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Some remarks on conservative symmetric-hyperbolic systems governing relativistic theories

by

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SUMMARY. — It is shown that for the covariant quasi-linear hyperbolic systems of conservation laws, endowed with a supplementary balance equation, which exhibit a special structure, some results obtained in a previous paper can be proved in a simpler way by choosing a field dependent congruence as time direction instead of a field independent fourvector. In the case of fluid dynamics it is shown that the suitable congruence is represented by the fluid fourvelocity.

RÉSUMÉ. — On montre que pour des systèmes hyperboliques quasi linéaires covariants de lois de conservation présentant une structure particulière, certains résultats obtenus dans un article précédent peuvent être démontrés de façon plus simple en choisissant comme direction temporelle une congruence dépendant des champs au lieu d'un quadrivecteur indépendant des champs. Dans le cas de la dynamique des fluides, on montre que la congruence adéquate est représentée par la quadrivitesse du fluide.

1. EXTENDED DEFINITION OF CONVEX COVARIANT DENSITY SYSTEM

Let us consider a quasi-linear system of N covariant equations:

$$A^{\alpha}(U)U_{\alpha} = f(U), \qquad (\alpha = 0, i;\ i = 1, 2, 3) \tag{1.1}$$

in which A^α are $N \times N$ matrices and f an N-dimensional vector, both functions of the field $\mathbf{U} \in \Omega \subseteq R^N$, $\mathbf{U}$ being a function of the generalized coordinates x^α characterizing a generic point x belonging to a fourdimensional Riemannian manifold V^4, the metric tensor of which g has signature $(+, -, -, -)$; $\mathbf{U}_\alpha$ denotes an N-vector the components of which are the covariant derivatives of the tensor components of $\mathbf{U}$. We recall the following definitions [*1*] [*2*] (for general references *see* [*1*]):

DEFINITION 1. — *Hyperbolic system.*

Let t_α be a time-like congruence: we shall say that a quasi-linear system (1.1) is hyperbolic in the time direction t_α iff:

a)
$$\det (t_\alpha A^\alpha) \neq 0 \tag{1.2}$$

b) for any space-like congruence n_α the eigenvalue problem:

$$(n_\alpha - \lambda t_\alpha)A^\alpha \boldsymbol{d} = 0 \tag{1.3}$$

has only real eigenvalues and N linearly independent eigenvectors.

DEFINITION 2. — *Symmetric-hyperbolic system.*

A hyperbolic system (1.1) is called symmetric-hyperbolic [*3*] [*4*] iff:

a) the coefficient matrices are symmetric, i. e.:

$$(A^\alpha)^T = A^\alpha \tag{1.4}$$

the superscript ^{T} denoting transposition;

b) the time derivative coefficient matrix:

$$A = A^\alpha t_\alpha \tag{1.5}$$

is positive definite.

DEFINITION 3. — *Convex covariant density system.*

A conservative system, i. e. a system (1.1) for which the coefficient matrices have the form $A^\alpha = \boldsymbol{\nabla} \mathbf{F}^\alpha$, $(\boldsymbol{\nabla} \equiv \partial/\partial \mathbf{U})$, possessing a supplementary conservation law:

$$\nabla_\alpha h^\alpha = g \tag{1.6}$$

with h^α, g functions of the field $\mathbf{U}$, will be called a convex covariant density system if the following four conditions are fulfilled:

i) There exists a unit time-like congruence t^α oriented towards the future:

$$t^\alpha t_\alpha = 1, \qquad t^0 > 0 \tag{1.7}$$

such that the vector:

$$\mathbf{U} = \mathbf{F}^\alpha t_\alpha \tag{1.8}$$

is a set of N independent variables and consequently can be chosen as a field;

ii) on introducing the fourvector:

$$h'^\alpha = \mathbf{U}' . \mathbf{F}^\alpha - h^\alpha \qquad (\textit{fourvector generating function}) \tag{1.9}$$

where $\mathbf{U}'$ (main field) is the set of N multipliers which combine eqs (1.1) into the scalar supplementary law (1.6), it results:

$$h'^{\alpha}\nabla t_{\alpha} = 0 \tag{1.10}$$

iii) if $\nabla t_{\alpha} \neq 0$ the quantity:

$$h' = h'^{\alpha} t_{\alpha} \tag{1.11}$$

must be positive;

iv) the covariant density:

$$h = h^{\alpha} t_{\alpha} \tag{1.12}$$

is a convex function of the field $\mathbf{U}$ defined by (1.8) on a convex domain $D \subseteq R^N$.

We point out that the present definition reduces to the one given in [*1*] when t_{α} is independent of the field $\mathbf{U}$, while it extends the previous definition when t_{α} is a function of $\mathbf{U}$. In the latter case condition *ii*) becomes equivalent to say that:

$$h'^{\alpha} = h' t^{\alpha} \tag{1.13}$$

thanks to (1.7) which imply through differentiation:

$$t^{\alpha}\nabla t_{\alpha} = 0 \tag{1.14}$$

It must be emphasized that the generality gained with the possibility of choosing a field dependent time direction t_{α} is payed by the very strong new conditions *iii*) and (1.13) which are not necessary if the time direction is a field independent congruence. But, as we shall see dealing with the relativistic fluid dynamics, when the time direction depends on the field the proof of convexity is easier.

In ref. [*1*] some results have been shown in the case $t_{\alpha} = \xi_{\alpha}$ with ξ_{α} independent of the field. Here we shall establish the same results for the case when t_{α} depends on the field and analyze the system of relativistic fluid dynamics.

2. SYMMETRIZATION OF THE SYSTEM

STATEMENT 1. — *A convex covariant density system is symmetric-hyperbolic and conservative respect to the main field* $\mathbf{U}'$.

Proof. — Compatibility between the system (1.1) and the supplementary law (1.6) is ensured by the existence of a vector of multipliers $\mathbf{U}'$, which is invariant respect to the choice of the field $\mathbf{U}$, such that:

$$\mathbf{U}'.\nabla\mathbf{F}^{\alpha} = \nabla h^{\alpha} \tag{2.1}$$

$$\mathbf{U}'.f = g \tag{2.2}$$

But identity (2.1) is equivalent through (1.9) to:

$$\mathbf{F}^\alpha = \nabla' h^\alpha, \qquad (\nabla' = \partial/\partial \mathbf{U}') \tag{2.3}$$

Then the conservative system (1.1) can be written in the form:

$$\mathrm{A}'^\alpha \mathbf{U}'_\alpha = f \tag{2.4}$$

with:

$$\mathrm{A}'^\alpha = \nabla'\nabla' h^\alpha \tag{2.5}$$

which is manifestly symmetric thanks to Schwarz theorem. Moreover the matrix A′ defined as:

$$\mathrm{A}' = t_\alpha \mathrm{A}'^\alpha \tag{2.6}$$

is positive definite thanks to convexity condition *iv*), (1.13) and (1.14). In fact the quadratic form:

$$\delta\mathbf{U}'.\mathrm{A}'\delta\mathbf{U}' = \delta\mathbf{U}'.(t_\alpha \nabla'\nabla' h'^\alpha)\delta\mathbf{U}'$$

through (1.13) becomes:

$$\delta\mathbf{U}'.\mathrm{A}'\delta\mathbf{U}' = \delta\mathbf{U}'.\nabla'\nabla' h'\delta\mathbf{U}' + h' t_\alpha \delta\mathbf{U}'.\nabla'\nabla' t^\alpha \delta\mathbf{U}' \tag{2.7}$$

On differentianting once and twice (1.7) we have:

$$t_\alpha \delta t^\alpha = 0, \qquad (\delta t^\alpha = \nabla' t^\alpha . \delta\mathbf{U}') \tag{2.8}$$

$$t_\alpha \delta\mathbf{U}'.\nabla'\nabla' t^\alpha \delta\mathbf{U}' + \delta t_\alpha \delta t^\alpha = 0 \tag{2.9}$$

which imply into (2.7):

$$\delta\mathbf{U}'.\mathrm{A}'\delta\mathbf{U}' = \delta\mathbf{U}'.\nabla'\nabla' h'\delta\mathbf{U}' - h'\delta t_\alpha \delta t^\alpha \tag{2.10}$$

which is manifestly positive, thanks to: *a*) convexity of $h(\mathbf{U})$ which is equivalent to the convexity of the Legendre conjugate $h'(\mathbf{U}')$; *b*) positivity of h'; *c*) eq. (2.8) which says that, t_α being time-like, δt_α must be space-like (or vanishing). We must emphasize that a theorem on global invertibility quoted in [*1*], ensures through the convexity of $h(\mathbf{U})$ that the mapping: $\mathbf{U} \leftrightarrow \mathbf{U}'$ is globally univalent on D and it results:

$$\mathbf{U}' = \nabla h, \qquad \mathbf{U} = \nabla' h' \tag{2.11}$$

The previous results (symmetry and positivity) imply that the system (2.4) is conservative and symmetric-hyperbolic in the time direction t_α.

3. INCREASING « ENTROPY » ACROSS THE SHCOKS

Let Γ be a two dimensional surface of V^4 of Cartesian equation:

$$\varphi(x^\alpha) = 0$$

representing a non characteristic shock wave-front. Then on Γ the Rankine-Hugoniot matching conditions hold [5]:

$$\varphi_\alpha[\mathbf{F}^\alpha] = 0, \qquad (\varphi_\alpha = \partial_\alpha \varphi) \tag{3.1}$$

where [ ] denotes the jump. Let us introduce the « generalized entropy »:

$$\eta = [h^\alpha \varphi_\alpha] \tag{3.2}$$

and the scalar parameter:

$$\sigma_* = - t_*^\alpha \varphi_\alpha \tag{3.3}$$

the meaning of notations being the same as in [1].

STATEMENT 2. — *If $h(\mathbf{U})$ is a convex function then η is an increasing function of σ_*, i. e. $\partial\eta/\partial\sigma_* > 0$.*

Proof. — Let us differentiate eq. (3.2) respect to φ_β:

$$\partial\eta/\partial\varphi_\beta = h^\beta - h_*^\beta + \varphi_\alpha \nabla h^\alpha \varphi_\beta . \partial\mathbf{U}/\partial\varphi_\beta \tag{3.4}$$

Now on differentiating respect to φ_β the Rankine-Hugoniot eqs. (3.1) and taking the dot product with ∇h we have:

$$\nabla h . (\mathbf{F}^\beta - \mathbf{F}_*^\beta) + \nabla h . \nabla \mathbf{F}^\alpha \varphi_\alpha \partial \mathbf{U}/\partial\varphi_\beta = 0 \tag{3.5}$$

which, taking account of the compatibility conditions (2.1), becomes:

$$\nabla h . (\mathbf{F}^\beta - \mathbf{F}_*^\beta) + \varphi_\alpha \nabla h^\alpha . \partial\mathbf{U}/\partial\varphi_\beta = 0 \tag{3.6}$$

Eq. (3.6) introduced into (3.4) yields:

$$\partial\eta/\partial\varphi_\beta = h^\beta - h_*^\beta - \nabla h . (\mathbf{F}^\beta - \mathbf{F}_*^\beta) \tag{3.7}$$

By contracting (3.7) with t_β^* and taking account of (1.9), (1.14) and (2.11) we reach:

$$t_\beta^* \partial\eta/\partial\varphi_\beta = - w - (t^\beta t_\beta^* - 1) h' \tag{3.8}$$

where:

$$w = h_* - h + \nabla h . (\mathbf{U} - \mathbf{U}_*) \tag{3.9}$$

Convexity of $h(\mathbf{U})$ ensures $w > 0$, $\forall \mathbf{U} \neq \mathbf{U}_*$; $\mathbf{U}, \mathbf{U}_* \in \mathrm{D}$. Moreover, since for any couple of unit time-like vectors oriented towards the future it results:

$$t_\beta t_*^\beta > 1 \tag{3.10}$$

and h' being positive for condition *iii*), it follows:

$$t_\beta^* \partial\eta/\partial\varphi_\beta < 0 \tag{3.11}$$

On evaluating the scalar (3.11) in the frame in which $t_*^0 = 1$, $t_*^i = 0$ locally, we have through (3.3):

$$\partial\eta/\partial\sigma_* > 0 \tag{3.12}$$

and, $\partial\eta/\partial\sigma_*$ being a scalar, its signature is independent of the frame. Analysis of k-shocks can be carried on in the same way as in [1].

4. η AS GENERATING FUNCTION OF THE SHOCK

STATEMENT 3. — *When the function η is known in terms of* $\mathbf{U}_*$ *and* φ_α, *then it results:*

$$\nabla_* \eta = [\mathbf{U}'] \cdot \nabla_* \mathbf{F}^\alpha_* \phi_\alpha \qquad (\text{on } \Gamma) \tag{4.1}$$

The proof is the same as in [*1*] since it does not involve the congruence t_α.

5. RELATIVISTIC BOUND OF THE SHOCK SPEED

STATEMENT 4. — *If it is assumed that the characteristic velocities* $\lambda^{(i)}$, $i = 1, 2, \ldots, N$, *eigenvalues of the problem:*

$$(n_\alpha - \lambda^{(i)} t_\alpha) A^\alpha \boldsymbol{d}^{(i)} = 0 \tag{5.1}$$

do not exceed the speed of light, then it follows that also the shock speeds are smaller then the velocity of light.

Proof. — Following [*1*] we must look for the intervals where the Jacobian matrix respect to $\mathbf{U}'$ of the l.h.s. of the Rankine-Hugoniot equations, written in the explicit form:

$$\mathbf{F}^\alpha \varphi_\alpha = \mathbf{F}^\alpha_* \varphi_\alpha \tag{5.2}$$

is positive or negative definite. On introducing the scalar shock speed:

$$\sigma = - t^\alpha \varphi_\alpha \tag{5.3}$$

and the space-like normal to the shock-front:

$$n_\alpha = \varphi_\alpha + \sigma t_\alpha \tag{5.4}$$

we have:

$$A' \varphi_\alpha = A'^\alpha (n_\alpha - \sigma t_\alpha)$$

Taking account that:

$$A'^\alpha = A^\alpha H' \tag{5.5}$$

with:

$$H' = \nabla' \nabla' h' \tag{5.6}$$

it follows from (5.1) and (5.5):

$$\det \{ A'^\alpha (n_\alpha - \lambda t_\alpha) \} = \det \{ A^\alpha (n_\alpha - \lambda t_\alpha) \} \det (H') = 0$$

Then λ are also the eigenvalues of $A'^\alpha n_\alpha$ respect to the matrix $A'^\alpha t_\alpha = A'$, which is positive definite; therefore, for a well known property of linear algebra, it follows that the Jacobian matrix $A'^\alpha \varphi_\alpha$ will be definite (negative or positive) if:

$$\sigma > \sup_{U' \in D'} \max_k \{ \lambda^{(k)} \} = M$$

$$\sigma < \inf_{U' \in D'} \min_k \{ \lambda^{(k)} \} = m$$

thanks to the fact that $A' = A'^{\alpha} t_{\alpha}$ is positive definite. We conclude that shocks take place only if:

$$m \leq \sigma \leq M$$

and if the characteristic velocities are supposed not to exceed the speed of light, also σ will behave like that.

6. RELATIVISTIC HYDRODYNAMICS

The system of equations governing relativistic hydrodynamics is a conservative systems of type (1.1) with [*1*]:

$$F^{\alpha} = \begin{vmatrix} T^{\alpha\beta} \\ ru^{\alpha} \end{vmatrix}, \qquad f = 0 \tag{6.1}$$

where:

$$T^{\alpha\beta} = rfu^{\alpha}u^{\beta} - pg^{\alpha\beta} \tag{6.2}$$

and the supplementary law (1.6) characterized by:

$$h^{\alpha} = - rSu^{\alpha}, \qquad g = 0 \tag{6.3}$$

In [*1*] it has been shown that the system has convex covariant density $h = h^{\alpha}\xi_{\alpha}$, ξ_{α} being a field independent congruence. In the Appendix 1, it has been pointed out that also the quantity $h^{\alpha}u_{\alpha} = - rS$ is convex respect to the field $\mathbf{U} = \mathbf{F}^{\alpha}u_{\alpha}$ but this circumstance did not seem to be related to symmetrization or other properties of the system. In the present paper we showed that the same properties established in [*1*] for $t^{\alpha} = \xi^{\alpha}$ hold also if we choose t^{α}, field dependent congruence. It is easily proved that the required conditions hold for the fluid. In fact, the main field $\mathbf{U}'$ is the same as in [*1*], being independent of the choice of $\mathbf{U}$:

$$\mathbf{U}' \equiv \left(- u_{\beta}/\Theta, \frac{G + 1}{\Theta} \right) \tag{6.4}$$

and the fourvector generating function results:

$$h'^{\alpha} = \frac{p}{\Theta} u^{\alpha} \tag{6.5}$$

and:

$$h' = \frac{p}{\Theta} > 0 \tag{6.6}$$

We must emphasize that even if the proof of convexity given in Appendix 1 of ref. [*1*] does not require the condition:

$$(\partial p/\partial \rho)_S < 1 \tag{6.7}$$

as it is required when a field independent congruence ξ_{α} is employed as time direction, the same condition (6.7) is necessary to ensure that the sonic waves travel slower than light, so that also the shock speed is bounded by the speed of light.

We point out also that the conditions *iii)* and *iv)* when the time congruence is field dependent become very strong conditions: in fact even if they hold for a perfect fluid, they are not fulfilled, e. g. for a charged fluid, even if in the latter case it is possible to prove the existence of a convex covariant density with the aid of a field independent time congruence [*6*].

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