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Regularly full logics and the uniqueness problem for observables

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ABSTRACT. — A full logic L is called regularly full if for any noncompatible elements $a, b \in L$ and any $\varepsilon > 0$ there exists a state m on L such that $m(a) = 1$, $m(b) \geqslant 1 - \varepsilon$. We show that the uniqueness problem for observables (as posed by S. Gudder [*4*] [*5*]) has a positive answer in regularly full logics.

RÉSUMÉ. — Une logique « complète » est dite « régulièrement complète » si pour tout couple $a, b \in L$ d'éléments non compatibles et tout $\varepsilon > 0$, il existe un état m sur L tel que $m(a) = 1$, $m(b) \geq 1 - \varepsilon$. On montre que le problème d'unicité pour les observables (tel qu'il est posé par S. Gudder [*4*] [*5*]) a une réponse positive dans une logique régulièrement complète.

1. INTRODUCTION

Suppose that the means of two bounded observables on a logic are equal in every state. Does this imply that the observables must be equal ? It does if the logic is that of projectors of a Hilbert space (*see* S. Gudder [*5*]) but one does not seem to know the answer for general logics. A partial result was obtained by S. Gudder [5] who proved that if one of the observables is « relatively simple » (has at most one limit point in its spectrum) then the answer is yes. He did not impose any particular condition on the logic. We bring another partial solution by pursuing the problem along a different line. We show that the answer is yes for arbitrary observables if we assume that the logic in question possesses a « relatively rich » set of states (see the abstract for a more rigorous definition).

2. LOGICS. BASIC NOTIONS AND RESULTS

DEFINITION 1.1. — A logic is a triple $(L, \geqslant, ')$, where L is a nonvoid set with a partial ordering $\geqslant$ and $'$ is a unary operation on L such that the following conditions are satisfied (the symbols $\vee$, $\wedge$ stand for the induced lattice-theoretic operations):

i) there is the least element 0 in L,

ii) if, $a, b \in L$, $a \leqslant b$ then $a' \geqslant b'$,

iii) if $a \in L$ then $(a')' = a$,

iv) if $a, b \in L$, $a \leqslant b$ then $b = a \vee (b \wedge a')$,

v) if $\{ a_i \mid i \in N \}$ is a sequence of elements of L and if $a_i \leqslant a_j'$ for any distinct $i, j \in N$ then

the least upper bound $\bigvee_{i=1}^{\infty} a_i$ exists in L.

We shall simply write L instead of $(L, \geqslant, ')$ and we shall reserve the letter L for logics. The σ-algebras and the lattice of projectors of a Hilbert space may serve for basic examples of logics.

DEFINITION 1.2. — Two elements $a, b \in L$ are called orthogonal if $a \leqslant b'$. More generally, two elements $a, b \in L$ are called compatible (in symbols: $a \leftrightarrow b$) if there are three mutually orthogonal elements a_1, b_1, c such that $a = a_1 \vee c, b = b_1 \vee c$.

Let us now recall basic properties of logics.

PROPOSITION 1.3. — (1) There is the greatest element $1 = 0'$ in any logic L.

(2) If $a \leftrightarrow b$ then $a \vee b$, $a \wedge b$ exist in L and $a \wedge b = 0$ if and only if $a \leqslant b'$.

(3) If $a \leftrightarrow b$ then $a \leftrightarrow b'$ and *vice versa*.

(4) If $a_i \leqslant a_{i+1}'$ $i \in N$ and if $a_i \leftrightarrow b$ for any $i \in N$ then $\bigvee_{i=1}^{\infty} a_i \leftrightarrow b$.

Proof. — See S. Gudder [5] for (1), (2) ,(3). The details of (4) can be found in J. Brabec, P. Pták [1].

DEFINITION 1.4. — A state on a logic L is a mapping $m: \mathrm{L} \to \langle 0, 1 \rangle$ such that

i) $m(1) = 1$,

ii) if $\{ a_i \mid i \in \mathrm{N} \}$ is a sequence of mutually orthogonal elements of L then $m\left(\bigvee_{i=1}^{\infty} a_i \right) = \sum_{i=1}^{\infty} m(a_i)$.

Let us denote by $\mathscr{S}(\mathrm{L})$ the set of all states on a logic L, It is known (*see* F. W. Shultz [*6*]) that $\mathscr{S}(\mathrm{L})$ may be either a void set or a singleton or an infinite (strongly convex) set. Naturally, the logics important within the quantum theories are supposed to have a reasonably rich set of states.

DEFINITION 1.5. — A logic is called quite full if the following implication holds true: If $a \nleq b$ then there is a state $m \in \mathscr{S}(\mathrm{L})$ such that $m(a) = 1$, $m(b) \neq 1$.

The latter definition is a standard one (*see* S. Gudder [*4*] [*5*]). We strengthen quite fullness to some extent and arrive thus at a type of logics we shall be interested in throughout the present paper.

DEFINITION 1.6. — A logic L is called regularly full if $\mathscr{S}(\mathrm{L})$ fulfils the following requirements:

i) if $a \in \mathrm{L}$, $a \neq 0$ then there exists a state $m \in \mathscr{S}(\mathrm{L})$ such that $m(a) = 1$,

ii)) if $a \nleftrightarrow b$ and we are given an ε, $\varepsilon > 0$ then there is a state $m \in \mathscr{S}(\mathrm{L})$ such that $m(a) = 1$, $m(b) \geqslant 1 - \varepsilon$.

PROPOSITION 1.7. — Any regularly full logic is quite full.

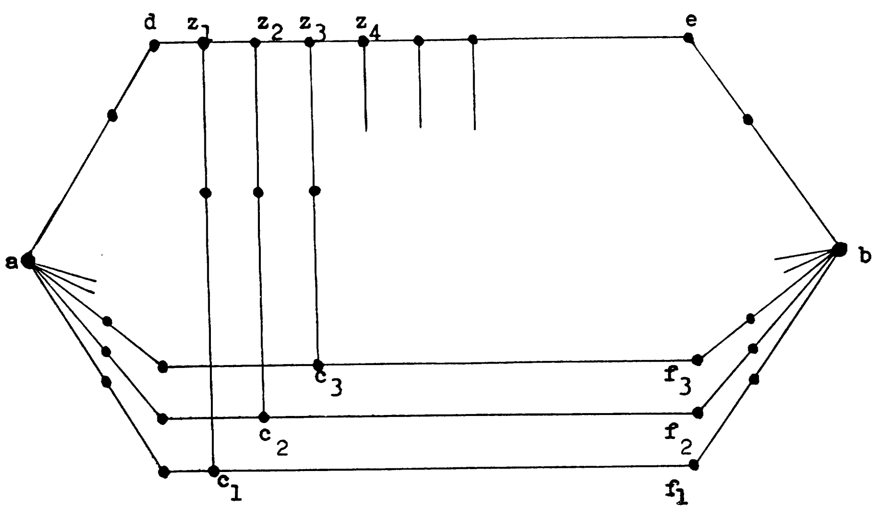
Proof. — Let L be a regularly full logic and let us suppose that $a \nleq b$, $a, b \in \mathrm{L}$. If $a \leftrightarrow b$ then there exist mutually orthogonal elements a_1, b_1, c such that $a = a_1 \vee c$, $b = b_1 \vee c$. Since $a \nleq b$ then $a_1 \neq 0$ and we have a state $m \in \mathscr{S}(\mathrm{L})$ such that $m(a_1) = 1$. It follows that $m(a) = 1$ and $m(b) = 0$.

If $a \nleftrightarrow b$ then $a \nleftrightarrow b'$ and we have a state $m \in \mathscr{S}(\mathrm{L})$ such that $m(a) = 1$, $m(b') \geqslant 1/2$. Since $m(b') = 1 - m(b)$, we see that $m(b) < 1/2$ and the proof is finished.

Let us consider the class of regularly full logics. Due to Gleason's theorem (*see* A. Gleason [*2*]), the Hilbert space logic L(H), dim H $\geqslant 3$, is not regularly full. Any quite full σ-algebra is evidently regularly full. More generally, any representable logic (so called σ-class, *see* S. Gudder [*5*]) is regularly full. Even more generally, the large class of logics in which any two non-compatible elements have both « probability 1 » in a state consists of regularly full logics. But there are regularly full logics outside the latter class as the following example shows.

Example. — There is a regularly full logic L with two noncompatible elements $a, b \in L$ such that $m(a) = 1 = m(b)$ does not hold for any $m \in \mathcal{S}(L)$.

We determine the example by a Greechie diagram (*see* the figure below). We allow ourselves to assume that the reader is familiar with the interpretation of the diagram (*see* R. Greechie [*3*], F. W. Shultz [*6*], etc.). Let us only recall that the « points » of the diagram are the atoms of the logic and those « points » belonging to an « abscissa » are atoms of a certain Boolean subalgebra. The logic is thus built up of « Boolean blocks » by « identifying » those atoms which are common to two blocks. A state on L is then such an evaluation of the atoms that any sum over all values of the atoms of a Boolean block is 1.



We first claim that the logic L determined by the above Greechie diagram is regularly full. One sees easily that if we take two noncompatible atoms $p, q \in L$, $(p, q) \neq (a, b)$ then there exists a state $m \in \mathcal{S}(L)$ such that $m(p) = 1 = m(q)$. Now let we are given ε, $0 < \varepsilon < 1$. One can check that there is a state $m \in \mathcal{S}(L)$ such that $m(a) = 1$, $m(b) = 1 - \varepsilon$. Indeed, we put $m(e) = 0$ and we further set $m(f_i) = \varepsilon$, $m(c_i) = 1 - \varepsilon$ for any $i = 1, 2, 3, \ldots$.

It suffices now to evaluate the atoms z_i, $i = 1, 2, 3, \ldots$ so that $\sum_{i=1}^{\infty} m(z_i) = 1$.

(We can do it even in such a manner that only finately many z_i have a nonzero evaluation.)

On the other hand, if $m(a) = 1 = m(b)$ for a state $m \in \mathcal{S}(L)$ then

$$m(d) = m(e) = 0 \qquad \text{and} \qquad m(c_i) = 1$$

for any $i = 1, 2, 3, \ldots$. Hence $m(z_i) = 0$ for any $i = 1, 2, 3, \ldots$ and this implies that $m(d) + m(e) + \sum_{i=1}^{\infty} m(z_i) = 0$ which is absurd.

3. OBSERVABLES. THE UNIQUENESS PROBLEM

We shall show now that the problem of the uniqueness of observables has a positive answer for regularly full logics. Let us denote by B(R) the σ-algebra of Borel subsets of the real line R and recall the notion of observable (*see* V. S. Varadarajan [7], S. Gudder [5], etc.).

DEFINITION 2.1. — An observable on a logic L is a mapping $x : \mathrm{B(R)} \to \mathrm{L}$ such that

i) $x(\mathrm{R}) = 1$,
ii) $x(\mathrm{R} - \mathrm{A}) = x(\mathrm{A})'$ for any $\mathrm{A} \in \mathrm{B(R)}$,
iii) $x\left(\bigcup_{i=1}^{\infty} \mathrm{A}_i\right) = \bigvee_{i=1}^{\infty} x(\mathrm{A}_i)$ for any collection $\{ \mathrm{A}_i \mid \mathrm{A}_i \in \mathrm{B(R)},\ i \in \mathrm{N} \}$.

An observable is called bounded if there is a bounded set A, $\mathrm{A} \in \mathrm{B(R)}$ such that $x(\mathrm{A}) = 1$.

Let L be a logic and let x be a bounded observable on L. Take a state $m \in \mathscr{S}(\mathrm{L})$. Then $m_x = m \circ x$ is a probability measure on B(R) and the integral $m(x) = \int \lambda . m_x(d\lambda)$ exists. Suppose now that L is quite full and x, y are bounded observables on L. The question is: If the equality $m(x) = m(y)$ is fulfilled for all $m \in \mathscr{S}(\mathrm{L})$, does it imply that $x = y$? We show that it indeed does provided L is regularly full. Let us state first a lemma.

LEMMA 2.2. — Let L be a quite full logic and let x, y be two observables. If $m(x) = m(y)$ for any $m \in \mathscr{S}(\mathrm{L})$ and if $x(-\infty, r) \leftrightarrow y\langle r, +\infty)$ for any $r \in \mathrm{R}$, then $x = y$.

Proof. — We shall show that $x(-\infty, r) = y(-\infty, r)$ for any $r \in \mathrm{R}$. This will suffice since if two observables agree on all generators of B(R), they have to agree on the entire B(R).

Put $c = x(-\infty, r) \wedge y\langle r, +\infty)$. If $c \neq 0$ then there is a state $m \in \mathscr{S}(\mathrm{L})$ such that $m(c) = 1$. Since $m[x(-\infty, r)] = 1$ and m is countably additive, there must exist an s, $s < r$ such that $m[x(-\infty, s\rangle] > 0$. Therefore $m(x) \leqslant r . m[x(s, r)] + s . m[x(-\infty, s\rangle] < r . m[x(s, r)] + r . m[x(-\infty, s\rangle] = r$.

On the other hand, $m(y) \geqslant r$ which is absurd. It follows that $c = 0$ and therefore $x(-\infty, r) \leqslant y\langle r, +\infty)' = y(-\infty, r)$. The inequality

$$y(-\infty, r) \geqslant x(-\infty, r)$$

derives dually. The proof is finished.

THEOREM 2.3. — Let L be a regularly full logic and let x, y be bounded observables. If $m(x) = m(y)$ for any $m \in \mathscr{S}(\mathrm{L})$ then $x = y$.

Proof. — Suppose that $x \neq y$. According to Lemma 2.2, there exists an $r \in \mathrm{R}$ such that $x(-\infty, r) \not\leftrightarrow y\langle r, +\infty)$. Since

$$\bigvee_{n=1}^{\infty} x(-\infty, r - 1/n) = x(-\infty, r).$$

then there exists a number $n \in \mathrm{N}$ such that $x(-\infty, r-1/n) \not\leftrightarrow y\langle r, +\infty)$. Choose real numbers K, ε such that $y\langle -\mathrm{K}, \mathrm{K}\rangle = 1$ and $\varepsilon(r + \mathrm{K}) < 1/n$. Take a state $m \in \mathscr{S}(\mathrm{L})$ satisfying $m[x(-\infty, r-1/n)] = 1, m[y\langle r, +\infty)] \geqslant 1-\varepsilon$. Then $m(x) < r - 1/n$ and $m(y) \geqslant (1-\varepsilon)r - \varepsilon\mathrm{K} = r - \varepsilon(r + \mathrm{K})$. We see that $m(x) < m(y)$ which is a contradiction. This completes the proof.

Our result extends the area of logics where the uniqueness problem has a positive answer and thus supports the conjecture that there is a positive answer in general. It seems to be desirable to find a common proof of our result and the proof of the « uniqueness theorem » for Hilbert logic. We are not able to do so for the time being.

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