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On the possibility of general relativistic oscillations II

by

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ABSTRACT. — The configuration of a spherically symmetric distribution of matter is investigated. Conditions which are necessary and sufficient are given for the density to be uniform. The motion of the configuration is investigated with a view to oscillations using different physical conditions and initial conditions. It is shown that for many classes of solutions, these conditions are sufficient to conclude that oscillations are not possible. Particularly it is found that for a gaseous mass, oscillations are *not* possible. For other classes of solutions, it is shown that the conditions necessary for the solutions to be physically acceptable, are consistent with oscillations. However, these conditions are in general not sufficient.

RÉSUMÉ. — On étudie la configuration d'une distribution de matière à symétrie sphérique. On donne des conditions nécessaires et suffisantes pour que la densité soit uniforme. On étudie le mouvement d'une telle configuration avec en vue ses oscillations, sous différentes conditions physiques et avec différentes conditions initiales. On montre que pour de nombreuses classes de solutions, ces conditions sont suffisantes pour conclure à l'impossibilité des oscillations. On trouve en particulier que les oscillations ne sont pas possibles pour une masse gazeuse. Pour d'autres classes de solutions, on montre que les conditions nécessaires pour que les solutions soient physiquement acceptables sont compatibles avec des oscillations. Par contre, ces conditions ne sont en général pas suffisantes.

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1. INTRODUCTION

Using a special form of the metric G , C. C. McVittie has investigated the radial motions of a spherically symmetric mass distribution under the influence of gravitation and pressure gradient [1]. By imposing certain symmetry conditions, instead of assuming a particular equation of state,

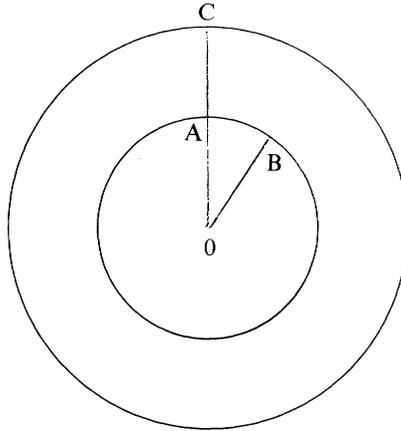


FIG. 1.

McVittie was able to solve Einstein's field equations for the coefficients of the metric. The metric used by McVittie belongs to a class investigated by Thompson and Whitrow [2], (Shear-free motion). The relation between the components of the metric tensor seems physically plausible because this relation implies a constant value for the ratio of the distances AB and AC , as measured by a local observer, where A , B are neighbouring particles on a sphere $R = \text{constant}$ and A , C are neighbouring particles on the same radius vector through 0 .

Mansouri [3] and Glass [4] have shown that if the metric is of the Thompson-Whitrow type, an equation of state *cannot* exist. But not all of McVittie's solutions satisfy an equation of state. Assuming isotropic pressure, McVittie was able to give four different classes of analytical solutions of Einstein's equations for the interior of the mass distribution. These four classes of solutions have been investigated restricting the scale-function to have the value unity at the initial moment [5], [6]. To avoid complications, different integration constants which appeared were also restricted to take particular but rather arbitrary values. In this paper it is shown that these last two restrictions are much too stringent. Classes of physically very important and interesting solutions may be lost. An

example is a physically acceptable oscillating model found by Nariai [7]. Instead of assuming the scale-function to have the value unity at the initial moment and giving particular values to integration constants, specific relations will be assumed connecting the initial value of the scale-function and the integration constant.

Different physical conditions and initial conditions are also introduced: to fit the internal solution to an external vacuum Schwarzschild solution it is necessary to put the pressure equal to zero at the boundary. It will be required that the mass is at rest at the initial moment. The following three restrictions are used for the boundary and initial moment: the density gradient with respect to radial coordinate will be demanded to be non-positive. The acceleration must be negative. The density cannot take negative values. We shall also require the acceleration to be positive at the boundary at the bounce.

The purpose of this paper is to investigate in detail the four classes of analytical solutions with regard to oscillatory motions.

2. BASIC EQUATIONS

With co-moving coordinates the metric within the mass is written [1]:

$$ds^2 = y^2 dt^2 - \frac{R_0^2 S^2 e^\eta}{c^2} [dr^2 + f^2(d\theta^2 + \sin^2 \theta d\phi^2)] \quad (1)$$

where R_0 is a constant, c is the speed of light, S and f are dimensionless functions of t and r , respectively, and y and η are dimensionless functions of the variable z defined by:

$$e^z = \frac{Q}{S} \quad (2)$$

where Q is another function of r . It is also shown in [1] that

$$y = 1 - \frac{1}{2} \frac{d\eta}{dz} \quad (3)$$

If the pressure is isotropic, then it is shown in [1] that Q , f and y satisfy three differential equations:

$$\frac{Q_{rr}}{Q} - \frac{Q_r f_r}{Q f} = a \left(\frac{Q_r}{Q} \right)^2 \quad (4)$$

$$\frac{f_{rr}}{f} - \frac{f_r^2}{f^2} + \frac{1}{f^2} = b \left(\frac{Q_r}{Q} \right)^2 \quad (5)$$

$$y_{zz} + (a - 3 + y)y_z + y[a + b - 2 - (a - 3)y - y^2] = 0 \quad (6)$$

where a and b are constants.

It is shown in [I] that (4) and (5) are always integrable in terms of elementary functions, but (6) is only so integrable in four special cases, denoted in [I] by equations (A.26) to (A.29):

$$(A.26) \quad b = \frac{-(6a^2 - 11a + 4)}{25} \quad (7)$$

$$y_z = \frac{a-3}{5}y + \frac{1}{2}y^2$$

$$(A.27) \quad b = 2 - a \quad (8)$$

$$y_z = -(a-3)y - y^2$$

$$(A.28) \quad a = 3, \text{ any } b$$

$$y_z = -\frac{1}{2}(b+1) + \frac{1}{2}y^2 \quad (9)$$

$$(A.29) \quad \text{any } a, \text{ any } b \quad (10)$$

$$y_z = (a+b-2) - (a-3)y - y^2$$

Of these (A.27) is a subcase of (A.29). These four cases form the class of solutions which will be investigated in this paper.

Equations (A.33) and (A.34) in [I] give for the density, ρ , and the pressure, p , respectively:

$$8\pi G\rho = 3\left(\frac{S_r}{S}\right)^2 + \frac{c^2 e^\eta}{R_0^2 S^2} \left\{ 3\frac{1-f_r^2}{f^2} - 6(1-y)\frac{f_r Q_r}{fQ} - [2b-2y_z+(1-y)(2a-1-y)]\frac{Q_r^2}{Q^2} \right\} \quad (11)$$

$$8\pi Gp/c^2 = \frac{1}{y} \left\{ -\frac{2S_u}{S} - (3y-2)\frac{S_r^2}{S^2} - \frac{c^2 e^{-\eta}}{R_0^2 S^2} \left[y\frac{1-f_r^2}{f^2} + 2(y^2-y-y_z)\frac{f_r Q_r}{fQ} + (1-y)(y^2-y-2y_z)\frac{Q_r^2}{Q^2} \right] \right\} \quad (12)$$

To fit the internal solution to an external vacuum Schwarzschild solution, it is necessary to put the pressure equal to zero at the boundary:

$$p_b \equiv 0 \quad (13)$$

Henceforth boundary values will be denoted by the suffix b .

In [6] it was shown that the condition (13) gives the following ordinary differential equation of first order for S_r^2 :

$$\frac{d}{dS}(S_r^2) + H(S)S_r^2 + J(S) = 0 \quad (14)$$

where the definitions of H and J may be found in [6]. In [5] it was found that without loss of generality one could put:

$$Q_b = 1 \tag{15}$$

The mass will be assumed to be at rest at the initial moment, *i. e.*:

$$S_t(0) = 0 \tag{16}$$

This completes the summary of the general theory.

3. THE GRADIENTS

The pressure gradient with respect to the co-moving coordinate r , p_r , may be found directly from (12), but alternatively it may be derived using the covariant derivative of the energy momentum tensor with respect to r .

Thus:

$$p_r/c^2 = -(\rho + p/c^2) \frac{y_z Q_r}{y Q} \tag{17}$$

The derivative of the density with respect to time t , ρ_t , may be found directly from (11), but alternatively it may be derived using the covariant derivative of the energy momentum tensor with respect to t .

Thus:

$$\rho_t = -3y \frac{S_t}{S} (\rho + p/c^2) \tag{18}$$

If the physical restriction $p_r < 0$ is imposed, and we demand an increasing density for the contraction period, *i. e.* $S_t < 0$, from (17) and (18) we obtain the following relations:

$$y > 0 \tag{19}$$

$$y_z Q_r > 0 \tag{20}$$

In passing we also note that (2) and (17) immediately yield:

$$\ln y = - \int \frac{d(p/c^2)}{\rho + p/c^2} \tag{21}$$

From (21) we conclude that an equation of state could exist if and only if ρ and p were both functions of y alone. As was shown by Taub [8], if the motion is shear-free and an equation of state exists, then McVittie's requirements on the coefficients of the metric are fulfilled. But from the results of Mansouri [3] and Glass [4], *i. e.* the non-existence of an equation of state, we conclude that ρ and p cannot both be functions of y alone.

The density gradient with respect to radial coordinate r may be found from (11):

$$8\pi G\rho_r = \frac{2c^2 e^{-\eta} Q_r}{R_0^2 S_2 Q} \{ y_z - [(a+b-2) - (a-3)y - y^2] \} \\ \times \left\{ 5 \frac{f_r Q_r}{f Q} + (2a-1) \frac{Q_r^2}{Q^2} \right\} \quad (22)$$

4. UNIFORM DENSITY

From (22) it is immediately obtained that the density is uniform, *i. e.* $\rho = \rho(t)$, if and only if one of the following three equations is fulfilled:

$$y_z = (a+b-2) - (a-3)y - y^2 \quad (23)$$

$$Q = \text{constant} \quad (24)$$

$$5 \frac{f_r Q_r}{f Q} + (2a-1) \frac{Q_r^2}{Q^2} = 0 \quad (25)$$

(23) is the most interesting case because it represents one of the classes of analytical solutions for the function y , *i. e.* (A.29). This result is consistent with previous results by McVittie and Stabell [9] and by Bonnor and Faulkes [10]. They showed that in their case the density is a function of time alone. Both cases are subcases of (A.29). The case treated by McVittie and Stabell is the case $a \neq 1$, $a \neq 3$, $b = 2 - a$. The case treated by Bonnor and Faulkes is the case $b = 0$, $f = r$.

If (24) is fulfilled, then the pressure gradient (17) and the condition (13) for the pressure at the boundary yield that the pressure is always and everywhere zero. The condition (2) for y and η yields that y is a function of time alone and the metric (1) may be written in the following form:

$$ds^2 = y^2(t)dt^2 - R_0^2 S^2 G(r) I(t) [dr^2 + f^2(d\theta^2 + \sin^2 \theta d\phi^2)] \quad (26)$$

where G is a positive function of r alone and

$$I(t) = e^{2 \int \frac{S_t}{S[y(t)-1]} dt}$$

It was further proved in [1] that when $Q_r = 0$, f may be written:

$$f = \frac{1}{C} \sin Cr, \quad r, \quad \frac{1}{C} \sinh Cr$$

where C is a positive constant. (26) is of course one of the forms of a Robertson-Walker-metric for pressureless dust.

The equation (25) may be written:

$$f = gQ^{\frac{1-2a}{5}} \quad (27)$$

where g is a positive constant.

5. ON THE POSSIBILITY OF OSCILLATIONS

5.1 $y = \text{constant}$.

Using equations (2), (3) and the condition (13), $p_b \equiv 0$, from (14) the following differential equation is obtained for the scale-function S:

$$S_r^2 = -\frac{c^2}{R_0^2} NS^{2-3y}(S^y - S_0^y) \tag{28}$$

where N is a constant and S_0 is the initial value of S. Henceforth initial values will be denoted by the suffix 0. Since the function on the right hand side of (28) is monotonic, oscillatory motion is not possible.

The four different classes of analytical solutions (A.26) to (A.29) will now be investigated with regard to oscillatory motion as far as a definite conclusion is not already drawn in [6].

5.2 A.26.

$$a \neq 3, y \neq \frac{2}{5}(3 - a)$$

(7) then yields the following equation:

$$y = -\frac{2}{5}k(a - 3) \frac{e^{\frac{a-3}{5}z}}{1 + ke^{\frac{a-3}{5}z}} \tag{29}$$

where the integration constant k should be positive to avoid a singular metric. Differentiating (29) with respect to z and using the condition (20) it is found that

$$Q_r < 0 \tag{30}$$

The condition that the density gradient (22) must be non-positive may be written:

$$8\pi G\rho_r = \frac{3c^2 e^{-\eta} Q_r}{R_0^2 S^2 Q} \left[y + \frac{2}{5}(a - 3) \right]^2 \left[5 \frac{f_r Q_r}{f Q} + (2a - 1) \frac{Q_r^2}{Q^2} \right] \leq 0 \tag{31}$$

From (31) we note the interesting result that in this case the non-positiveness of the density gradient is only dependent upon the variable r , not upon the variable y .

The condition $(\rho_r)_b \leq 0$ may now be written:

$$5B_2 + (2a - 1)B_3 \geq 0 \tag{32}$$

The constants B_1, B_2, B_3 , are defined in [6]. The following equation for the scale-function is obtained:

$$S_i^2 = -U \frac{c^2 e^\varepsilon}{R_0^2} \left(X^2 + 2 \frac{V}{U} X + \frac{W}{U} \right) \frac{S^2}{(1+X)^6} \quad (33)$$

where

$$X = kS^{\frac{3-a}{5}} \quad (34)$$

$$U = B_1 - \frac{2(2a-1)}{5} B_2 - \frac{(2a-1)^2}{25} B_3 \quad (35)$$

$$V = B_1 - \frac{2(a+2)}{5} B_2 - \frac{2a-1}{5} B_3 \quad (36)$$

W and ε are arbitrary integration constants. From (33) the following necessary oscillatory conditions are then obtained:

$$U > 0, V < 0, W > 0 \quad (37)$$

From (37) it is easily obtained:

$$5B_2 + (2a-1)B_3 > 0 \quad (38)$$

consistent with (32), but the strict condition $(\rho_r)_b = 0$ is thus not compatible with oscillatory motions.

Defining a constant Z by

$$Z = B_1 - 2B_2 + \frac{1}{25}(2a-1)(2a-11)B_3 \quad (39)$$

the condition $\rho_{b,0} \geq 0$ may now be written:

$$Z + UX_0 \geq 0 \quad (40)$$

The restriction that the acceleration must be negative at the boundary at the initial moment, *i. e.* $(S_{tt})_{b,0} < 0$ may be written using (12) for the pressure and the condition (13), $p_b \equiv 0$:

$$V + UX_0 > 0 \quad (41)$$

If the strict condition $\rho_{b,0} = 0$ is imposed, then the following inequality is easily derived from (40) and (41):

$$5B_2 + (2a-1)B_3 > 0 \quad (42)$$

consistent with (32) and (38).

In [6] it was shown that when $k = 1$ and $S_0 = 1$ oscillations are not possible. It will be shown below that oscillations are not possible in the more general case $kS_0^{\frac{3-a}{5}} = 1$.

The condition $\rho_{b,0} \geq 0$ yields:

$$Z + U \geq 0 \quad (43)$$

But from the definitions of U, V, Z the following equation is obtained:

$$Z + U = 2V \tag{44}$$

In this case the necessary oscillatory conditions (37) and the condition $\rho_{b,0} \geq 0$ are incompatible. Oscillatory motions are thus not possible in this subcase. The restriction $kS_0^{\frac{3-a}{5}} = 1$ for case (A.26) is from now on relaxed.

If one makes the requirement $\rho_b \equiv 0$, the following equation for the scale function is obtained from (11):

$$S_t^2 = -U \frac{c^2 e^{\epsilon}}{R_0^2} \left(X^2 + 2 \frac{V}{U} X + \frac{Z}{U} \right) \frac{S^2}{(1+X)^6} \tag{45}$$

This equation is quite similar to equation (33) and again the necessary oscillatory conditions ($U > 0, V < 0, Z > 0$) and the equation (44) are incompatible. Hence oscillatory motions are not possible.

In the passing we note that equation (45) is a special case of equation (33), and for a proper choice of an integration constant the condition $p_b \equiv 0$ thus yields $\rho_b \equiv 0$, *i. e.* the matter is gaseous. The possibility of constructing a time dependent gaseous model is left for a later investigation

$a = 3$.

(A.26) yields the following equation:

$$y = \frac{1}{k - \frac{1}{2}z} \tag{46}$$

Using condition (20) it is now found that:

$$Q_r > 0 \tag{47}$$

In our case $a = 3, b = -1$, equation (5) may be solved for f and Q by the same method specified in [6], page 351, where the same equation was solved when $a = 5, b = -4$. The results are the same and are given in [6] by equations denoted by (65) to (73) in that paper. Irrespective of the integration constant P found in [6] it is seen that one must impose the restriction $q > 0$. This yields immediately:

$$Q_r < 0 \tag{48}$$

in contradiction to (47). For this model the condition $p_r < 0$ thus cannot be fulfilled, and we discard it as physically unacceptable.

5.3 A.27.

$a \neq 3, y \neq 3 - a.$

(8) then yields the following equation:

$$y = \frac{(3 - a)ke^{(3-a)z}}{1 + ke^{(3-a)z}} \quad (49)$$

where the integration constant k should be positive to avoid a singular metric. The following differential equation is obtained for the scale-function:

$$S_t^2 = \frac{c^2 e^\varepsilon}{R_0^2} S^2 (\alpha_1 X^3 + \beta_1 X^2 + \gamma_1 X + \delta_1) \quad (50)$$

where

$$X = kS^{a-3} \quad (51)$$

ε and α_1 are arbitrary integration constants, and $\beta_1, \gamma_1, \delta_1$ are constants defined in the following way:

$$\beta_1 = 3\alpha_1 - [B_1 - 2(a - 2)B_2 - (a - 2)^2 B_3] \quad (52)$$

$$\gamma_1 = 3\alpha - 2 [B_1 - (a - 1)B_2 - (a - 2)B_3] \quad (53)$$

$$\delta_1 = \alpha_1(B_1 - 2B_2 - B_3) \quad (54)$$

Necessary oscillatory conditions are then that the polynomial on the right hand side of (50) has at least two positive roots and that the polynomial is positive between these two roots.

The condition that the density must be positive at the boundary at the initial moment, *i. e.* $\rho_{b,0} > 0$, may now be written:

$$[B_1 - 2(a - 2)B_2 - (a - 2)^2 B_3]X_0^2 + 2[B_1 - (a - 1)B_2 - (a - 2)B_3]X_0 + [B_1 - 2B_2 - B_3] > 0 \quad (55)$$

In [6] it was shown that when $k = 1$ and $S_0 = 1$ oscillations are impossible. It will be shown below that oscillations are not possible in the more general case $kS_0^{a-3} = 1$. Because the mass is at rest at the initial moment, *i. e.* $(S_t)_0 = 0$ it must be the case that:

$$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = 0 \quad (56)$$

(55) and (56) now yield:

$$\alpha_1 > 0 \quad (57)$$

Inspection of (50) then shows that oscillations are only possible if:

$$\alpha_1 + \beta_1 + \gamma_1 > 0 \quad (58)$$

$$\alpha_1 + \beta_1 < 0 \quad (59)$$

Combining (56), (57), (58), (59) the following relation is obtained:

$$\beta_1 + \delta_1 < 0 \quad (60)$$

But (60) is equivalent to the following inequality which cannot be fulfilled:

$$(a - 3)^2 B_3 < 0 \tag{61}$$

Hence, oscillatory motions are not possible in this case.

$a = 3$.

In this case (8) yields the following equation:

$$y = \frac{1}{z + k} \tag{62}$$

where k is an integration constant. Using the equation (13), $p_b \equiv 0$, the following differential equation is now obtained for the scale-function:

$$S_t^2 = -\frac{c^2 e^{\epsilon}}{R_0^2} S^2 [AX^3 + (B_1 - 2B_2 - B_3)X^2 + 2(B_2 + B_3)X - B_3] \tag{63}$$

where

$$X = k - \ln S \tag{64}$$

and A is an arbitrary integration constant. The definition (2) of z and the condition (15) for Q_b yield then, since y must be positive:

$$X > 0 \tag{65}$$

A necessary oscillatory condition is then:

The polynomial on the right hand side of (63) has at least two positive roots and the polynomial is positive between these two roots.

The physical condition that the density must be positive at the boundary at the initial moment, *i. e.* $\rho_{b,0} > 0$, may now be written:

$$B_1 - 2\left(1 - \frac{1}{X_0}\right)B_2 - \left(1 - \frac{1}{X_0}\right)^2 B_3 > 0 \tag{66}$$

If one puts $k = 1$ and $S_0 = 1$ as in [5] and [6], (66) turns into:

$$B_1 > 0 \tag{67}$$

But from the oscillatory condition it may be easily seen now that:

$$B_1 < 0 \tag{68}$$

in contradiction to (67).

Hence, no oscillatory motions are possible in this subcase.

5.4 A. 28.

In [6] it was found that oscillations are only possible if $b > -1$.

In [5] δ is defined by $\delta^2 = b + 1$.

$y \neq \pm \delta$.

(9) then yields the following equation:

$$y = \delta \frac{1 - ke^{\delta z}}{1 + ke^{\delta z}} \quad (69)$$

where the integration constant k should be positive to avoid a singular metric. Differentiating (69) with respect to z and using the condition (20) it is found that:

$$Q_r < 0 \quad (70)$$

If we restrict the analysis to the case $kS_0^{-\delta} = 1$ the condition that the density gradient (22) must be non-positive at the boundary at the initial moment *i. e.* $(\rho_r)_{b,0} \leq 0$, may now be written:

$$B_2 + B_3 \leq 0 \quad (71)$$

Using as before the condition $(S_{tt})_{b,0} < 0$, we now obtain:

$$B_2 + B_3 > 0 \quad (72)$$

in contradiction to (71).

This model must thus be discarded as physically unacceptable. The case $k = 1$, $S_0 = 1$ discussed in [5] and [6] is a subcase of this case. The restriction $kS_0^{-\delta} = 1$ for case (A.28) is from now on relaxed.

In [5], page 379, we find the following way of dealing with $b \neq 0$, suggested by W. B. Bonnor. A coordinate transformation can be found which changes the (t, r) of equation (1) into (t^*, r^*) and leaving the form of (1) unchanged. It turns out that the transformation can be chosen so that the constant b in (5) is zero. However, the transformation cannot be used when $b = 2 - a$, so for the case (A.27) this line of attack is abandoned. In the remaining investigation of case (A.28), full advantage is taken of this transformation, so that we only have to deal with the simple case $b = 0$, *i. e.* $\delta = \pm 1$.

$\delta = +1$.

The condition $(\rho_r)_{b,0} \leq 0$ again reads:

$$B_2 + B_3 \leq 0 \quad (73)$$

The restriction $(S_{tt})_{b,0} < 0$ may be written:

$$(1 - X_0^2)B_1 + 4X_0^2B_2 + 4X_0^2B_3 > 0 \quad (74)$$

where

$$X_0 = k/S_0$$

The condition $\rho_{b,0} \geq 0$, may be written:

$$-(1 + X_0)X_0B_1 + 4X_0^2B_2 + 4X_0^2B_3 \leq 0 \quad (75)$$

Combining (73) and (74) it is then easily found that:

$$B_1 > 0 \tag{76}$$

Combining (73), (74), (75) it is also easily found that:

$$k < S_0 \tag{77}$$

This is clearly not compatible with putting $k = 1, S_0 = 1$ as it is done in [5] and [6].

In this case the following equation for the scale-function is obtained.

$$S_t^2 = -\frac{c^2 e^\epsilon}{R_0^2} B_1 \frac{S^4}{(S+k)^6} (S - S_0)(S - S_1) \tag{78}$$

where

$$S_0 = \frac{-D + \sqrt{D^2 - 4B_1 E}}{2B_1} \tag{79}$$

$$S_1 = \frac{-D - \sqrt{D^2 - 4B_1 E}}{2B_1} \tag{80}$$

$$E = k^2(B_1 - 4B_2 - 4B_3) \tag{81}$$

D is an integration constant fulfilling the condition $D^2 > 4B_1 E$. From (78) it is seen that necessary oscillatory conditions may be written:

$$B_1 > 0 \tag{82}$$

which is consistent with (76).

$$B_1 - 4B_2 - 4B_3 > 0 \tag{83}$$

$$D < 0 \tag{84}$$

Using (82) and (83) in (75), it is then obtained that:

$$\rho_{b,0} > 0 \tag{85}$$

i. e. the density at the boundary at the initial moment cannot be zero. From (82) it is immediately seen that only column 1 of Table I in [4] is compatible with oscillations. Table I also yields:

$$\alpha\beta > 0 \tag{86}$$

and for convenience define:

$$Y = \alpha^{-2} - \beta^2 > 0 \tag{87}$$

$\alpha > 0, \beta > 0$.

The condition $(\rho_r)_{b,0} \leq 0$ now yields:

$$Y < 0 \tag{88}$$

in contradiction to (87).

This model must thus be discarded as physically unacceptable.

$$\alpha < 0, \beta < 0.$$

Oscillatory condition from the scale-function equation:

$$Y < -2\beta + 1 \quad (89)$$

Non-positive density gradient, $(\rho_r)_{b,0} < 0$:

$$Y \leq -2\beta \quad (90)$$

Negative acceleration, $(S_{tt})_{b,0} < 0$:

$$Y > -2\beta + 1 - \frac{1}{X_0^2} \quad (91)$$

Positive acceleration at the boundary at the bounce:

$$Y < -2\beta + 1 - \frac{1}{X_{\text{bounce}}^2}$$

A necessary, but not sufficient requirement for a physically acceptable model is then that (β, Y) lies in the shaded area of Figure 2.

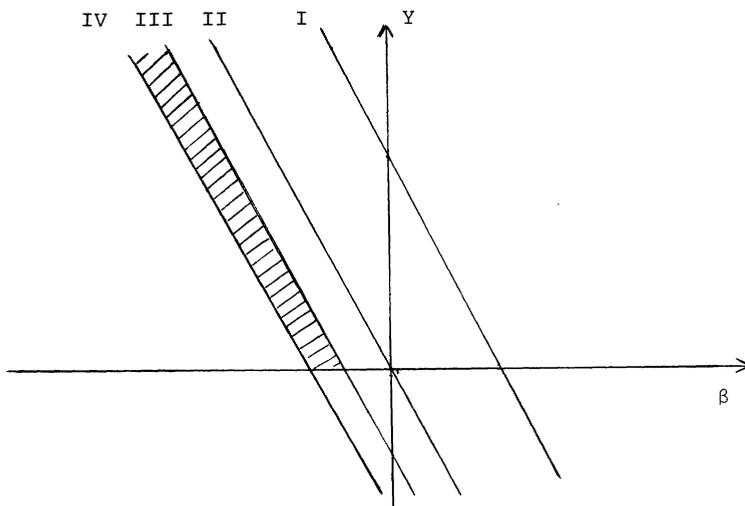


FIG. 2. — Inequalities Diagram. I is $Y = -2\beta + 1$, II is $Y = -2\beta$,
 III is $Y = -2\beta + 1 - \frac{1}{x_{\text{bounce}}^2}$, IV is $Y = -2\beta + 1 - \frac{1}{x_0^2}$.

$$\delta = -1.$$

The analysis and principal results are very much the same as in the previous case, $\delta = 1$, so we do not write it out in detail.

The oscillatory model found by Nariai [7] is a special case of this solution.

5.5 A.29.

(22) yields that in this case the density is uniform. Since the general non-singular solution for a time-dependent shear-free sphere of uniform density is given by Gupta [11], we will not give a detailed analysis for this case. A detailed investigation of the possibilities given by Gupta's work will be published elsewhere.

When we put $b = 0$, (10) yields the following equation:

$$y = \frac{1 + (2 - a)ke^{(1-a)z}}{1 + ke^{(1-a)z}} \tag{92}$$

The following equation for the scale-function is then obtained:

$$S_t^2 = -\frac{c^2 e^e (1 + X)^3}{R_0^2 S} \int \left[\frac{\omega_1}{(1 + X)} + \frac{\omega_3}{(1 + X)^2} + \frac{\omega_3}{(1 + X)^3} + \frac{\omega_4 X}{(1 + X)^2} + \frac{\omega_5 X}{(1 + X)^3} + \frac{\omega_6 X}{(1 + X)^4} + \frac{\omega_7 X^2}{(1 + X)^3} + \frac{\omega_8 X^2}{(1 + X)^4} + \frac{\omega_9 X^3}{(1 + X)^4} \right] dS \tag{93}$$

where $\omega_1, \dots, \omega_9$ are constants implicitly defined and $X = kS^{a-1}$. From (93) it seen that in this case S_t^2 cannot be expressed in elementary functions of S if a is not specified. Hence, it is impossible to find necessary oscillatory conditions from (93). We also note from (18) that:

$$\rho_b \equiv 0 \Rightarrow p_b \equiv 0 \tag{94}$$

for a non-static model.

A method used in [6] when the case (A.29) was investigated in that paper, is in contradiction to (94).

The results in [6] is in fact not correct. But in view of the new result in this paper, *i. e.* $\rho_r = 0$ for the case (A.29), those mistakes are not important any more.

6. SUMMARY

A group of solutions of Einstein's field equations found by McVittie [1] is investigated. It is found that for one analytical class, (A.29), of this group, the density is uniform. With regard to oscillatory motions the following results are obtained:

- A) If the density at the boundary is zero throughout the motion, *oscillations are not possible.*
- B) If $y = \text{constant}$, the differential equation for the scale-function yields that *oscillations are not possible.*
- C) Case (A.27), $S_0 = 1, k = 1, a = 3$ or $kS_0^{a-3} = 1, a \neq 3$: oscillations are *not possible.*

- D) Case (A.26), $kS_0^{\frac{3-a}{5}} = 1$, $\rho_{b,0} \geq 0$: oscillation is *not* possible.
 E) Case (A.26), $a = 3$, $p_r < 0$ cannot be fulfilled.
 F) Case (A.28), $kS_0^{-\delta} = 1$, conditions $(S_{tt})_{b,0} < 0$ and $(\rho_r)_{b,0} \leq 0$ are in contradiction.
 G) Case (A.28), $\alpha > 0$, $\beta > 0$: oscillations are *not* possible.
 H) Case (A.28), $\alpha < 0$, $\beta < 0$: it is possible to fulfill all the conditions $(\rho_r)_{b,0} \leq 0$, $\rho_{b,0} > 0$, $(S_{tt})_{b,0} < 0$ and the necessary oscillatory conditions from the equation for the scale-function. The oscillatory model found by Nariai belongs to this case.

But it must be stressed that though these conditions are necessary, they are not sufficient to secure a physically acceptable oscillatory model.

7. A COMMENT ON OSCILLATORY MOTION IN EINSTEIN-CARTAN THEORY

Using a so-called « classical » description of spin in Einstein-Cartan theory for a spherical symmetric mass distribution Kuchowicz [12] has shown that the expressions for both the energy density ρ and the pressure p will differ from the corresponding expressions in general relativity only by an additive term proportional to \bar{Q}^2 . \bar{Q} is here the only non-vanishing component of the torsion tensor. Kuchowicz also gives arguments why \bar{Q} should be put equal to zero at the boundary, *i. e.* $\bar{Q}_b = 0$.

Following Kuchowicz and using (1) as the metric for the spherical mass distribution, the following result is immediately obtained: Since all our conclusions concerning oscillatory motions are based on pressure, density, density gradient and acceleration, all taken at the boundary, *all our results concerning oscillatory motions are also valid in this version of Einstein-Cartan theory.*

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