \{ t \in [T_0, \infty) : |\langle E_1^{(1)}\beta_n^{(1)}, \phi^{(1)}(t) \rangle|^2 > \varepsilon'/4N \}$$

and

$$B_n = \{ t \in [T_0, \infty) : |\langle E_2^{(2)}\eta_n^{(2)}, \phi^{(2)}(t) \rangle|^2 > \varepsilon'/4N \}.$$

As a consequence of inequality (2) we have $\bigcup_{n=1}^{N}(A_n \cup B_n) = [T_0, \infty)$, and at least one of the sets A_n, B_n must contain at least K points of the form T_0, $T_0 + T_1$, $T_0 + 2T_1, \ldots, T_0 + 2NKT_1$ (since there are 2N sets which contain these $2NK + 1$ points). Without loss of generality we may assume that the set A_n for some fixed n contains the points

$$T_0 + m_1T_1, \qquad T_0 + m_2T_1, \ldots, T_0 + m_KT_1,$$

where $0 \leq m_1 < m_2 < \ldots < m_K \leq 2NK$ and $m_j \in \mathbb{Z}$. If we define $\gamma_j^{(1)} = U^{(1)}(0, T_0 + m_jT_1)E_1^{(1)}\beta_n^{(1)}$, then we have

$$|\langle \gamma_j^{(1)}, \phi^{(1)}(0) \rangle| = |\langle E_1^{(1)}\beta_n^{(1)}, \phi^{(1)}(T_0 + m_jT_1) \rangle| > (\varepsilon'/4N)^{1/2} \equiv \varepsilon'' \quad (3)$$

for $1 \leq j \leq K$, by the definition of A_n. However, the vectors $\gamma_j^{(1)}$ form an almost orthogonal set in the sense that

$$|\langle \gamma_j^{(1)}, \gamma_l^{(1)} \rangle| < \delta \quad (4)$$

for $1 \leq l < j \leq K$.

Since K is arbitrarily large, δ is arbitrarily small, $\varepsilon'' > 0$ is independent of K and δ, and $\| \gamma_j^{(1)} \| \leq 1$ for $1 \leq j \leq K$, we can conclude that $\| \phi^{(1)}(0) \|$ is arbitrarily large by virtue of inequalities (3) and (4). This contradicts our assumption $\| \phi(0) \| = 1$, and the proposition is proved. ■

Remark. — In the last step of the above proof we used the fact that no vector ψ of norm 1 can have a large (i. e. greater than ε'' in absolute value) inner product with a large number (K) of elements $\gamma_n^{(1)}$ of an almost orthogonal set of vectors whose norms are at most 1. This fact is intuitively obvious, but we have not been able to find a simple proof of it. The only proof we know involves the use of Gram-Schmidt orthogonalization on the $\gamma_n^{(1)}$'s to produce an orthogonal set of vectors, all of which have large (i. e. bounded below in absolute value by $\varepsilon''' > 0$ which depends on ε'' and δ) inner products with ψ. This proof is extremely tedious!

PROPOSITION 3. — Let $g_\varepsilon^{(1)}(H_0)$ be the operator $e^{-\varepsilon H_0}$ on $\mathscr{H}^{(1)}$ and let $g_\varepsilon^{(2)}(H_0)$ be the operator $e^{-\varepsilon H_0}$ on $\mathscr{H}^{(2)}$. Let $\chi_1^{(1)} : \mathscr{H}^{(1)} \to \mathscr{H}^{(1)}$ and

$\chi_2^{(2)} : \mathscr{H}^{(2)} \to \mathscr{H}^{(2)}$ be multiplication by characteristic functions of bounded subsets of $\mathbb{R}^n$. Assume V_1 and V_2 satisfy condition (1). For each $\phi \in \mathscr{H}$ the corresponding $\phi^{(1)} \in \mathscr{H}^{(1)}$ and $\phi^{(2)} \in \mathscr{H}^{(2)}$ satisfy

$$\lim_{\varepsilon \to 0} \{ \sup_{t > 0} \| \chi_1^{(1)}(1 - g_\varepsilon^{(1)}(H_0))U^{(1)}(t, 0)\phi^{(1)} \| \} = 0$$

and

$$\lim_{\varepsilon \to 0} \{ \sup_{t > 0} \| \chi_2^{(2)}(1 - g_\varepsilon^{(2)}(H_0))U^{(2)}(t, 0)\phi^{(2)} \| \} = 0 .$$

Proof. — Consider only the first result. Proof of the second is similar.

$$\begin{aligned} &\| \chi_1^{(1)}(1 - g_\varepsilon^{(1)}(H_0))U^{(1)}(t, 0)\phi^{(1)} \| \\ &\qquad \leq \| \chi_1^{(1)}(1 - g_\varepsilon^{(1)}(H_0))e^{-itH_0}\phi^{(1)} \| \\ &\qquad + \left\| \chi_1^{(1)}(1 - g_\varepsilon^{(1)}(H_0)) \int_0^t e^{-isH_0}V^{(1)}(t - s)U^{(1)}(t - s, 0)\phi^{(1)} ds \right\| . \end{aligned}$$

The first term on the right hand side tends to zero as $\varepsilon \to 0$ because $(1 - g_\varepsilon^{(1)}(H_0))$ commutes with e^{-itH_0} and $(1 - g_\varepsilon^{(1)}(H_0))$ converges strongly to zero as $\varepsilon \to 0$. The second term is dominated by

$$\begin{aligned} &\int_0^t \| \chi_1^{(1)}(1 - g_\varepsilon^{(1)}(H_0))e^{-isH_0}V^{(1)}(t - s)U^{(1)}(t - s, 0)\phi^{(1)} \| ds \\ &\qquad \leq \int_0^\infty \| V^{(1)}(t - s)e^{isH_0}(1 - g_\varepsilon^{(1)}(H_0))\chi_1^{(1)} \| ds . \qquad (5) \end{aligned}$$

Condition (1) forces V_1 and V_2 to belong to $L^r(\mathbb{R}^n)$ for all r in a neighborhood of n (*see* [*15*], page 154). Consequently we can use a standard argument (*see* [*10*], pages 154-155) to prove that the integrand on the right hand side of (5) is dominated by a fixed L^1 function of s, independent of ε and t. Briefly, that argument goes as follows: The operator $\chi_1^{(1)}$ is bounded from L^2 to L^p for $1 \leq p \leq 2$. The operator $(1 - g_\varepsilon^{(1)}(H_0))e^{isH_0}$ maps L^p into L^q for $1 \leq p \leq 2$ and $1/p + 1/q = 1$. For suitable values of q, $V^{(1)}(t-s)$ maps L^q into L^2 with uniformly bounded norm. By keeping track of the s and ε dependence one sees that the integrand on the right hand side of (5) is dominated by a fixed L^1 function of s, independent of t and ε. Furthermore, these estimates and a density argument show that we need only prove that the right hand side of (5) tends to zero as $\varepsilon \to 0$ under the additional assumption that V_1 and V_2 belong to $L^2(\mathbb{R}^n)$. Henceforth we make this assumption.

By dominated convergence, we need only show that the integrand on the right hand side of (5) tends to zero pointwise as $\varepsilon \to 0$. This integrand is bounded by the Hilbert-Schmidt norm

$$\begin{aligned} &\| V^{(1)}(t - s)e^{isH_0}(1 - g_\varepsilon^{(1)}(H_0))\chi_1^{(1)} \|_{H-S} \\ &\qquad = \left[\iint | V^{(1)}(t - s, y)K_\varepsilon(s, x - y)\chi_1^{(1)}(x) |^2 dxdy \right]^{1/2} , \qquad (6) \end{aligned}$$

where

$$K_\varepsilon(s, x - y) \\ = (2\pi i s)^{-n/2} \exp \{ i \mid x-y \mid^2/2s \} - (2\pi i(s - i\varepsilon))^{-n/2} \exp \{ i \mid x-y \mid^2/2(s-i\varepsilon) \}.$$

Applying the dominated convergence theorem again, one sees that the right hand side of (6) tends to zero as $\varepsilon \to 0$ for each t and s. This proves the proposition ■

Proof of Theorem 1. — Let $\varepsilon > 0$. Choose δ by Proposition 3 so small that $\| \chi_1^{(1)}(1-g_\delta^{(1)}(H_0))\phi^{(1)}(t) \|$ and $\| \chi_2^{(2)}(1-g_\delta^{(2)}(H_0))\phi^{(2)}(t) \|$ are both less than $\varepsilon/2$. Since $\chi_1^{(1)}g_\delta^{(1)}(H_0)$ and $\chi_2^{(2)}g_\delta^{(2)}(H_0)$ are Hilbert-Schmidt, and hence, compact operators, it follows from Proposition 2 that

$$\liminf_{t\to\infty} \{ \| \chi_1^{(1)}g_\delta^{(1)}(H_0)\phi^{(1)}(t) \| + \| \chi_2^{(2)}g_\delta^{(2)}(H_0)\phi^{(2)}(t) \| \} = 0 .$$

Thus,

$$\liminf_{t\to\infty} \{ \| \chi_1^{(1)}\phi^{(1)}(t) \| + \| \chi_2^{(2)}\phi^{(2)}(t) \| \} < \varepsilon .$$

This implies the $t \to \infty$ part of the theorem. The $t \to -\infty$ part is similar. ■

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(Manuscrit reçu le 28 janvier 1982)