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L. MANGIAROTTI
M. MODUGNO

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Section A :
Physique théorique.

Some results on the calculus of variations on jet spaces (*)

by

L. MANGIAROTTI (**) and M. MODUGNO (***)

ABSTRACT. — The basic object is a fibered manifold $p: E \to M$ and the framework is that of jet spaces. Given a Lagrangian form Λ on JE, we work with the space $\{ \Lambda \}$ of variational forms associated to Λ. It is this space which is important in the calculus of variations. We study a new operator $\mathscr{E}_\Lambda$ (defined only on Ker $\vartheta \subset$ TJE where ϑ is the fundamental 1-form) canonically associated to $\{ \Lambda \}$. This operator is well suited for studying critical sections and functorial properties. The so called Euler-Lagrange operator E_Λ appears as an extension of $\mathscr{E}_\Lambda$. Variational symmetries are introduced as morphisms of a category whose objects are the variational forms $\{ \Lambda \}$. The uniqueness of the Poincaré-Cartan form Θ_Λ is proved under certain circumstances. Various interesting relations between Λ, E_Λ and Θ_Λ are investigated.

RÉSUMÉ. — L'objet fondamental est une variété avec un espace fibré $p : E \to M$ et le cadre est celui des espaces de jets. Étant donnée une forme Lagrangienne Λ sur JE, on considère l'espace $\{ \Lambda \}$ des formes variationnelles associées à Λ, qui est l'espace important pour le calcul des variations. On étudie un nouvel opérateur $\mathscr{E}_\Lambda$ (défini seulement sur Ker $\theta \subset$ TJE, où θ est la 1-forme fondamentale), canoniquement associé à $\{ \Lambda \}$. Cet opérateur est bien adapté à l'étude des sections critiques et des propriétés fonctorielles. L'opérateur d'Euler-Lagrange E_Λ apparaît comme extension de $\mathscr{E}_\Lambda$. Les symétries variationnelles sont introduites

(*) Work performed under the auspices of C. N. R. (Gruppo Nazionale per la Fisica Matematica).

(**) Permanent address: Istituto di Matematica, Università, Camerino (MC).
(***) Permanent address: Istituto di Matematica Applicata, Università, Firenze.

comme morphismes d'une catégorie dont les objets sont les formes variationnelles de $\{ \Lambda \}$. On montre l'unicité de la forme Θ_Λ de Poincaré-Cartan dans certaines circonstances. On étudie quelques relations intéressantes entre Λ, E_Λ et Θ_Λ.

INTRODUCTION

In recent years much work has been done on the calculus of variations on jet spaces [*2*] [*3*] [*4*] [*6*] [*7*] [*9*] [*10*] [*11*] and [*12*]. Following this stream, the present paper emphasizes both the geometrical basis of the theory as well as its functorial nature. Moreover, it contains novelties in the treatment as well as new results.

The basic geometric object is a fibered manifold $p : E \to M$ and the basic spaces are the first two jet spaces, i. e. JE and J^2E. In Section 1 we recall the two main facts about the geometry of jet spaces, namely the affine structures and the fundamental 1-form ϑ. Then the subject of infinitesimal contact transformations is developed in a new way. These transformations are important because a « variational problem » is « infinitesimally functorial » with respect to them. A deeper analysis at all orders of jet spaces will appear in a related paper [*8*].

In Section 2 we introduce Lagrangian forms Λ on JE, i. e. horizontal forms with respect to the canonical projection $JE \to M$ (in physical terms, these are Lagrangian densities). Affine Lagrangian forms are studied in detail. Then we define the space $\{ \Lambda \}$ of variational forms associated with a Lagrangian form Λ (already used in [*10*] in the context of affine bundles). It is $\{ \Lambda \}$ which is important in the Calculus of Variations. Various properties of $\{ \Lambda \}$ in relation with automorphisms $\varphi : E \to E$ (for brevity we consider only automorphisms of E though many properties hold in a more general context) as well as infinitesimal contact transformations are demonstrated. Moreover, the uniqueness of the Poincaré-Cartan form is proved under certain circumstances.

A main result in Section 3 is the introduction of a new operator $\mathscr{E}_\Lambda$ (defined only on $\mathrm{Ker}\, \vartheta \subset TJE$) canonically associated to the set $\{ \Lambda \}$. This operator is well suited for studying critical sections and functorial properties. The classical Euler-Lagrange operator E_Λ is an extension of $\mathscr{E}_\Lambda$. Various interesting relations between Λ, E_Λ and Θ_Λ are investigated. Variational symmetries are introduced as morphisms of a category whose objects are the variational forms $\{ \Lambda \}$. However, this is only a first insight into such an intricate question [*2*] [*4*] [*6*] and [*11*]. The analysis of the subject, also with respect to Noether's theorem, will be pursued in a future work.

In Section 4 we treat of critical sections. The functorial nature of a variational problem is exhibited in a very direct way. A new characterization of critical sections in terms of $\mathscr{E}_\Lambda$ is given.

1. NOTATION AND PRELIMINARY RESULTS

a) All manifolds and maps will be C^∞. The notation used is that of modern differential geometry. The basic object is a fibered manifold $p : E \to M$ (i. e. p is a surjective submersion), $\dim M = m, \dim E = m + l$. Fibered charts are denoted by (x^α, y^i), $1 \leq \alpha \leq m$, $1 \leq i \leq l$. Induced fibered charts on the 1-jet space JE are denoted by $(x^\alpha, y^i, y^i_\alpha)$. The canonical projection $JE \to E$ is denoted by p_E. Then $p_M = p \circ p_E$ is the canonical projection $JE \to M$. If s is a (local) section of E, then $js : M \to JE$ is the 1-jet prolongation of s. If $\varphi : E \to E$ is an automorphism of E (over the diffeomorphism $\varphi_M : M \to M$), then $J\varphi : JE \to JE$ is the 1-jet prolongation of φ. Recall that $J\varphi$ is characterized by $J\varphi \circ js = j(\varphi_* s) \circ \varphi_M$ for any section s of E, $\varphi_* s = \varphi \circ s \circ \varphi_M^{-1}$.

There are two basic properties concerning JE. First, JE is an affine bundle over E with vector bundle the tensor product bundle $T^*M \otimes_E VE$ [3]. Here T and V denote the tangent and vertical functors, respectively ($VE = \operatorname{Ker} Tp \subset TE$). As a rule, obvious pull-backs will be omitted.

Second, there is a (first order) canonical inclusion over $JE \to E$, namely

$$c : JE \times_M TM \to TE \tag{1.1}$$

characterized by $c \circ (js, u) = Tjs \circ u$ for any section $s : M \to E$ and any vector field $u : M \to TM$. It induces the canonical projection

$$\vartheta : JE \times_E TE \to VE, \tag{1.2}$$

the so called structure 1-form (*see* [2] and [3]), whose local expression is

$$(x^\alpha, y^i, y^i_\alpha ; \dot{y}^i) \circ \vartheta = (x^\alpha, y^i, y^i_\alpha ; \dot{y}^i - y^i_\alpha \dot{x}^\alpha) . \tag{1.3}$$

By means of the tangent map $Tp_E : TJE \to TE$, ϑ can also be considered as a linear morphism $\vartheta : TJE \to VE$ over $p_E : JE \to E$. Hence ϑ is a 1-form on JE valued over VE whose local expression is

$$\vartheta^i = dy^i - y^i_\alpha dx^\alpha . \tag{1.4}$$

In the sequel, the subbundle $\Delta = \operatorname{Ker} \vartheta \subset TJE$ will be used. A local basis of Δ is given by $(\partial_\alpha + y^i_\alpha \partial_i, \partial^\alpha_i)$. Here $(\partial_\alpha, \partial_i, \partial^\alpha_i)$ are the local fields associated to the fibered charts $(x^\alpha, y^i, y^i_\alpha)$. Let us note that ϑ (and hence Δ) is invariant with respect the automorphisms $\varphi : \mathrm{E} \to \mathrm{E}$, i. e.

$$\vartheta \circ \mathrm{TJ}\varphi = \mathrm{V}\varphi \circ \vartheta . \tag{1.5}$$

In the usual pull-back notation, (1.5) is written as $\varphi^*\vartheta = \vartheta$.

All that we have said holds for higher order jet spaces [*3*] [*4*] and [*8*]. Nevertheless we need only to consider the 2-jet space $\mathrm{J^2E}$ (because the Euler-Lagrange equation is of the second order). Induced fibered charts on $\mathrm{J^2E}$ are denoted by $(x^\alpha, y^i, y^i_\alpha, y^i_{\alpha\beta})$. Note that there is a canonical injection $\mathrm{J^2E} \subset \mathrm{J(JE)}$.

The 2-jet space $\mathrm{J^2E}$ is an affine bundle over JE with vector bundle $(\bigvee_2 \mathrm{T^*M}) \otimes_{\mathrm{JE}} \mathrm{VE}$, where $\bigvee_2$ denotes the second order symmetric tensor product.

From the second order canonical inclusion

$$c : \mathrm{J^2E} \times_{\mathrm{M}} \mathrm{TM} \to \Delta \subset \mathrm{TJE}, \tag{1.6}$$

we get the (second order) structure 1-form on $\mathrm{J^2E}$ which can be considered as a linear morphism $\mathrm{TJ^2E} \to \mathrm{VJE}$ over the canonical projection $\mathrm{J^2E} \to \mathrm{E}$ (here $\mathrm{VJE} = \mathrm{Ker\,T}p_{\mathrm{M}} \subset \mathrm{TJE}$). Its local expression is

$$\vartheta^i = dy^i - y^i_\alpha dx^\alpha, \qquad \vartheta^i_\alpha = dy^i_\alpha - y^i_{\alpha\beta}dx^\beta . \tag{1.7}$$

b) There are distinguished vector fields on JE, namely the so called infinitesimal contact transformations (in short : i. c. t.) [*2*] [*4*]. They are important in the calculus of variations, essentially because a « variational problem » is « infinitesimally functorial » with respect to them. We give here a new treatment of these vector fields (compare with [*2*] and [*4*]). A comprehensive analysis of the subject, involving jet spaces of any order, will appear in [*8*].

DEFINITION. — A vector field u on JE is an i. c. t. (of first order) iff $\mathrm{L}_u\Delta \subset \Delta$ (here L_u is the Lie derivative with respect to u).

Let $u = u^\alpha\partial_\alpha + u^i\partial_i + u^i_\alpha\partial^\alpha_i$ be the local expression of u. Then it is easily proved that u is an i. c. t. iff

$$u^i_\alpha = \partial_\alpha u^i + y^j_\alpha\partial_j u^i - y^i_\beta(\partial_\alpha u^\beta + y^j_\alpha\partial_j u^\beta), \qquad \partial^j_\alpha u^i - y^i_\beta\partial^j_\alpha u^\beta = 0 . \tag{1.8}$$

There is an interesting way of characterizing the i. c. t. which is based on the existence of a canonical projection $r : \mathrm{JTE} \to \mathrm{TJE}$ [*8*]. Let us remark preliminarily that JTE is an affine bundle over the fibered product $\mathrm{JE} \times_{\mathrm{M}} \mathrm{JTM}$ with vector bundle (the pull-back over $\mathrm{JE} \times_{\mathrm{M}} \mathrm{JTM}$ of) VJE. Also TJE is an affine bundle over $\mathrm{JE} \times_{\mathrm{M}} \mathrm{TM}$ with vector bundle (the pull-back over $\mathrm{JE} \times_{\mathrm{M}} \mathrm{TM}$ of) VJE.

PROPOSITION. — There is an unique affine morphism r: JTE $\to$ TJE over the canonical projection JE $\underset{M}{\times}$ JTM $\to$ JE $\underset{M}{\times}$ TM such that

i) $r \circ \mathrm{JT}s = \mathrm{T}js$ for any section $s : \mathrm{M} \to \mathrm{E}$,
ii) its fiber derivative Dr restricts to the identity over the fibers.
The local expression of r is

$$(x^\alpha, y^i, y^i_\alpha, \dot{x}^\alpha, \dot{y}^i, \dot{y}^i_\alpha) \circ r = (x^\alpha, y^i, y^i_\alpha, \dot{x}^\alpha, \dot{y}^i, \dot{y}^i_\alpha - y^i_\beta \dot{x}^\beta_\alpha). \tag{1.9}$$

PROPOSITION. — A vector field u on JE is an i. c. t. iff we have $u = r \circ \mathrm{J}(\mathrm{T}p_{\mathrm{E}} \circ u)$. If u_{E} is a vector field on E, then $u = r \circ \mathrm{J}u_{\mathrm{E}}$ is the unique i. c. t. projectable over u_{E} (1-jet prolongation of u_{E}). Moreover, in the particular case when u_{E} is projectable over M, u is the infinitesimal generator of JF_t, where F_t is the flux of u_{E}.

If $u_{\mathrm{E}} = u^\alpha \partial_\alpha + u^i \partial_i$ is the local expression of u_{E}, then the local expression of the 1-jet prolongation of u_{E} is

$$u = u^\alpha \partial_\alpha + u^i \partial_i + u^i_\alpha \partial^\alpha_i, \tag{1.10}$$

where u^i_α is given by (1.8).

It is obvious how the previous definition works on $\mathrm{J}^2\mathrm{E}$. If

$$u = u^\alpha \partial_\alpha + u^i \partial_i + u^i_\alpha \partial^\alpha_i + u^i_{\alpha\beta} \partial^{\alpha\beta}_i$$

is the local expression of a vector field on $\mathrm{J}^2\mathrm{E}$, it follows from (1.7) that u is a second order i. c. t. iff

$$\begin{cases} u^i_\alpha = \partial_\alpha u^i + y^j_\alpha \partial_j u^i + y^j_{\alpha\beta} \partial^\beta_j u^i - y^i_\beta(\partial_\alpha u^\beta + y^j_\alpha \partial_j u^\beta + y^j_{\alpha\gamma} \partial^\gamma_j u^\beta), \\ u^i_{\alpha\beta} = \partial_\alpha u^i_\beta + y^j_\alpha \partial_j u^i_\beta + y^j_{\alpha\gamma} \partial^\gamma_j u^i_\beta - y^i_{\beta\gamma}(\partial_\alpha u^\gamma + y^j_\alpha \partial_j u^\gamma + y^j_{\alpha\lambda} \partial^\lambda_j u^\gamma), \\ \partial^{\alpha\beta}_j u^i - y^i_\gamma \partial^{\alpha\beta}_j u^\gamma = 0, \qquad \partial^{\alpha\beta}_j u^i_\gamma - y^i_{\gamma\lambda} \partial^{\alpha\beta}_j u^\lambda = 0. \end{cases} \tag{1.11}$$

It follows easily from (1.8) and (1.11) that if u is projectable over a vector field u_{E} on E and also over a vector field on JE, then this projection is the 1-jet prolongation of u_{E}. As in the last proposition, it is clear that (1.11) gives us the local expression of the 2-jet prolongation of a vector field u_{E} on E.

2. VARIATIONAL FORMS

a) Let us recall that p_{M} denotes the canonical projection JE $\to$ M. A p_{M}-horizontal m-form on JE is called a Lagrangian form on JE. An m-form Λ on JE is a Lagrangian form iff its local expression is

$$\Lambda = \mathscr{L}\omega, \qquad \omega = dx^1 \wedge \ldots \wedge dx^m, \tag{2.1}$$

where $\mathscr{L}$ is a local function on JE.

If the manifold M is orientable, then the choice of a volume form on M induces a bijection between Lagrangian forms and functions on JE. In physical terms, Λ is a Lagrangian density while $\mathscr{L}$ is a Lagrangian.

Let us make some remarks about Lagrangian forms of the affine type. Clearly, Λ is an affine map (over E) iff can be written as

$$\mathscr{L} = \pi_i^\alpha y_\alpha^i + \lambda, \tag{2.2}$$

where π_i^α and λ are local functions on E. To go further, we need the fact that the affine bundle $p_E : JE \to E$ has a canonical « linearizing bundle » $L \to E$.

To see this, note that the product bundle $E \times \mathbb{R} \to E$ is a vector subbundle of $T^*M \underset{E}{\otimes} TM$ in a canonical way. Hence L is the vector subbundle of $T^*M \underset{E}{\otimes} TE$ which is projected on $E \times \mathbb{R}$ by $id_{T^*M} \underset{E}{\otimes} Tp$. Moreover, from the canonical injection $JE \hookrightarrow T^*M \underset{E}{\otimes} TE$ (cf. (1.1)) which takes its values in L, we get a canonical affine injection $j : JE \to L$. Then the local expression of the unique $\tilde{\Lambda} \in L^* \underset{E}{\otimes} \overset{m}{\bigwedge} T^*M$ such that $\Lambda = \tilde{\Lambda} \circ j$ is

$$\tilde{\Lambda} = \left(\pi_i^\alpha \partial_\alpha \otimes dy^i + \lambda \partial_\alpha \otimes dx^\alpha\right) \otimes \omega . \tag{2.3}$$

Since there is a canonical injection of

$$L^* \underset{E}{\otimes} \overset{m}{\bigwedge} T^*M \quad \text{into} \quad T^*E \underset{E}{\otimes} \overset{m-1}{\bigwedge} T^*M,$$

from Λ, by using the projection Tp, we get an m-form Λ_E on E whose local expression is

$$\Lambda_E = \pi_i^\alpha dy^i \wedge \omega_\alpha + \lambda\omega, \qquad \omega_\alpha = \partial_\alpha \lrcorner\, \omega , \tag{2.4}$$

where $\lrcorner$ denotes the inner product.

Now let ρ be a p-horizontal $(m - 1)$-form on E and let

$$\rho = \rho^\alpha \omega_\alpha , \tag{2.5}$$

be its local expression (ρ^α are local functions on E). Then by using the functor J we get an affine Lagrangian form (of the so called divergence type: (cf. [*1*] [*5*] and [*6*]), namely $\Lambda = \operatorname{div} \rho$, whose local expression is

$$\mathscr{L} = (\partial_i \rho^\alpha) y_\alpha^i + \partial_\alpha \rho^\alpha . \tag{2.6}$$

In terms of the form Λ_E we have $\Lambda_E = d\rho$. Hence Λ_E is closed. Conversely, if Λ is an affine Lagrangian form such that $d\Lambda_E = 0$, Λ is locally a Lagrangian form of the divergence type (as can easily be seen).

Let Λ be a Lagrangian form on JE. By taking the fiber derivative $D\Lambda$ (with respect to E), followed by an obvious canonical contraction, we get an $(m-1)$-form Ω_Λ on JE valued on V*E, that is

$$\Omega_\Lambda : JE \to V^*E \underset{E}{\otimes} \overset{m-1}{\wedge} T^*M, \tag{2.7}$$

whose local expression is

$$\Omega_\Lambda = \pi_i^\alpha dy^i \otimes \omega_\alpha, \qquad \pi_i^\alpha = \partial_i^\alpha \mathscr{L}. \tag{2.8}$$

The map Ω_Λ is the so called Legendre transformation associated to Λ [*2*]. Note that Λ is affine iff $D\Omega_\Lambda = 0$.

The map Ω_Λ can also be viewed, in a canonical way, as an m-form on JE. For this, it is sufficient to observe that the projection ϑ, as defined in (1.2), allows us to construct the canonical injection

$$V^*E \underset{JE}{\otimes} \overset{m-1}{\wedge} T^*M \hookrightarrow T^*E \underset{JE}{\otimes} \overset{m-1}{\wedge} T^*E. \tag{2.9}$$

It is clear that Ω_Λ is a p_E-horizontal m-form on JE whose local expression is

$$\Omega_\Lambda = \pi_i^\alpha \vartheta^i \wedge \omega_\alpha, \tag{2.10}$$

as follows from (2.8) and (2.9).

Let $\varphi : E \to E$ be an automorphism. Then $(J\varphi)^*\Lambda$ is again a Lagrangian form and it is easily proved that

$$(J\varphi)^*\Omega_\Lambda = \Omega_{(J\varphi)^*\Lambda}. \tag{2.11}$$

Moreover, let u_E be a vector field on E, projectable on M, and let u be the 1-jet prolongation of u_E. Then the infinitesimal version of (2.11) is

$$L_u \Omega_\Lambda = \Omega_{L_u\Lambda}. \tag{2.12}$$

b) Now we consider variations of Lagrangian forms. The basic tool is the subbundle $\Delta \subset TJE$ [*10*].

DEFINITION. — Let Λ be a Lagrangian form on JE. A variation of Λ is a p_E-horizontal m-form Θ on JE such that

i) $\Theta \mid \Delta = \Lambda \mid \Delta,$

ii) $d\Theta \mid \Delta = 0.$

Let $\{\Lambda\}$ be the set of variations of Λ. Clearly $\{\Lambda\}$ is an affine space (over $\mathbb{R}$) with vector space the space $[\overline{\Theta}]$ of the p_E-horizontal m-form $\overline{\Theta}$ on JE such that $\overline{\Theta} \mid \Delta = 0$ and $d\overline{\Theta} \mid \Delta = 0$. Note that $\Lambda \in \{\Lambda\}$ iff $\Omega_\Lambda = 0$. There is a canonical form $\Theta_\Lambda \in \{\Lambda\}$, namely the so called Poincaré-Cartan form

$$\Theta_\Lambda = \Lambda + \Omega_\Lambda. \tag{2.13}$$

The assignment $\Lambda \mapsto \{\Lambda\}$ is functorial, i. e. if $\varphi : E \to E$ is an automorphism, we find that

$$(2.14)\qquad (J\varphi)^*\{\Lambda\} = \{(J\varphi)^*\Lambda\}.$$

In particular, from (2.11) and (2.13) it follows that

$$(2.15)\qquad (J\varphi)^*\Theta_\Lambda = \Theta_{(J\varphi)^*\Lambda}.$$

If u is an i. c. t. on JE, we have

$$(2.16)\qquad L_u\{\Lambda\} \subset \{L_u\Lambda\}.$$

More particularly, if u is the 1-jet prolongation of a vector field u_E on E, projectable on M, from (2.12) and (2.13) we get the infinitesimal version of (2.15), namely

$$(2.17)\qquad L_u\Theta_\Lambda = \Theta_{L_u\Lambda}.$$

Note that, since $\Theta_\Lambda = 0$ iff $\Lambda = 0$ (as is easily seen), from (2.17) it follows that

$$(2.18)\qquad L_u\Theta_\Lambda = 0 \Leftrightarrow L_u\Lambda = 0.$$

c) At least in some particular cases, we can say more about the set of variational forms $\{\Lambda\}$. For example, in Classical Analytical Mechanics one is interested to the case in which $M = \mathbb{R}$, the absolute time. Another case which occurs often in practice is that in which the fibers of E are 1-dimensional. In both these cases $[\overline{\Theta}]$ is trivial.

PROPOSITION. — Let Λ be a Lagrangian form on JE. Then if $\dim M = 1$ or also if $\dim E = m + 1$, the Poincaré-Cartan form is the unique variational form of Λ.

Proof. — Let $\dim M = 1$ and let $\overline{\Theta}$ be a p_E-horizontal 1-form on JE such that $\overline{\Theta} \,|\, \Delta = 0$. Then locally we must have

$$(2.19)\qquad \overline{\Theta} = f_i\vartheta^i,$$

where f_i are local functions on JE. The result follows now from

$$(2.20)\qquad \partial_i^1 \lrcorner\, d\overline{\Theta} = -f_i dx^1 + (\partial_i^1 f_j)\vartheta^j.$$

The case $\dim E = m + 1$ is treated in the same way.

Apart from these two cases, in general $[\overline{\Theta}]$ is not trivial. For example, let $M = \mathbb{R}^2$ and let $E = \mathbb{R}^2 \times \mathbb{R}^2$. Then the 2-form on JE

$$(2.21)\qquad \overline{\Theta} = (y_1^1 y_2^2 - y_2^1 y_1^2)dx^1 \wedge dx^2 + \vartheta^1 \wedge dy^2 + dy^1 \wedge \vartheta^2 - dy^1 \wedge dy^2,$$

belongs to $[\overline{\Theta}]$.

3. VARIATIONAL STRUCTURES

a) The concept of variational form leads us to introduce, in a natural way, m-forms defined on Δ. This is done in the following proposition.

PROPOSITION. — Let Λ be a Lagrangian form on JE. Then there exists a unique m-form $\mathscr{E}_\Lambda$ defined on Δ and valued on V*E such that

$$(3.1) \qquad (js)^*(u_0 \lrcorner\, d\Theta) = - \langle (js)^*\mathscr{E}_\Lambda, \vartheta \circ u_0 \circ js \rangle ,$$

where $\Theta \in \{ \Lambda \}$, $s : M \to E$ is any section and u_0 is any vector field on JE. The local expression of $\mathscr{E}_\Lambda$ is

$$(3.2) \qquad (\mathscr{E}_\Lambda)_i = [d\pi_i^\alpha \wedge \omega_\alpha - (\partial_i \mathscr{L})\omega] \mid \Delta .$$

Proof. — The uniqueness of $\mathscr{E}_\Lambda$ follows easily from (3.1). To prove the existence, we use the well known formula

$$d\Theta(u_0, u_1, \ldots, u_m) = \sum_{i=0}^{m} (-1)^i u_i \cdot \Theta(u_0, \ldots, \hat{u}_i, \ldots, u_m) + \sum_{0 \le i < j \le m} (-1)^{i+j}\Theta([u_i, u_j], u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_m),$$

where $u_1, \ldots, u_m$ are vector fields on JE belonging to Δ. Let us note preliminarily that

$$(3.4) \qquad d\Theta(\partial_\alpha, u_1, \ldots, u_m) = - y_\alpha^i d\Theta(\partial_i, u_1, \ldots, u_m),$$

since $d\Theta \mid \Delta = 0$ and the vector fields $w_\alpha = \partial_\alpha + y_\alpha^i \partial_i \in \Delta$.

By putting $u_0 = \partial_i^\alpha$, $u_\alpha = w_\alpha$ into (3.9) and by using the properties of Θ, it follows that

$$(3.5) \qquad \pi_i^\alpha = (-1)^{\alpha-1}\Theta(\partial_i, w_1, \ldots, \hat{w}_\alpha, \ldots, w_m),$$

because $[\partial_i^\alpha, w_\beta] = \partial_\beta^\alpha \partial_i$.

Let now $s : M \to E$ be a section and let

$$(3.6) \qquad u_\alpha = w_\alpha + (\partial_{\alpha\beta}^2 s^i)\partial_i^\beta \in \Delta, \qquad s^i = y^i \circ s .$$

By putting again these vector fields in (3.9), since $[u_\alpha, u_\beta] = 0$, we get

$$(3.7) \quad (js)^*(u_0 \lrcorner\, d\Theta)(\partial_1, \ldots, \partial_m) = d\Theta(u_0, u_1, \ldots, u_m) \circ js$$
$$= (js)^*\left[u_0 \mathscr{L} + \sum_{\alpha=1}^{m} (-1)^\alpha u_\alpha \cdot \Theta(u_0, w_1, \ldots, \hat{w}_\alpha, \ldots, w_m) + \sum_{\alpha=1}^{m} (-1)^\alpha \Theta([u_0, u_\alpha], w_1, \ldots, \hat{w}_\alpha, \ldots, w_m) \right],$$

where we have used the properties that Θ is p_E-horizontal and that $\Theta \mid \Delta = \Lambda \mid \Delta$. By putting $u_0 = \partial_i$ into (3.7) and by taking into account (3.5), it follows that

$$(3.8)\quad (js)^*(\partial_i \lrcorner\, d\Theta)(\partial_1, \ldots, \partial_m) = -(js)^*[d\pi_i^\alpha \wedge \omega_\alpha - (\partial_i\mathscr{L})\omega](\partial_1, \ldots, \partial_m).$$

Finally, if the local expression of u_0 is $u_0 = u^\alpha\partial_\alpha + u^i\partial_i + u^i_\alpha\partial^\alpha_i$, since $d\Theta \mid \Delta = 0$, we get from (3.4) and (3.8)

$$(3.9)\quad (js)^*(u_0 \lrcorner\, d\Theta) = -(js)^*[d\pi_i^\alpha \wedge \omega_\alpha - (\partial_i\mathscr{L})\omega](u^i - y^i_\alpha u^\alpha) \circ js.$$

Hence the proof is complete.

The functoriality of the assignment $\Lambda \to \mathscr{E}_\Lambda$ can easily be proved. Let $\varphi : \mathrm{E} \to \mathrm{E}$ be an automorphism (over $\varphi_\mathrm{M} : \mathrm{M} \to \mathrm{M}$). Then by taking the pull-back of both sides of (3.1) by means of φ_M and by using (1.5), we have

$$(3.10)\quad (js^*)^*\{[(J\varphi)^*u_0] \lrcorner\, d(\mathrm{J}\varphi)^*\Theta\} = -\langle (js^*)^*\varphi^*\mathscr{E}_\Lambda, \vartheta \circ (\mathrm{J}\varphi)^*u_0 \circ js^* \rangle,$$

where $s^* = \varphi^*s$. By recalling (2.14) and the uniqueness property of $\mathscr{E}_{(\mathrm{J}\varphi)^*\Lambda}$, as stated before, (3.10) implies that

$$(3.11)\qquad \varphi^*\mathscr{E}_\Lambda = \mathscr{E}_{(\mathrm{J}\varphi)^*\Lambda}.$$

b) Let Λ be a Lagrangian form on JE. Let us recall that $\mathscr{E}_\Lambda$ is an m-form defined only on $\Delta \subset \mathrm{TJE}$ and valued in $\mathrm{V^*E}$. Then the second order canonical inclusion (1.6) induces the operator

$$(3.12)\qquad \mathrm{E}_\Lambda : \mathrm{J^2E} \to \mathrm{V^*E} \underset{\mathrm{E}}{\otimes} \overset{m}{\bigwedge} \mathrm{T^*M}.$$

It is characterized by

$$(3.13)\qquad \mathrm{E}_\Lambda \circ j^2 s = (js)^*\mathscr{E}_\Lambda,$$

for any section $s : \mathrm{M} \to \mathrm{E}$. This second order operator is the Euler-Lagrange operator. Its local expression, as follows from (3.2) and (3.13), is

$$(3.14)\qquad \mathrm{E}_\Lambda = [(\partial^\beta_j\pi^\alpha_i)y^j_{\alpha\beta} + (\partial_j\pi^\alpha_i)y^j_\alpha + \partial_\alpha\pi^\alpha_i - \partial_i\mathscr{L}]dy^i \otimes \omega.$$

Note that E_Λ is an affine map over JE.

Following the procedure as in (2.9), it is clear that E_Λ can be viewed as an $(m+1)$-form on $\mathrm{J^2E}$ whose local expression is

$$(3.15)\qquad \mathrm{E}_\Lambda = [(\partial^\beta_j\pi^\alpha_i)y^j_{\alpha\beta} + (\partial_j\pi^\alpha_i)y^j_\alpha + \partial_\alpha\pi^\alpha_i - \partial_i\mathscr{L}]dy^i \wedge \omega.$$

The functoriality of $\Lambda \mapsto \mathrm{E}_\Lambda$ can be seen in this way. If $\varphi : \mathrm{E} \to \mathrm{E}$ is an automorphism, by taking the pull-back of both sides of (3.13) by means of φ^*, we get

$$(3.16)\qquad (\varphi^*\mathrm{E}_\Lambda) \circ j^2 s^* = (js^*)^*\mathscr{E}_{(\mathrm{J}\varphi)^*\Lambda}, \qquad s^* = \varphi^* s,$$

where we have used (3.11). Hence the uniqueness property of $\mathrm{E}_{(\mathrm{J}\varphi)^*\Lambda}$ implies that

$$(3.17)\qquad \varphi^*\mathrm{E}_\Lambda = \mathrm{E}_{(\mathrm{J}\varphi)^*\Lambda}.$$

The following proposition gives the infinitesimal version of (3.17) in terms of i. c. t.

PROPOSITION. — Let u_E be a vector field on E, projectable on M. Then we have

$$L_u E_\Lambda = E_{L_u\Lambda}. \tag{3.18}$$

where with u on the right and the left side of (3.18) we denote the 1 and 2-jet prolongation of u_E, respectively.

Proof. — We use the formula

$$d\Theta_\Lambda = - E_\Lambda + d_V\Omega_\Lambda, \tag{3.19}$$

where d_V is the vertical derivation on jet spaces [*12*]. Formula (3.19) can be proved by a straigtforward calculation. The local expression of $d_V\Omega_\Lambda$ is

$$d_V\Omega_\Lambda = (\partial_j\pi_i^\alpha)\vartheta^j \wedge \vartheta^i \wedge \omega_\alpha + (\partial_j^\beta\pi_i^\alpha)\vartheta_\beta^j \wedge \vartheta^i \wedge \omega_\alpha. \tag{3.20}$$

Since the following commutation relation holds

$$L_u d_V\Omega_\Lambda = d_V L_u\Omega_\Lambda, \tag{3.21}$$

from (3.19) we get

$$d\Theta_{L_u\Lambda} = - L_u E_\Lambda + d_V\Omega_{L_u\Lambda}, \tag{3.22}$$

where we have used (2.12) and (2.17). Hence the result follows by comparing (3.22) with

$$d\Theta_{L_u\Lambda} = - E_{L_u\Lambda} + d_V\Omega_{L_u\Lambda}. \tag{3.23}$$

c) There are some useful relations between Λ, E_Λ and Θ_Λ. Note that $\mathscr{E}_\Lambda = 0$ is equivalent to $E_\Lambda = 0$, as follows from (3.13). From (3.1) it follows that $d\Theta_\Lambda = 0$ implies $E_\Lambda = 0$ (and also that Λ is affine). Also the converse is true. Hence we have

$$d\Theta_\Lambda = 0 \Leftrightarrow \Lambda \text{ is affine and } E_\Lambda = 0. \tag{3.24}$$

Another statement equivalent to (3.24) is: Λ is affine and Λ_E is closed (cfr. (2.4)).

Let us remark that if dim M = 1 or also if dim E = $m+1$, then $E_\Lambda = 0$ implies that Λ is affine. It follows that also $d\Theta_\Lambda = 0$. However, apart from these special cases, in general $E_\Lambda = 0$ does not implies that Λ is affine [*6*].

d) Given a Lagrangian form Λ, intuitively, the « variational symmetries of Λ » are the morphisms of a category whose objects are the variational forms $\{ \Lambda \}$. This is made precise by the following definition.

DEFINITION. — Let Λ be a Lagrangian form on JE. Then an automorphism $\varphi : E \to E$ is said to be a variational symmetry (in short: a v. s..) of Λ iff there is some variational form $\Theta \in \{ \Lambda \}$ such that

$$(J\varphi)^* d\Theta \in d\{ \Lambda \}. \tag{3.25}$$

Note that by taking the automorphisms $\varphi : E \to E$ such that

$$(J\varphi)^* \Theta \in \{ \Lambda \}, \tag{3.26}$$

for some $\Theta \in \{ \Lambda \}$, we get a subcategory of the previous one. It is clear that if φ is a v. s. of Λ, then φ is a symmetry of the Euler-Lagrange operator E_Λ, i. e. $\varphi^* E_\Lambda = E_\Lambda$. This follows from (3.1) in the same way used to prove (3.10).

The existence of a canonical variational form of Λ, namely the Poincaré-Cartan Θ_Λ, allows us to consider some distinguished v. s. By recalling (2.15) and the equivalence between $\Theta_\Lambda = 0$ and $\Lambda = 0$, it follows that

$$\varphi \in \mathrm{Aut}\{ \Theta_\Lambda \} \Leftrightarrow \varphi^* \Lambda = \Lambda. \tag{3.27}$$

Hence these v. s. are just the symmetries of Λ. Moreover, by recalling (3.24), we get

$$\varphi \in \mathrm{Aut}\{ d\Theta_\Lambda \} \Leftrightarrow (J\varphi)^* \Lambda - \Lambda \quad \text{is affine and} \quad \varphi^* E_\Lambda = E_\Lambda. \tag{3.28}$$

Note that if $\dim M = 1$ or also if $\dim E = m + 1$, then a symmetry of E_Λ is also an automorphism of $d\Theta_\Lambda$.

We finish by considering briefly the infinitesimal symmetries. Let u_E be a vector field on E projectable on M_1 and let u be its 1-jet prolongation. Then u_E is said to be an infinitesimal variational symmetry (in short: i. v. s.) of a Lagrangian form Λ iff

$$L_u d\Theta = 0, \tag{3.29}$$

for some $\Theta \in \{ \Lambda \}$. It is clear that if u_E is an i. v. s. of Λ, then u (the 2-jet prolongation of u_E) is an infinitesimal symmetry of the Euler-Lagrange operator E_Λ, i. e. $L_u E_\Lambda = 0$, as follows from (3.1) and (3.18).

Clearly, we get i. v. s. of Λ by taking those vector fields u_E on E, projectable on M, such that

$$L_u \Theta = 0, \tag{3.30}$$

for some $\Theta \in \{ \Lambda \}$ (as before u is the 1-jet prolongation of u_E).

The infinitesimal versions of (3.27) and (3.28) are

$$L_u \Theta_\Lambda = 0 \Leftrightarrow L_u \Lambda = 0, \tag{3.31}$$

and

$$L_u d\Theta_\Lambda = 0 \Leftrightarrow L_u \Lambda \quad \text{is affine and } L_u E_\Lambda = 0, \tag{3.32}$$

respectively.

4. CRITICAL SECTIONS

a) In order to exibit the functorial nature of a variational problem in the most direct way, we start with the following definition.

DEFINITION. — Let $s : M \to E$ be a section. A variation of s is a pair $(P, (\psi_t)_* s)$ where

i) $P \subset M$ is an orientable m-dimensional compact submanifold of M with boundary (denoted by ∂P),

ii) ψ_t is a one-parameter family of automorphisms $\psi_t : E \to E$ (over $\chi_t : M \to M$) such that

$$\psi_t \mid p^{-1}(\partial P) = id_{p^{-1}(\partial P)} \quad \text{for any } t. \tag{4.1}$$

The existence of ψ_t follows easily by considering, for example, vertical fields on E with compact support. Note that (4.1) implies that

$$\chi_t \mid \partial P = id_{\partial P}, \qquad (\psi_t)_* s \mid \partial P = s \mid \partial P \quad \text{for any} \quad t. \tag{4.2}$$

Variations of s behave in the right way under automorphisms of E. In fact, let $\varphi : E \to E$ be an automorphism (over $\varphi_M : M \to M$) and let $(P, (\psi_t)_* s)$ be a variation of s. Then

$$\varphi^*(P, (\psi_t)_* s) = (N, \varphi^*(\psi_t)_* s), \tag{4.3}$$

where $N = \varphi_M^{-1}(P)$, is a variation of $\varphi^* s$ since

$$\varphi^*(\psi_t)_* s = (\varphi_t)_* \varphi^* s, \qquad \varphi_t = \varphi^* \psi_t = \varphi_M^{-1} \circ \psi_t \circ \varphi_M . \tag{4.4}$$

DEFINITION. — Let Λ be a Lagrangian form on JE. A section $s : M \to E$ is said to be critical (with respect to Λ) iff the function

$$t \mapsto \int_{\chi_t(P)} j[(\psi_t)_* s]^* \Lambda \tag{4.5}$$

is stationary at $t = 0$ for any variation $(P, (\psi_t)_* s)$ of s.

Let $\varphi : E \to E$ be an automorphism (over $\varphi_M : M \to M$). The $\varphi^* s$ is a critical section of the Lagrangian form $(J\varphi)^* \Lambda$ iff s is a critical section of Λ, as follows from (4.4)

$$\begin{aligned} \int_{\chi_t(P)} j[(\psi_t)_* s]^* \Lambda &= \int_{\varphi_M^{-1}(\chi_t(P))} \varphi_M^* j[(\psi_t)_* s]^* \Lambda \\ &= \int_{\lambda_t(N)} (J\varphi \circ j[\varphi^*(\psi_t)_* s])^* \Lambda = \int_{\lambda_t(N)} j[(\varphi_t)^* \varphi^* s]^* (J\varphi)^* \Lambda , \end{aligned} \tag{4.6}$$

where $\lambda_t = \varphi_M^* \chi_t$ is the diffeomorphism induced by φ_t.

b) Given a Lagrangian form Λ, critical sections of Λ can be characterized by using the form $\mathscr{E}_\Lambda$ as follows.

PROPOSITION. — Let Λ be a Lagrangian form on JE. Then a section $s : M \to E$ is critical (with respect to Λ) iff $(js)^*\mathscr{E}_\Lambda = 0$.

Proof. — The derivative of the function (4.5) at $t = 0$ is

$$\text{(4.7)} \qquad D\left(\int_{\chi_t(P)} j[(\psi_t)_* s]^*\Lambda\right)\Big|_{t=0} = \int_P (js)^* L_u \Lambda\,,$$

where u is the 1-jet prolongation of u_E, the vector field on E determined by ψ_t. This result follows by applying the change of variables formula to (4.5), i. e.

$$\text{(4.8)} \qquad \int_{\chi_t(P)} j[(\psi_t)_* s]^*\Lambda = \int_P (js)^*(J\psi_t)^*\Lambda\,.$$

Note that u is just the vector field on JE determined by the one parameter family $J\psi_t$.

By using successively (2.16), the well known identity

$$L_u\Theta = u \lrcorner\, d\Theta + d(u \lrcorner\, \Theta)$$

and Stokes' theorem, (4.7) can be written as

$$\text{(4.9)} \qquad \int_P (js)^* L_u\Lambda = \int_P (js)^*(u \lrcorner\, d\Theta) + \int_{\partial P} (js)^*(u \lrcorner\, \Theta)\,,$$

where $\Theta \in \{\Lambda\}$. The last integral vanishes since Θ is p_E-horizontal and $u_E \mid p^{-1}(\partial P) = 0$. The result follows now from (3.1).

By choosing $\Theta \in \{\Lambda\}$, from (3.1) we get another characterization of critical sections, namely a section s is critical iff

$$\text{(4.10)} \qquad (js)^*(u \lrcorner\, d\Theta) = 0\,,$$

for any vector field u on JE (or also for any i. c. t. u projectable over E). In terms of the Euler-Lagrange operator E_Λ, the condition for s to be critical is the well known one, namely $E_\Lambda \circ j^2 s = 0$, as follows from (3.13).

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