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Section A :
Physique théorique.

Propagation of chaos for Burgers' equation

par

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ABSTRACT. — Burger's equation is obtained as a mean field limit of suitable diffusion processes.

RÉSUMÉ. — On obtient l'équation de Burgers comme limite du champ moyen de processus de diffusion convenables.

1. INTRODUCTION

In studying a class of parabolic equations, McKean [*1*] posed a problem of propagation of chaos for the Burgers' equation:

$$\frac{\partial}{\partial t} p = -\frac{1}{2}\frac{\partial}{\partial x} p^2 + \frac{1}{2}\frac{\partial^2}{\partial x^2} p, \qquad t \geqslant 0, \qquad x \in \mathbb{R}^1 \tag{1.1}$$

Let us spend some words about it.

It is well known that the initial value problem associated to Eq.. 1.1 is exactly solvable by means of the Hopf-Cole transformation [*2*], which yields:

$$p(x, t) = -\frac{\partial}{\partial x} \ln \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} \left(\exp\left[-\frac{(x-y)^2}{2t} \right] \right) \exp\left[- \int_{-\infty}^{y} q(z) dz \right] \tag{1.2}$$

where $q(x) = \lim_{t \to 0^+} p(x, t)$.

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Consider now the diffusion process solution of the following equation:

$$x(t) = x + b(t) + \frac{1}{2} \cdot \int_0^t dsp(x(s), s) \qquad (1.3)$$

where b is a standard one-dimensional Brownian motion. Then $p(x, t)$ is the distribution of $x(t)$, assuming $q \geqslant 0$ to be the distribution of $x = x(0)$. This follows by considering eq. (1.1) as Kolmogorov forward equation for p and interpreting p^2 as the product of the drift and distribution itself.

Let $\{ x_i \}_{i=1}^n$ be the positions of a n-particles system satisfying:

$$dx_i = \frac{1}{2n} \sum_{i \neq j} \delta(x_i - x_j)dt + db_i, \qquad i = 1, \ldots, n \qquad (1.4)$$

where b_i are n independent one-dimensional standard Brownian motions and δ denotes the Dirac distribution.

(Notice that the processes solutions of (1.4) could make sense in terms of the local time spent in coincidence by particles i and j). Then, denoting by $\rho_t(x_1 \ldots x_n)$ the distribution of the n particles system at time t (assumed symmetric at time zero and hence symmetric at all times) we have:

$$\frac{\partial}{\partial t} \rho_t = - \frac{1}{2n} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\partial}{\partial x_i} \delta(\mathrm{X}_i - \mathrm{X}_j)\rho_t + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \rho_t \,. \qquad (1.5)$$

Formally integrating by parts both sides w. r. t. the last $n-k$ variables, we have for the k-particles distribution ρ_t^k:

$$\begin{aligned} \frac{\partial}{\partial t} \rho_t^k(x_1 \ldots x_n) &= \frac{1}{2} \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} \rho_t^k(x_1 \ldots x_k) \\ &- \left(\frac{n-k}{2n}\right) \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \int dx_{\mathrm{K}+1} \delta(x_i - x_{k+1}) \rho_t^{k+1}(x_1 \ldots x_{k+1}) \\ &- \frac{1}{2n} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\partial}{\partial x_i} \delta(x_i - x_j) \rho_t^k(x_1 \ldots x_k) \end{aligned} \qquad (1.6)$$

Taking the formal limit $n \to \infty$, Eq. (1.6) reduces to:

$$\begin{aligned} \frac{\partial}{\partial t} \rho_t^k(x_1 \ldots x_k) &= \frac{1}{2} \sum_{i=1}^{k} \frac{\partial^2}{\partial \mathrm{X}_i^2} \rho_t^k(x_1 \ldots x_k) \\ &\qquad - \frac{1}{2} \sum_{i=1}^{k} \frac{\partial}{\partial \mathrm{X}_i} \rho_t^{k+1}(x_1 \ldots x_i \ldots x_k x_i) \end{aligned} \qquad (1.7)$$

If the initial distribution factorizes, i. e. $\rho_0^n = q \times \ldots \times q$, then dynamics (1.4) create correlations and ρ_t^n is no more a product measure. But we expect this factorization to be restored in the limit $n \to \infty$. Indeed an inspection of Eq. (1.7) shows that it is satisfied by a k-fold product of p_t, where p_t is solution of the Burger's equation with initial distribution q. The processes $x_1(t), \ldots, x_K(t)$ are also expected to converge in the limit $n \to \infty$ to k independent copies of the Burger's process (1.3). Such feature, called propagation of chaos in kinetic theory, has been rigorously proved in [*1*] for a class of non linear parabolic equations, with sufficiently smooth coefficients. The difficulties arising in the Burgers' problem are, of course, connected with the presence of δ functions in (1.4), making difficult also a rigorous position of the problem.

We approach this problem in a different simpler way. First we regularize the diffusion processes (1.4) replacing δ by a C^∞ approximation δ_ε. In this case everything may be proved for a limiting equation that resembles the Burgers' equation: the bilinear term $\frac{1}{2}\frac{\partial}{\partial x}p^2$ is replaced by $\frac{1}{2}\frac{\partial}{\partial x}p\cdot(p * \delta_\varepsilon)$. Then, a proof of convergence may be tried removing the cutoff $\varepsilon = \varepsilon(n)$ simultaneously to the limit $n \to \infty$, in such a way that the field

$$\frac{1}{2n}\cdot\sum_{i=1}^{n}\delta_{\varepsilon(n)}(x - x_i),$$

generated by the particles, is not so dramatic to deal with.

This kind of procedure is inspired to the one used in [*3*], modelling the Navier-Stokes equation by the vortex dynamics, but the presence here of a more singular kernel requires different « ad hoc » techniques.

In Section 2 we discuss the convergence of solutions of the regularized Burgers' equation to the true solution in the limit $\varepsilon \to 0$. In doing this, the explicit knowledge of regularity properties of the limit solutions will be used. In Section 3 we deduce the propagation of chaos by a diagonal limiting procedure, making use of results in Section 2.

2. AN APPROXIMATION THEOREM

Given the Burgers' problem:

$$\begin{cases} \dfrac{\partial}{\partial t}p(x,t) = -\dfrac{1}{2}\dfrac{\partial}{\partial x}p^2(x,t) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}p(x,t) \\ p(x,0) = q(x) \geqslant 0, (t,x) \in \mathbb{R}^+ \times \mathbb{R} \quad \text{and} \quad \displaystyle\int q(x)dx = 1, \end{cases} \qquad (2.1)$$

we set as its regularized version the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t} p^\varepsilon(x, t) = -\dfrac{1}{2}\dfrac{\partial}{\partial x} u_\varepsilon(x, t) p^\varepsilon(x, t) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} p^\varepsilon(x, t) \\ p^\varepsilon(x, 0) = q(x); \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R} \quad \text{and} \quad \varepsilon > 0 \end{cases} \tag{2.2}$$

where $u_\varepsilon(x, t) = (\delta_\varepsilon * p_t^\varepsilon)(x)$, $\delta_\varepsilon(x) = \dfrac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/2\varepsilon}$, $*$ denotes, as usual, convolution and $p_t^\varepsilon = p^\varepsilon(.\,, t)$.

We can write explicitly the solution of problem (2.1):

$$p(x, t) = -\frac{\partial}{\partial x} \ln \left[\int_{-\infty}^{+\infty} dy \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \exp\left(-\int_{-\infty}^{y} q(z)dz \right) \right]. \tag{2.3}$$

(2.3) is a consequence of the well known Hopf-Cole transformation [2], $p(x, t) = -\dfrac{\partial}{\partial x} \ln v(x, t)$, yielding for $v(x, t)$ the heat equation. Furthermore, because of the particular form of eq. (2.1) and solution (2.3), it is easy to deduce for $p(x, t)$ the following inequalities:

$$\begin{aligned} &a) \qquad \| p_t \|_2 \leqslant \| q \|_2 \\ &b) \qquad \| p_t \|_1 = \| q \|_1 \\ &c) \qquad \left\| \frac{\partial}{\partial x} p_t \right\|_\infty \leqslant 2 \| q \|_\infty^2 + \| q' \|_\infty . \end{aligned} \tag{2.4}$$

where $\| \cdot \|_p$ $1 \leqslant p \leqslant \infty$ denote, as usual, norms in $L^p(\mathbb{R})$ space and $p_t = p(.\,, t)$.

Note that Eqs. (2.1) and (2.2) can be read as forward equations for the probability distribution densities of two diffusion processes satisfying following stochastic differential equations:

$$\begin{aligned} &a) \qquad dx(t) = \frac{1}{2} p(x, t)dt + db(t) \\ &b) \qquad dx^\varepsilon(t) = \frac{1}{2} u_\varepsilon(x, t)dt + db(t) \end{aligned} \tag{2.5}$$

where $b(t)$ is a standard brownian motion.

In virtue of this fact, $(2.4)_b$ is simply the conservation of probability and $(2.4)_a$ follows by the inequality $\dfrac{d}{dt} \| p_t \|_2^2 \leqslant 0$, that is a direct consequence of Eq. (2.1). Finally by Hopf-Cole transformation we have: $p' = (v'^2 - vv'')v^{-2}$, where $'$ denotes the spatial derivative. Moreover, since v satisfies the heat equation, we have:

$$v'(x, t) = \int \frac{\exp[-(x-y)^2/2t]}{\sqrt{2\pi t}} \cdot q(y)v(y, 0)dy$$

$$v''(y, t) = -\int \frac{\exp[-(x-y)^2/2t]}{\sqrt{2\pi t}} \cdot \{ q'(y)v(y, 0) + q(y)v'(y, 0) \} dy \tag{2.6}$$

Hence $|v'(x,t)| \leqslant \|q\|_\infty \cdot |v(x,t)|$ and

$$|v''(x,t)| \leqslant \|q'\|_\infty \cdot |v(x,t)| + \|q\|_\infty \cdot |v'(x,t)|,$$

which imply $(2.4)_c$.

Let $\mathcal{M}([0,T])$ be the space of continuous functions from $[0,T]$ to $\mathcal{M}$, the space of all Borel probability measures on $\mathbb{R}$, endowed with the weak convergence topology. We define the following map S_q, $q \in \mathcal{M}$, from $\mathcal{M}([0,T])$ into itself:

$$S_q : \mu = (\mu_t)_{t\in[0,T]} \to S_q\mu = (S_q\mu)_t, \qquad t \in [0,T]$$

$$(S_q\mu)_t(A) = \int P_\mu(t, A \mid x, 0)dq(x) \tag{2.7}$$

where $P_\mu(t, . \mid x, 0)$ is the probability transition family associated to the diffusion process $x^\varepsilon_\mu(t)$ satisfying $dx^\varepsilon_\mu(t) = u^\varepsilon_\mu(x,t)dt + db(t)$ where

$$u^\varepsilon_\mu(x,t) = \int \delta_\varepsilon(x-y)\mu_t(dy).$$

Because of the smoothness of δ_ε one can prove the existence of a unique fixed point for the map S_q [1] [3] and hence a unique solution for the problem (2.2) can be constructed.

Let $\hat{p}(k,t)$ and $\hat{p}^\varepsilon(k,t)$ be the Fourier transforms (characteristic functions) of $p(x,t)$ and $p^\varepsilon(x,t)$, solutions of the initial value problems (2.1) and (2.2). We have:

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{p}(k,t) = -\dfrac{k^2}{2}\hat{p}(k,t) - i\dfrac{k}{2}\displaystyle\int dh\hat{p}(h,t)\hat{p}(k-h,t) \\ \hat{p}(k,0) = \hat{q}(k), \qquad k \in \mathbb{R} \end{cases} \tag{2.8}$$

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{p}^\varepsilon(k,t) = -\dfrac{k^2}{2}\hat{p}^\varepsilon(k,t) - i\dfrac{k}{2}\displaystyle\int dhe^{-h^2\varepsilon/2}\hat{p}^\varepsilon(h,t)\hat{p}^\varepsilon(k-h,t) \\ \hat{p}^\varepsilon(k,0) = \hat{q}(k), \qquad k \in \mathbb{R} \end{cases} \tag{2.9}$$

We will give now the main theorem of this section:

THEOREM 2.1. — Let $p(x,t)$ and $p^\varepsilon(x,t)$ be the solutions of problems (2.1) and (2.2), with initial datum $q(x)$, bounded with its first derivative. Then $p^\varepsilon(.,t) \to p(.,t)$, as $\varepsilon \to 0$, in the sense of weak convergence of distribution functions and uniformly in t on compact sets.

To prove Theorem 2.1 we need the following Proposition:

PROPOSITION 2.1. — In the hypotheses of Theorem 2.1, for all $T > 0$, $\hat{p}^\varepsilon(.,t)$ converges to $\hat{p}(.,t)$, as $\varepsilon \to 0$, in the $L^2(\mathbb{R})$ sense, uniformly in $t \in [0,T]$.

In the proof of Proposition 2.1 we will make use of the following Lemma, whose proof is shifted at the end of the Section.

LEMMA 2.1. — Let $\hat{p}^{\varepsilon}(.,t)$ be the solution of (2.9). Then, fixed $r > 4$, there exists a positive constant M_0 such that:

$$\sup_{t\in[0,\hat{t}]} \| \hat{p}^{\varepsilon}_t \|_2^2 \leqslant 4 \cdot \| q \|_2^2 \tag{2.10}$$

where
$$\hat{t} = \hat{t}(\| q \|_2) = 1/(2^{r+1} \cdot M_0 \cdot \| q \|_2^r).$$

Proof of Proposition 2.1. — Fixed an arbitrary $T > 0$, first we shall prove $L^2(\mathbb{R})$ convergence up to the time $\tilde{t} \leqslant T$, for which we have an ε-uniform estimate of $\sup_{t\in[0,\tilde{t}]} \| \hat{p}^{\varepsilon}_t \|_2^2$.

Then, in virtue of Lemma 2.1, we get $L^2(\mathbb{R})$ convergence for small times.

Finally, regularity properties of the limiting solution allow us to enlarge the convergence interval up to T.

We suppose:

$$\sup_{t\in[0,t]} \| \hat{p}^{\varepsilon}_t \|_2^2 \leqslant 4 \| q \|_2^2 \tag{2.11}$$

By (2.8) and (2.9) we have:

$$a)\quad \hat{p}(k,t) = e^{-\frac{1}{2}k^2 t}\hat{q}(k) - \frac{ik}{2}\int_0^t dse^{-\frac{1}{2}k^2(t-s)}\int dh\hat{p}(h,s)\hat{p}(k-h,s)$$

$$b)\quad \hat{p}^{\varepsilon}(k,t) = e^{-\frac{1}{2}k^2 t}\hat{q}(k) - \frac{ik}{2}\int_0^t dse^{-\frac{1}{2}k^2(t-s)}\int dhe^{-h^2\varepsilon/2}\hat{p}^{\varepsilon}(h,s)\hat{p}^{\varepsilon}(k-h,s) \tag{2.12}$$

from which it follows:

$$| \hat{p}^{\varepsilon}(k,t) - \hat{p}(k,t) | \leqslant \int_0^t dse^{-\frac{1}{2}k^2(t-s)} | k |/2 \left| \int dh[\hat{p}(h,s)\hat{p}(k-h,s) - e^{-h^2\varepsilon/2}\hat{p}^{\varepsilon}(h,s)\hat{p}^{\varepsilon}(k-h,s)] \right| \tag{2.13}$$

Moreover:

$$\begin{aligned}
&\| \hat{p}_t - \hat{p}^{\varepsilon}_t \|_2^2 \\
&\leqslant \int dk\frac{k^2}{4}\left(\int_0^t dse^{-\frac{1}{2}k^2(t-s)}\left|\int dh[\hat{p}(h,s)\hat{p}(k-h,s) - e^{-h^2\varepsilon/2}\hat{p}^{\varepsilon}(h,s)\hat{p}^{\varepsilon}(k-h,s)]\right|\right)^2 \\
&\leqslant \frac{3}{4}\int dkk^2\left(\int_0^t dse^{-\frac{1}{2}k^2(t-s)}\left|\int dh[\hat{p}(h,s)\hat{p}(k-h,s) - e^{-h^2\varepsilon/2}\hat{p}(h,s)\hat{p}(k-h,s)]\right|\right)^2 \\
&+ \frac{3}{4}\int dkk^2\left(\int_0^t dse^{-\frac{1}{2}k^2(t-s)}\left|\int dhe^{-h^2\varepsilon/2}\hat{p}(h,s)[\hat{p}(k-h,s) - \hat{p}^{\varepsilon}(k-h,s)]\right|\right)^2 \\
&+ \frac{3}{4}\int dkk^2\left(\int_0^t dse^{-\frac{1}{2}k^2(t-s)}\left|\int dhe^{-h^2\varepsilon/2}\hat{p}^{\varepsilon}(k-h,s) - \hat{p}(h,s) - \hat{p}^{\varepsilon}(h,s)]\right|\right)^2
\end{aligned}$$

$$\leqslant \frac{3}{4}\int dkk^2\left[\int_0^t dse^{-\frac{1}{2}k^2(t-s)}\,|(\hat{p}_s^2 - p_s(\hat{\delta}_\varepsilon * p_s))|\right]^2$$
$$+\frac{3}{4}\int dkk^2\left[\left(\int_0^t dse^{-\frac{k^2}{2}(t-s)}\|\hat{p}_s\|_2\,\|\hat{p}_s-\hat{p}_s^\varepsilon\|_2\right)^2\right.$$
$$\left.+\left(\int_0^t dse^{-\frac{k^2}{2}(t-s)}\|\hat{p}_s^\varepsilon\|_2\cdot\|\hat{p}_s-\hat{p}_s^\varepsilon\|_2\right)^2\right]. \quad (2.14)$$

Hence, by Hölder inequality

$$\|\hat{p}_t^\varepsilon - \hat{p}_t\|_2^2 \leqslant \frac{3}{4}\cdot\varepsilon\cdot\sup_{s\in[0,\tilde{t}]}\left\|\frac{\partial}{\partial x}p_s\right\|_\infty^2\cdot\|q\|_1^2\int dkk^2\left(\int_0^t dse^{-\frac{k^2}{2}(t-s)}\right)^2$$
$$+3/4(\|q\|_2^2+\sup_{s\in[0,\tilde{t}]}\|\hat{p}_s^\varepsilon\|_2^2)\int dkk^2\cdot\left(\int_0^t dse^{-\frac{k^2}{2}r'(t-s)}\right)^{2/r'}\cdot\left(\int_0^t ds\|\hat{p}_s^\varepsilon-\hat{p}_s\|_2^r\right)^{2/r}$$

where r and r' and conjugate exponents. (2.15)

Since $r' < 4/3$, we have:

$$\int dkk^2\cdot\left(\int_0^t dse^{-\frac{k^2}{2}r'(t-s)}\right)^{2/r'} \leqslant \int_{|K|\leqslant 1} dkk^2t^{2/r'}$$
$$+\int_{|K|>1} dkk^{2-4/r'}\left(\frac{2}{r'}\right)^{2/r'} = \frac{2}{3}T^{2/r'} + \frac{r'}{4-3r'}\left(\frac{2}{r'}\right)^{2/r'} \equiv C(r') \quad (2.16)$$

where $C(r')$ is defined by the last equality.

Therefore, by $(2.4)_c$ and (2.11) we have:

$$\|\hat{p}_t^\varepsilon - \hat{p}_t\|_2^r \leqslant (\varepsilon C_1)^{r/2} + \overline{C}_1\int_0^t ds\|\hat{p}_s^\varepsilon - \hat{p}_s\|_2^r \quad (2.17)$$

where $C_1 = \text{const.}\cdot\|q\|_1^2\cdot(2\cdot\|q\|_\infty^2 + \|q'\|_\infty)^2$ and $\overline{C}_1 = \text{const.}\cdot\|q\|_2^r$. Finally, by Gronwall's Lemma:

$$\|\hat{p}_t^\varepsilon - \hat{p}_t\|_2\cdot \leqslant \sqrt{\varepsilon C_1}\cdot\exp[(\overline{C}_1/r)\cdot\tilde{r}], \qquad t\in[0,\tilde{t}]. \quad (2.18)$$

Now we show the $L^2(\mathbb{R})$ convergence for all times by proving the following estimate: there exists $\overline{\varepsilon}$ such that for all $\varepsilon \leqslant \overline{\varepsilon}$:

$$\sup_{t\in[0,T]}\|\hat{p}_t^\varepsilon\|_2 \leqslant 2\|q\|_2 \quad (2.19)$$

We denote by $\{\Delta_i\}_{i=1}^n$ a partition of $[0,T]$ with non-overlapping intervals $\Delta_i = [t_{i-1}, t_i]$, such that $|t_{i-1}-t_i| \leqslant \hat{t}/2^r$ ($\hat{t}$ given by Lemma 1) for each $i = 1, \ldots, n$, $t_0 = 0$ and $t_n = T$. By $(2.4)_a$, Lemma 1, (2.11) and (2.18), we have:

$$\|\hat{p}_t^\varepsilon\|_2 \leqslant \|\hat{p}_t\|_2 + \|\hat{p}_t^\varepsilon-\hat{p}_t\|_2 \leqslant \|q\|_2+\sqrt{\varepsilon C_1}\cdot\exp(\overline{C}\hat{t}/r^{2r})$$
$$\leqslant 2\|q\|_2, \qquad t\in\Delta_1 \quad (2.20)$$

for $0 < \varepsilon \leqslant \varepsilon_1$, ε_1 sufficiently small.

Now we prove (2.19) by induction on the intervals Δ_i. Namely, assuming the existence of an ε_j for which:

$$\sup_{t\in[0,t_j]} \| \hat{p}_t^\varepsilon \|_2 \leqslant 2 \| q \|_2, \qquad \varepsilon \leqslant \varepsilon_j \tag{2.21}$$

we show that there exists ε_{j+1} such that

$$\sup_{t\in[0,t_{j+1}]} \| \hat{p}_t^\varepsilon \|_2 \leqslant 2 \| q \|_2, \qquad \varepsilon \leqslant \varepsilon_{j+1} \tag{2.22}$$

In fact, for $t \in \Delta_{j+1}$:

$$\| \hat{p}_t^\varepsilon \|_2 \leqslant \| \hat{p}_t \|_2 + \| \hat{p}_t^\varepsilon - \hat{p}_t \|_2 \tag{2.23}$$

and

$$\begin{aligned}\| \hat{p}_t^\varepsilon - \hat{p}_t \|_2^2 &\leqslant 2 \| \hat{p}_{tj}^\varepsilon - \hat{p}_{tj} \|_2^2 + 2 \int dk \frac{k^2}{4} \left(\int_{t_j}^t dse^{-K^2(t-s)/2} \right. \\ &\quad \cdot \left. \left| \int dh [\hat{p}(h,s)\hat{p}(k-h,s) - e^{-h^2\varepsilon/2}\hat{p}^\varepsilon(h,s)\hat{p}^\varepsilon(k-h,s)] \right| \right)^2 \\ &\leqslant 2(\| \hat{p}_{tj}^\varepsilon - \hat{p}_{tj} \|_2^2 + C_1\varepsilon) + \frac{3}{2}(\| q \|_2^2 + \sup_{s\in\Delta_{j-1}} \| \hat{p}_s^\varepsilon \|_2^2) \\ &\quad \cdot \int dk k^2 \left(\int_{t_j}^t dse^{-k^2 r'(t-s)/2} \right)^{2/r'} \left(\int_{t_j}^t ds \| \hat{p}_s^\varepsilon - \hat{p}_s \|_2^r \right)^{2/r}\end{aligned} \tag{2.24}$$

where, in the last step, we have used the same arguments as in the estimates (2.14) and (2.15).

In virtue of the inductive hypothesis, (2.11) holds with $\tilde{t} = t_j$ and hence by (2.18) $\| \hat{p}_{t_j}^\varepsilon - \hat{p}_{t_j} \|_2^2 \leqslant$ const. ε.

Furthermore, by Lemma 1 (noticing that $\hat{t}(2 \| q \|_2) = \hat{t}(\| q \|_2)/2^r = \hat{t}/2^r$) and (2.21), we have:

$$\sup_{t\in\Delta_{j+1}} \| \hat{p}_t^\varepsilon \|_2^2 \leqslant 4 \| \hat{p}_{t_j}^\varepsilon \|_2^2 \leqslant 16 \| q \|_2^2 \tag{2.25}$$

Thus

$$\| \hat{p}_t^\varepsilon - \hat{p}_t \|_2^r \leqslant (\varepsilon C_{j+1})^{r/2} + \overline{C}_{j+1} \int_{tj}^t ds \| \hat{p}_s^\varepsilon - \hat{p}_s \|_2^r \tag{2.26}$$

for suitable constants C_{j+1} and $\overline{C}_{j+1}$ depending only on q. By Gronwall's Lemma and (2.23), we get:

$$\| \hat{p}_t^\varepsilon \|_2 \leqslant \| q \|_2 + \text{const.} \sqrt{\varepsilon} \tag{2.27}$$

and hence (2.22).

Finally, choosing $\overline{\varepsilon} = \min_{j=1,\ldots,n} \varepsilon_j$, we obtain (2.19). □

Proof of Theorem 2.1. — It is enough to show that $\hat{p}_t^\varepsilon$ converges pointwise to $\hat{p}_t$, as $\varepsilon \to 0$, uniformly in t on compact sets (See Th. 26.3, p. 303 [*4*]).

By (2.13) we have:

$$| \hat{p}(k,t) - \hat{p}^\varepsilon(k,t) | \leqslant \frac{|k|}{2} \int_0^t ds e^{-\frac{k^2}{2}(t-s)} \int dh [| \hat{p}(h,s)\hat{p}(k-h,s) - e^{-h^2\varepsilon/2}\hat{p}(h,s)\hat{p}(k-h,s) | + | \hat{p}(h,s)\hat{p}(k-h,s) - \hat{p}(h,s)\hat{p}^\varepsilon(k-h,s) | + | \hat{p}(h;s)\hat{p}^\varepsilon(k-h,s) - \hat{p}^\varepsilon(h,s)\hat{p}^\varepsilon(k-h,s) |] \tag{2.28}$$

and hence, by Schwarz inequality (2.4) and (2.19), we get:

$$| \hat{p}(k,t) - \hat{p}^\varepsilon(k,t) | \leqslant \varepsilon \cdot \mathrm{T} \frac{|k|}{2} \| q \|_1 \cdot (2 \| q \|_\infty^2 + \| q' \|_\infty) + 5 \frac{|k|}{2} \int_0^t ds \| q \|_2^2 \cdot \| \hat{p}_s - \hat{p}_s^\varepsilon \|_2^2 . \tag{2.29}$$

So the thesis follows by (2.29) and $\mathrm{L}^2(\mathbb{R})$ convergence of $\hat{p}_t^\varepsilon$ to $\hat{p}_t$. □

Proof of Lemma 2.1. — By $(2.12)_b$ we can write:

$$| \hat{p}^\varepsilon(k,t) | \leqslant e^{-\frac{k^2}{2}t} | \hat{q}(k) | + \frac{|k|}{2} \int_0^t ds e^{-\frac{k^2}{2}(t-s)} \left| \int dh e^{-h^2\varepsilon/2} \hat{p}^\varepsilon(h,s)\hat{p}^\varepsilon(k-h,s) \right| \tag{2.30}$$

So, we have

$$\| \hat{p}_t^\varepsilon \|_2^2 \leqslant 2 \| q \|_2^2 + \int dk \frac{k^2}{2} \left(\int_0^t ds e^{-\frac{k^2}{2}(t-s)} \| \hat{p}_s^\varepsilon \|_2^2 \right)^2 . \tag{2.31}$$

and hence, by Hölder inequality:

$$\| \hat{p}_t^\varepsilon \|_2^2 \leqslant 2 \| q \|_2^2 + \int dk \frac{k^2}{2} \left(\int_0^t ds e^{-\frac{k^2}{2}r'(t-s)} \right)^{2/r'} \left(\int_0^t ds \| \hat{p}_s^\varepsilon \|_2^{2r} \right)^{2/r} \tag{2.32}$$

Choosing $r' < 4/3$ and est imating the integral in the r. h. s. of (2.32) as in Proposition 1 (2.16), we get:

$$\| \hat{p}_t^\varepsilon \|_2^r \leqslant 2^{r-1} \| q \|_2^r + 2^{\frac{r}{2}-1} \left[\left(\frac{2}{r'} \right)^{2/r'} r'/2(4-3r') + 1/3 \cdot t^{2/r'} \right]^{r/2} \cdot \int_0^t ds \| \hat{p}_s^\varepsilon \|_2^{2r} \tag{2.33}$$

Let us set :

$$y(t) = \| \hat{p}_t^\varepsilon \|_2^r \quad \text{and} \quad y_0 = \| \hat{q} \|_2^r \tag{2.34}$$

Then it is possible to find a positive constant M_0 such that, by (2.33) we can write:

$$y(t) \leqslant 2^{r-1}y_0 + \mathrm{M}_0(1 + t^{r-1}) \int_0^t dsy(s)^2 \tag{2.35}$$

Let $\mathrm{X}(t, y_0)$ be the solution of the following initial value problem:

$$\begin{cases} \dot{x}(t) = \mathrm{M}_0(1 + \hat{t}^{r-1})x^2(t) \\ x(0) = 2^{r-1}y_0 \end{cases} \tag{2.36}$$

namely

$$x(t, y_0) = \frac{y_0 2^{r-1}}{1 - y_0 2^{r-1} M_0 (1 + \hat{t}^{r-1}) t} \tag{2.37}$$

Choosing $\hat{t} = 1/2^{r+1} M_0 y_0$ one realizes that by (2.37), (2.35) and (2.34) we get (2.10), since $1 - 2^{r-1} y_0 M_0 (1 + \hat{t}^{r-1}) t \geqslant 1/2$. □

3. A MEAN FIELD LIMIT

For any positive integer n and $\varepsilon > 0$, we define the processes $\{ x_i^{\varepsilon,n}(t) \}_{i=1}^n$, solutions of the following stochastic differential equations:

$$dx_i^{\varepsilon,n}(t) = \frac{1}{2} U_i^{\varepsilon}(x^{\varepsilon,n}(t)) ds + db_i(t) \tag{3.1}$$

where

$$U_i^{\varepsilon}(x^{\varepsilon,n}(t)) = \frac{1}{n} \sum_{j=1}^{n} \delta_\varepsilon (x_i^{\varepsilon,n}(t) - x_j^{\varepsilon,n}(t)) \tag{3.2}$$

$x^{\varepsilon,n}(t) = \{ x_i^{\varepsilon,n}(t) \}_{i=1}^n$ and, finally, $\{ b_i(t) \}_{i=1}^\infty$ are mutually independent standard brownian motions.

We denote by $X^{\varepsilon,n}(t) = \{ x_i^{\varepsilon,n}(t) \}_{i=1}^r = \{ x_i^{\varepsilon,n}(t, X^n) \}_{i=1}^n$ the unique (but for stochastic equivalence) stochastic process solution of (3.1); starting from $X^n = \{ x_i \}_{i=1}^n$ at time zero. X^n are supposed to be mutually independent random variables, distributed via $q(x)dx$.

We denote by $(\Omega, \Sigma, \mathbb{P})$ the sample space in which all $\{ b_i(t) \}_{i=1}^\infty$ are supposed to be realized, together with infinitely many independent copies of the Burgers' process, satisfying $dx_i(t) = p(x_i(t), t)dt + db_i$ (see def. $(2.5)_a$). Then we have:

THEOREM 3.1. — There exists a sequence $\varepsilon = \varepsilon(n)$, $\varepsilon(n) \to 0$ when $n \to \infty$, such that $x_i^{\varepsilon(n),n}(t)$ converges to $x_i(t)$, for all integers i, as $n \to \infty$, in the $L^2((\Omega, \Sigma, \mathbb{P}))$ norm, uniformly in $t \in [0, T]$.

Proof. — By using the same arguments as in [*1*], we can show that $x_i^{\varepsilon,n}(t) \to x_i^{\varepsilon}(t)$, as $n \to \infty$, where $\{ x_i^{\varepsilon}(t) \}_{i=1}^\infty$ are infinite mutually independent copies of the process solution of $(2.5)_b$, in the sense of $L^2(\Omega, \Sigma, \mathbb{P})$ norm. For this reason we only give a sketch of the proof. Fixed $\varepsilon > 0$, $x^{\varepsilon,n}(t)$ is a Cauchy sequence. This is a consequence of the following inequality:

$$\mathbb{E}(| x_i^{\varepsilon,n}(t) - x_i^{\varepsilon,n}(t) |^2) \leqslant \frac{4T}{\varepsilon n} \cdot \exp(4T^2/\varepsilon^2) \tag{3.3}$$

with $m > n$ and $t \in [0, T]$.

$$s \in [0, T]$$

$$\exp \{ 2T^2 \cdot (2 \| q \|_\infty + \| q' \|_\infty)^2 \} \quad (3.6)$$

Finally, by (3.3) and (3.6) we get:

$$\mathbb{E}(|x_i^{\varepsilon,n}(t)-x(t)|^2)\leqslant 2\mathbb{E}(|x_i^{\varepsilon,n}(t)-x^{\varepsilon}(t)|^2)+2\mathbb{E}(|x^{\varepsilon}(t)-x(t)|^2)$$
$$\leqslant \frac{8\mathrm{T}}{\varepsilon h}\cdot \exp\,[4\mathrm{T}^2/\varepsilon^2] + 8\mathrm{T}^2 0(\varepsilon)\exp(2\mathrm{T}^2(2\,\|q\|_\infty + \|q'\|_\infty)^2) \quad (3.7)$$

where $0(\varepsilon) = \{\varepsilon\cdot(2\,\|q\|_\infty + \|q'\|_\infty)^2 \sup_{s\in[0,\mathrm{T}]} \|p_s^\varepsilon - p_s\|_2^2\}$.

Therefore, it is possible to find a sequence $\varepsilon = \varepsilon(n)$, going to zero when $n \to \infty$, in such a way that $\mathbb{E}(|x_i^{\varepsilon(n),n}(t) - x(t)|^2) \to 0$ as $n \to \infty$. □

Comments and remarks.

The above results can be improved, by proving propagation of chaos in a different way. Suppose the processes $\{x_i^{\varepsilon,n}(t)\}_{i=1}^n$ start, almost surely, from the points $\{x_i\}_{i=1}^n$ at time zero. Suppose that such points simulate the initial distribution $q(x)dx$ in the sense that, for all bounded and continuous functions f, it is:

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \to \int q(x)f(x)dx, \qquad n \to \infty. \qquad (3.8)$$

Then

$$\mu_t^n(f) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(f(x_i^{\varepsilon,n}(t))), \qquad t > 0 \qquad (3.9)$$

converges to $\int p(x,t)f(x)dx$, where $p(.,t)$ is the solution of the Burger's equation with initial datum q, for a suitable choice of $\varepsilon = \varepsilon(n)$, $\varepsilon(n) \to 0$ as $n \to \infty$.

The proof of this statement is the same as the one given in [*3*] in the context of Navier-Stokes equation, thus we do not repeat it here. We finally remark that the regularized finite dimensional model we have introduced in this paper, is directly connected to the Burgers' equation also in a different limit, not involving explicit scaling of the interaction.

Let $\{x_i(t)\}_{i=1}^n$ be a n-particles system satisfying:

$$dx_i(t) = \sum_{j=1}^{n} \delta_\varepsilon(x_i(t) - x_j(t))dt + db_i(t) \qquad (3.10)$$

Suppose we rescale the variables in the following way:

$$x_i^n(t) = \frac{1}{n}\,x_i(tn) \qquad (3.11)$$

Then

$$dx_i^n(t) = \frac{1}{n}\sum_{j=1}^{n} \delta_{\varepsilon/n^2}(x_i^n(t) - x_j^n(t))dt + \sqrt{\frac{1}{n}}\,db_i(t), \qquad (3.12)$$

having used the scaling properties:

$$\delta_\varepsilon(rn) = \frac{1}{n}\delta_{\varepsilon/n^2}(r), \qquad \sqrt{1/n}\cdot b_i(t/n) = b_i(t)\,.$$

Thus, if $1/n\cdot\sum_{i=1}^{n} f(x_i^n(0)) \to \int f(x)q(x)dx$ as $n \to \infty$, we expect $x_i^n(t)$ to converge to $x(t)$, for fixed ε and $n \to \infty$, where $x(t)$ is the solution of:

$$\frac{dx(t)}{dt} = p(x(t), t) \tag{3.13}$$

and $p(x, t)$ is a solution of inviscid Burgers' equation. Moreover such solution, among the weak solutions of the inviscid Burgers' equation, is expected to be the weak limit, for $\sigma \to 0$, of the solution of the viscid Burgers' equation, with diffusion coefficient σ.

Unfortunately our estimates do not allow us to prove the above conjecture in a general framework.

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