ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 40, nº 1 (1984), p. 21-34
<http://www.numdam.org/item?id=AIHPA_1984__40_1_21_0>

Ann. Inst. Henri Poincaré,
Vol. 40, n° 1, 1984, p. 21-34. *Physique théorique*

A gauge theoretical approach to space-time structures

by

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ABSTRACT. — We discuss a general gauge theoretical scheme leading to space-time structures. This is applied to the Poincaré group, the Galilei group and its central extension as gauge groups.

It is shown how theories of gravitation can be formulated as Yang-Mills type gauge theories with a Goldstone field.

Dealing with the central extension of the Galilei group we obtain as a by-product a characterization of Newtonian connections in terms of a generalized torsion.

RÉSUMÉ. — On discute un schéma général de théories de jauge conduisant à des structures d'espace-temps. On l'applique aux cas où l'on prend pour groupe de jauge le groupe Poincaré, le groupe de Galilée, ou son extension centrale. On montre comment les théories de la gravitation peuvent être formulées comme théories de jauge du type de Yang-Mills avec un champ de Goldstone. Dans le cas de l'extension centrale du groupe de Galilée, on obtient comme sous produit une caractérisation des connections Newtoniennes en termes de torsion généralisée.

INTRODUCTION

It is well-known that theories of gravitation based on a Lorentz metric and a metric compatible linear connection show an internal Lorentz symmetry. This enables us to deal with fields which are representations of the Lorentz group in an arbitrary space-time. Quantum theory, however, is dominated by the Poincaré group, i. e. the inhomogeneous Lorentz group. From this point of view it is therefore convenient to look for a way to introduce the Poincaré group in a general space-time (cf. also [*1*]). Such a way is provided by the theory of fiber bundles which enables us to extend the internal Lorentz symmetry to an internal Poincaré symmetry.

An orthodox gauge theoretical approach starts with a Poincaré bundle and a connection on this bundle ([*1*]-[*4*]). The physical fields are cross sections of bundles associated with the (principal) Poincaré bundle. The space-time structure is then obtained as a secondary concept via the introduction of a « Goldstone field » (cf. [*5*] and the references given there for the sense in which we use this notion here).

In the prerelativistic case we meet with a similar structure. The covariant formulation of Newton's theory of gravitation leads to a Galilei structure (i. e. a Galilei subbundle of the bundle of linear frames) and Galilei connections [*6*] so that the theory admits the homogeneous Galilei group as an internal symmetry group. Quantum mechanics deals with projective (ray) representations of the inhomogeneous Galilei group, respectively representations of the central extension of the Galilei group (cf. e. g. [*7*]). The fiber bundle geometry opens a way to introduce the inhomogeneous Galilei group respectively its central extension in a general Galilei space-time. Again, we can start with a principal fiber bundle with structure group the inhomogeneous Galilei group or its central extension and derive the space-time structure via a symmetry breaking procedure from the gauge theoretical framework.

The present work is understood as a first step to the program indicated above, concentrating mainly on the geometry of the gauge field and leaving aside the question how to deal with matter fields (cf. [*1*] for the case of the Poincaré group).

In order to display some very general features we present a gauge theoretical scheme for deriving space-time structures which in particular includes a treatment of the Poincaré group ([*2*]-[*4*]), the affine group [*8*] and the prerelativistic groups.

Dealing with the central extension of the Galilei group we obtain an extended canonical 1-form on a Galilei structure which transforms according to a five-dimensional representation of the homogeneous Galilei group (if the base manifold is four-dimensional). Due to the fact that in the prerelativistic case we actually need a mass-momentum-energy tensor [*9*] instead of the familiar (relativistic) energy-momentum tensor as source for the geometry, the five-dimensional representation of the homogeneous Galilei group appears to be of special interest (cf. also [*10*]-[*11*]). Associated with the extended canonical 1-form is an extended torsion form in terms of which we derive a characterization of a class of Galilei connections, the Newtonian connections [*6*]. A similar result has been obtained in [*12*] with different methods. Newtonian connections arise through the limit relation: Einstein's gravitation $\rightarrow$ Newton's gravitation (cf. [*13*] and references given there).

Section 1 presents the general framework we are dealing with. The following three sections treat respectively the Poincaré group, the inhomogeneous Galilei group and its central extension as specific examples. Here we contrast the « explicit » symmetry breaking procedure outlined in section 1 with an « implicit » symmetry breaking where a Goldstone field is used to construct quantities which transform according to a representation of the homogeneous (Lorentz respectively Galilei) group under the action of the full group.

1. THE GENERAL FRAMEWORK

Let $\hat{G}$ be a Lie group and G a closed subgroup. We assume that the Lie algebra $\hat{\mathcal{G}}$ of $\hat{G}$ admits a linear subspace V such that [*14*]

$$\hat{\mathcal{G}} = V + \mathcal{G} \qquad \text{(direct sum)} \tag{1.1}$$

and

$$\operatorname{ad}(G)V = V. \tag{1.2}$$

Furthermore, let $\hat{P}$ be a (principal) $\hat{G}$-bundle over a paracompact n-dimensional manifold M.

For many groups $\hat{G}$ such a bundle always admits a reduction of the structure group [*15*]

$$f : P \rightarrow \hat{P}$$

(with group homomorphism the inclusion map $G \hookrightarrow \hat{G}$) where P is a G-bundle over M. A well-known necessary and sufficient condition for a reduction to exist is the existence of a global cross section of the associated bundle

$$E = \hat{P} \times_{\hat{G}} (\hat{G}/G) \tag{1.3}$$

([*15*], p. 57). This is in particular the case if $\hat{G}/G$ is diffeomorphic with a Euclidean space $\mathbb{R}^m$ ([*15*], p. 58).

Given a reduction f and a connection form $\hat{\omega}$ on $\hat{P}$ we can decompose the pull-back of $\hat{\omega}$ to P as follows

$$f^* \hat{\omega} = \underbrace{\tilde{\omega}}_{\in \mathcal{G}} + \underbrace{\tilde{\psi}}_{\in V} \tag{1.4}$$

Using the properties of the reduction map and the structure of the Lie algebra $\hat{\mathcal{G}}$ one obtains (cf. [*15*], p. 83):

LEMMA 1. — (1) $\tilde{\omega}$ is a connection form on P.

(2) $\tilde{\psi}$ is a tensorial 1-form of type (ad, V) on P. □

In order to end up with the geometric structure of theories of gravitation we need contact with the bundle L(M) of linear frames on M. To make this possible we restrict G to be a subgroup of the general linear group:

$$G \subset GL(n, \mathbb{R}).$$

Defining a soldering form [*14*] on P to be a tensorial 1-form ψ of type (id, $\mathbb{R}^n$) with

$$\operatorname{rank}(\psi) = n \tag{1.5}$$

we get the following result which essentially has been mentioned in [*16*].

LEMMA 2. — Let G be any subgroup of GL $(n, \mathbb{R})$ and P a G-bundle over M. P is isomorphic with a G-structure (i. e. a G-subbundle of L(M)) if and only if a soldering form on P exists.

Proof. — If $\kappa : P \to \kappa(P) \subset L(M)$ is an isomorphism of principal bundles, then

$$\psi = \kappa^* \theta \tag{1.6}$$

is a soldering form on P where θ denotes the canonical 1-form on L(M) restricted to $\kappa(P)$.

Conversely, if ψ is a soldering form on P we define a map

$$\kappa : P \to L(M)$$

by

$$p \mapsto (\sigma^* \psi \,|_{\pi(p)})_{\text{dual}}$$

where σ is a local cross section through p. Since ψ is tensorial, κ does not depend on the choice of σ. The condition (1.5) ensures that $\sigma^* \psi \,|_{\pi(p)}$ is a coframe which determines a (dual) basis of the tangent space $T_{\pi(p)}M$, i. e. a pointof L(M) ($\pi : P \to M$ denotes the bundle projection). κ is easily shown to be an isomorphism onto its image. Moreover, κ is a reduction of the structure group GL $(n, \mathbb{R})$ of L(M) to G. □

The resulting problem is now whether we can use $\tilde{\psi}$ to construct a soldering form ψ on P. If this can be done we get the following situation:

$$\begin{array}{ccc} \hat{\omega} & \hat{P} & \\ \vdots & \uparrow f & \\ \vdots & P \to \kappa(P) \subset L(M) & \\ \downarrow & & \\ \tilde{\omega}, \psi & \dashrightarrow & \omega, \theta \\ & & (+\text{ additional fields}) \end{array}$$

with the connection form

$$\omega = (\kappa^{-1})^* \tilde{\omega}. \tag{1.7}$$

The cross section of E which determines the reduction map f is equivalently described by a tensorial 0-form Φ on $\hat{P}$ with values in $\hat{G}/G$. Φ and f are related by

$$\Phi \circ f(p) = G \qquad (\forall p \in P). \tag{1.8}$$

The soldering condition (1.5) cannot be fulfilled in general. To illustrate this fact consider the trivial bundle $P = M \times GL(n, \mathbb{R})$. It follows from Lemma 2 that P admits a soldering form if and only if M is parallelizable which is a severe restriction on the manifold M.

In the following sections we discuss the construction outlined above in some more detail for the Poincaré group, the inhomogeneous Galilei group and its central extension as examples for $\hat{G}$. Furthermore, the « Goldstone field » Φ [5] will be used to formulate theories of gravitation with an explicit $\hat{G}$ symmetry.

2. THE POINCARÉ GROUP

In this section we consider the case where $\hat{G}$ is the Poincaré group, i. e. the semidirect product of $0(n - 1, 1)$ with the group of translations in n dimensions. G is chosen to be the Lorentz group $0(n - 1, 1)$ and V the subspace of $\hat{\mathscr{G}}$ corresponding to the translations.

Using the $(n + 1)$-dimensional matrix representation of the Poincaré group

$$(\Lambda, t) = \begin{pmatrix} \Lambda & t \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

$$\Lambda \in 0(n - 1, 1), \qquad t \in \mathbb{R}^n$$

and the induced representation of the Lie algebra we find

$$\tilde{\psi} = \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

where ψ is tensorial of type (id, $\mathbb{R}^n$) on the Lorentz bundle P. Demanding the soldering condition (1.5), ψ determines a Lorentz structure on M which in turn defines a Lorentz (pseudo-Riemannian) metric g on M. ω is a linear connection compatible with g.

An essential point is that the « tetrad field » (g-orthonormal coframe field) arises from the translational part of the Poincaré connection on $\hat{P}$ (cf. also [2]-[4]).

The quotient space $\hat{G}/G$ can be identified with $\mathbb{R}^n$ in a natural way. Using this identification we obtain the following transformation rule for the Goldstone field Φ under the right action of $\hat{G}$ on $\hat{P}$:

$$\hat{\mathscr{R}}^*_{(\Lambda,t)}\begin{pmatrix} \Phi \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & t \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \Phi \\ 1 \end{pmatrix}. \tag{2.3}$$

This shows the existence of a gauge with respect to which Φ vanishes. This is also implied by (1.8) which in the case under consideration becomes

$$\Phi \circ f = 0 . \tag{2.4}$$

We conclude that Φ plays no physical role in the theory.
But it enables us to formulate a gravitational theory on $\hat{\mathrm{P}}$, i. e. with an internal $\hat{\mathrm{G}}$ symmetry. This will be outlined in the following.

For the covariant derivative of Φ with respect to $\hat{\omega}$ we get

$$\mathscr{R}^{*}_{(\Lambda,t)}\hat{\mathrm{D}}\Phi = \Lambda^{-1}\hat{\mathrm{D}}\Phi . \tag{2.5}$$

In particular, $\hat{\mathrm{D}}\Phi$ is invariant under internal translations. Furthermore,

$$f\,{}^{*}\hat{\mathrm{D}}\Phi = \psi \tag{2.6}$$

by use of (2.4). The condition (1.5) is therefore equivalent to

$$\operatorname{rank} \hat{\mathrm{D}}\Phi = n \tag{2.7}$$

(cf. [*2*]). Due to this condition, the set of 1-forms $\hat{\omega}$ and $\hat{\mathrm{D}}\Phi$ provide a parallelization of $\hat{\mathrm{P}}$. Each differential form on $\hat{\mathrm{P}}$ can be expressed in terms of this coframe field.

The curvature form of

$$\hat{\omega} = \left(\begin{array}{c|c} \hat{\omega}^{i}{}_{j} & \hat{\omega}^{i} \\ \hline 0 & 0 \end{array}\right) \tag{2.8}$$

is given by

$$\hat{\Omega} =. d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = \left(\begin{array}{c|c} \hat{\Omega}^{i}{}_{j} & \hat{\Omega}^{i} \\ \hline 0 & 0 \end{array}\right) \tag{2.9}$$

with

$$\hat{\Omega}^{i}{}_{j} = d\hat{\omega}^{i}{}_{j} + \hat{\omega}^{i}{}_{k} \wedge \hat{\omega}^{k}{}_{j} \tag{2.10}$$

$$\hat{\Omega}^{i} = d\hat{\omega}^{i} + \hat{\omega}^{i}{}_{j} \wedge \hat{\omega}^{j} . \tag{2.11}$$

In contrast to $\hat{\Omega}^{i}$, $\hat{\Omega}^{i}{}_{j}$ is invariant under translations. We define the curvature tensor by

$$\hat{\Omega}^{i}{}_{j} = \frac{1}{2}\mathrm{R}^{i}{}_{jkl}\hat{\mathrm{D}}\Phi^{k} \wedge \hat{\mathrm{D}}\Phi^{l} \tag{2.12}$$

and introduce a torsion form and a torsion tensor:

$$\hat{\Theta}^{i} = \hat{\mathrm{D}}^{2}\Phi^{i} = \frac{1}{2}\mathrm{Q}^{i}{}_{kl}\hat{\mathrm{D}}\Phi^{k} \wedge \hat{\mathrm{D}}\Phi^{l} . \tag{2.13}$$

Here the curvature and the torsion tensor are defined as 0-forms on $\hat{\mathrm{P}}$. It should be noticed, however, that they are invariant under internal translations. Using $\eta = \operatorname{diag}(-1, 1, \ldots, 1)$ we can therefore build (gauge invariant) scalars on $\hat{\mathrm{P}}$ from $\mathrm{R}^{i}{}_{jkl}$ and $\mathrm{Q}^{i}{}_{kl}$. In this way we arrive at all types of Lagrangians for the gravitational field which are possible in the usual framework formulated on L(M) with M being four-dimensional. As an example, the Einstein-Cartan Lagrangian on $\hat{P}$ reads (cf. [2])

$$L_{EC} = \eta_{ij}\hat{\Omega}^i{}_k \wedge * (\hat{D}\Phi^j \wedge \hat{D}\Phi^k) \tag{2.14}$$

where the $*$-operator is defined on the (horizontal) basis $\hat{D}\Phi^i$ in the usual way. Furthermore, a Lagrangian proposed in [17]-[18] can be written as follows:

$$L_H = \frac{1}{2\kappa}\hat{\Omega}^i{}_j \wedge * \hat{\Omega}^j{}_i - \frac{1}{2l^2}(\hat{\Theta}^i \wedge \hat{D}\Phi^j) \wedge * (\hat{\Theta}_j \wedge \hat{D}\Phi_i) \tag{2.15}$$

(l Planck length, κ a coupling constant).

The dynamical variables of the gravitational field are the Poincaré connection $\hat{\omega}$ and the Goldstone field Φ. Variation of the translational part of $\hat{\omega}$ is equivalent to variation of the « tetrad field » $\hat{D}\Phi$. Since Φ enters only through $\hat{D}\Phi$, the equation resulting from variation of Φ turns out to be a consequence of the « tetrad field equation ».

Imposing the condition

$$\hat{\Theta}^i = 0\,, \tag{2.16}$$

the Lorentz part of the connection $\hat{\omega}$ is uniquely fixed by the tetrad field (respectively the corresponding metric g). In this way we recover Einstein's theory from the Lagrangian (2.14).

Due to the introduction of the Goldstone field Φ we have been able to extend the internal Lorentz symmetry of gravitational theories to an internal Poincaré symmetry. The formulation presented here should be compared with that given in [5].

The availability of a Goldstone field is necessary for the minimal coupling procedure proposed in [4].

3. THE INHOMOGENEOUS GALILEI GROUP

In this section the inhomogeneous Galilei group takes the role of $\hat{G}$. A matrix representation of this group is given by

$$\left(\begin{array}{cc|c} 1 & 0 & \\ \vec{v} & W & t \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right) \tag{3.1}$$

with $\quad \vec{v} \in \mathbb{R}^{n-1}, \quad t \in \mathbb{R}^n, \quad W \in O(n-1)\,.$

Let G be the homogeneous Galilei group and V again the subspace of $\hat{\mathcal{G}}$ spanned by the generators of translations. Obviously, we can proceed by analogy with the treatment of the Poincaré group. In the case under consideration we end up with a Galilei structure on M which is characterized by a pair (Ψ, γ) of a nowhere vanishing 1-form

$$\Psi = \Psi_\mu dx^\mu \tag{3.2}$$

and a positive semi-definite symmetric tensor field

$$\gamma = \gamma^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \tag{3.3}$$

of rank $n - 1$ on M subject to

$$\gamma^{\mu\nu}\Psi_\nu = 0 \tag{3.4}$$

[6]. ω given by equation (1.7) is a Galilei connection, i. e.

$$\mathrm{D}\Psi_i = 0 \tag{3.5}$$

$$\mathrm{D}\gamma^{ij} = 0 \tag{3.6}$$

where the indices now refer to an arbitrary frame and D denotes the exterior covariant derivative associated with ω. In terms of these objects Newton's theory of gravitation can be formulated in a covariant way (cf. [6] [13] and references given there).

We denote a Galilei structure by $\mathrm{L}^{\Psi,\gamma}(\mathrm{M})$. On $\mathrm{L}^{\Psi,\gamma}(\mathrm{M})$ we have

$$\Psi_i = \delta_i^0 \tag{3.7}$$

$$\gamma^{ij} = \delta^i_{\mathrm{A}}\delta^j_{\mathrm{B}}\delta^{\mathrm{AB}} \tag{3.8}$$

(A, B $= 1, \ldots, n - 1$). For more details about the geometry and physics of Galilei structures we refer to [6].

The equations of section 2 apply equally well to the case under consideration if Λ is replaced in (2.1) and the subsequent equations by an element of the homogeneous Galilei group. The only exception is that we are not able to introduce a (invertable) $*$-operator since the homogeneous Galilei group does not allow a non-degenerate invariant metric on $\mathbb{R}^n$. The tensors on $\mathbb{R}^n$ which are invariant under this group are just given by (3.7) and (3.8).

Nevertheless if, for simplicity, we choose $n = 4$ then we can introduce « dual forms » using the totally antisymmetric 0-form ε_{ijkl} on $\hat{\mathrm{P}}$ defined by $\varepsilon_{0123} = 1$, e. g.

$$\varepsilon_{ij} = \frac{1}{2!}\, \varepsilon_{ijkl}\hat{\mathrm{D}}\Phi^k \wedge \hat{\mathrm{D}}\Phi^l \tag{3.9}$$

$$\varepsilon = \frac{1}{4!}\, \varepsilon_{ijkl}\hat{\mathrm{D}}\Phi^i \wedge \hat{\mathrm{D}}\Phi^j \wedge \hat{\mathrm{D}}\Phi^k \wedge \hat{\mathrm{D}}\Phi^l \tag{3.10}$$

With these quantities we can construct a counterpart to the Einstein-Cartan Lagrangian (2.14) on $\hat{\mathrm{P}}$:

$$\mathrm{L} = \hat{\Omega}^{ij} \wedge \varepsilon_{ij} = \mathrm{R}\varepsilon \tag{3.11}$$

where a dot indicates that an index has been raised with $\gamma^{ij}(\mathrm{R} = \mathrm{R}^{l}{}_{\cdot l}$, $\mathrm{R}_{kl} = \mathrm{R}^{i}{}_{kil})$. However, due to the use of the degenerate « metric », L is invariant under a change of the (linear part of the) connection by an arbitrary boost part:

$$\hat{\omega}^{i}{}_{j} \mapsto \hat{\omega}^{i}{}_{j} + \Psi_{j}\gamma^{ik}\mathrm{A}_{k} \qquad (3.12)$$

where A is any covector-valued 1-form. More generally, this applies to all scalars built from the geometrical quantities $\mathrm{R}^{i}{}_{jkl}$ and $\mathrm{Q}^{i}{}_{kl}$. For connections with vanishing torsion these scalars do not contain any information about the connection but only about the underlying Galilei structure.

It is therefore obvious that the variation of L with respect to $\hat{\omega}$, i. e.

$$\delta\mathrm{L} = \delta\hat{\omega}^{i}{}_{j} \wedge \hat{\mathrm{D}}\varepsilon_{i}{}^{j} + \delta\hat{\mathrm{D}}\Phi^{j} \wedge (\mathrm{R}\delta^{i}_{j} - 2\mathrm{R}^{i}_{\cdot j})\varepsilon_{i}\,, \qquad (3.13)$$

does not lead to satisfactory field equations. We meet with similar problems if we try to construct a matter Lagrangian. Of course, there are no problems with formulating field equations without a Lagrange formalism (cf. section 4).

4. THE CENTRAL EXTENSION OF THE GALILEI GROUP

4.1 A useful matrix representation of the central extension of the (inhomogeneous) Galilei group is given by [7]:

$$(\vec{v}, \mathrm{W}, c, \vec{b}, \lambda) = \begin{pmatrix} 1 & 0 & c & 0 \\ \vec{v} & \mathrm{W} & \vec{b} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}\vec{v}^{2} & \vec{v}^{t}\mathrm{W} & \lambda & 1 \end{pmatrix} \qquad (4.1)$$

with $\quad \vec{v}, \vec{b} \in \mathbb{R}^{n-1}, \quad c, \lambda \in \mathbb{R}, \quad \mathrm{W} \in \mathrm{O}(n-1)$.

Again, the homogeneous Galilei group plays the role of G. V is the $(n+1)$-dimensional subspace of $\hat{\mathscr{G}}$ spanned by the translation generators and the central generator. Using the matrix representation of the Lie algebra $\hat{\mathscr{G}}$ which is induced by the group representation (4.1) we find the following general form of a connection on $\hat{\mathrm{P}}$:

$$\hat{\omega} = \begin{pmatrix} 0 & 0 & \hat{\psi}^{0} & 0 \\ \hat{\phi}^{\mathrm{A}}_{\cdot} & \hat{\omega}^{\mathrm{A}}{}_{\mathrm{B}} & \hat{\psi}^{\mathrm{A}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \hat{\phi}_{\mathrm{A}} & \hat{\psi}^{n} & 0 \end{pmatrix} \qquad (4.2)$$

where the entries of the matrix are 1-forms and

$$\hat{\omega}^{AB} = \hat{\omega}^{A}{}_{C}\delta^{CB} = -\hat{\omega}^{BA} \tag{4.3}$$

(A, B $= 1, \ldots, n-1$). For the pull-back of $\hat{\omega}$ with respect to the reduction map f we obtain the form

$$f^{*}\hat{\omega} = \begin{pmatrix} 0 & 0 & \psi^{0} & 0 \\ \tilde{\phi}^{A} & \tilde{\omega}^{A}{}_{B} & \psi^{A} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \tilde{\phi}_{A} & \psi^{n} & 0 \end{pmatrix} \tag{4.4}$$

The vector-valued 1-form

$$\underset{\sim}{\psi} = \begin{pmatrix} \psi^{0} \\ \vec{\psi} \\ \psi^{n} \end{pmatrix} \tag{4.5}$$

on P is tensorial (cf. Lemma 1) and transforms according to the $(n+1)$-dimensional representation of the homogeneous Galilei group:

$$\begin{pmatrix} 1 & 0 & 0 \\ \vec{v} & \mathrm{W} & 0 \\ \frac{1}{2}\vec{v}^{2} & \vec{v}^{t}\mathrm{W} & 1 \end{pmatrix}. \tag{4.6}$$

The first n components of $\underset{\sim}{\psi}$ transform according to the usual n-dimensional representation, i. e. they constitute a tensorial 1-form of type (id, $\mathbb{R}^{n}$) on P. If we require this form to be of maximal rank we can apply Lemma 2. This leads to a Galilei structure $L^{\Psi,\gamma}(M)$ on which we obtain a Galilei connection ω and an « extended canonical 1-form »:

$$\underset{\sim}{\theta} = \begin{pmatrix} \theta^{0} \\ \vec{\theta} \\ \theta^{n} \end{pmatrix} \tag{4.7}$$

transforming according to the representation (4.6). Covariant differentiation of $\underset{\sim}{\theta}$ (with respect to ω) provides us with an « extended torsion form »

$$\underset{\sim}{\Theta} = \mathrm{D}\underset{\sim}{\theta}. \tag{4.8}$$

This can be used to characterize Newtonian connections [*6*]-[*12*], i. e. Galilei connections on $L^{\Psi,\gamma}(M)$ which are torsion-free and for which the curvature tensor satisfies

$$\mathrm{R}^{i}{}_{j}{}^{k}{}_{l} = \mathrm{R}^{k}{}_{l}{}^{i}{}_{j} \tag{4.9}$$

where indices $i, j, \ldots$ run from 0 to $n-1$ and are raised with the help of γ^{ij}.

THEOREM. — ω is a Newtonian connection if and only if

$$\underset{\sim}{\Theta} = \begin{pmatrix} 0 \\ \vec{0} \\ d\chi \end{pmatrix} \tag{4.10}$$

(locally) with a 1-form χ.

Proof. — With

$$\omega = (\omega^i{}_j) = \left(\begin{array}{c|c} 0 & 0 \\ \hline \omega^A & \omega^A{}_B \end{array}\right) \tag{4.11}$$

equation (4.10) becomes

$$D\theta^i = 0 \tag{4.12}$$

$$d\theta^n + \omega_A \wedge \theta^A = d\chi . \tag{4.13}$$

Locally the last equation is equivalent to

$$d(\omega_A \wedge \theta^A) = 0 . \tag{4.13'}$$

Using the structure equation

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \tag{4.14}$$

for the curvature form of ω and (4.12), the last equation is turned into

$$\Omega_{AO} \wedge \theta^A = 0 \tag{4.15}$$

which with the help of

$$0 = D^2\theta^i = \Omega^i{}_j \wedge \theta^j \tag{4.16}$$

is seen to be equivalent to (4.9). □

In contrast to the corresponding situation in the Lorentz case (cf. the remarks following (2.16)) the condition (4.10) does not fix a unique connection on the Galilei structure. We are still left with a whole class of Newtonian connections locally parametrized by a timelike unit vector field (observer) u, i. e.

$$u^i\Psi_i = 1 . \tag{4.17}$$

More precisely, given an observer field u, a Newtonian connection is uniquely determined by [*6*]

$$u^i\nabla_i u^j = 0 \tag{4.18}$$

$$\nabla^{[i}u^{j]} = 0 \tag{4.19}$$

with

$$Du^i = \nabla_j u^i \theta^j . \tag{4.20}$$

Conversely, for each Newtonian connection there exists (at least locally) such a nonrotating and freely falling observer [*19*].

We see that in order to fix a Newtonian gravitational field a Galilei structure has to be supplemented by a timelike vector field. In the Newtonian theory, this vector field actually plays the role of the gravitational potential.

4.2 Using the Goldstone field Φ we can formulate the theory with an internal $\hat{G}$ symmetry, i. e. on $\hat{P}$ .This will be shown in the following.

The quotient space $\hat{G}/G$ can be identified with $\mathbb{R}^{n+1}$ via

$$\hat{G}/G \to R^{n+1}$$

$$(\vec{v}, W, c, \vec{b}, \lambda) . G \mapsto \begin{pmatrix} c \\ \vec{b} \\ \lambda \end{pmatrix}.$$

The left action of $\hat{G}$ on $\hat{G}/G$ is then translated into the action of $\hat{G}$ on the vectors

$$\begin{pmatrix} c \\ \vec{b} \\ 1 \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+2}$$

through the representation (4.1).

Regarding Φ as $\mathbb{R}^{n+1}$-valued, (1.8) becomes

$$\Phi \circ f = 0 . \tag{4.21}$$

The covariant derivative of Φ transforms according to

$$\hat{\mathcal{R}}^*_{(\vec{v}, W, c, \vec{b}, \lambda)} \hat{D}\Phi = \begin{pmatrix} 1 & 0 & 0 \\ \vec{v} & W & 0 \\ \frac{1}{2}\vec{v}^2 & \vec{v}^t W & 1 \end{pmatrix}^{-1} \hat{D}\Phi \tag{4.22}$$

which shows that $\hat{D}\Phi$ is invariant under internal translations and the central group. Using (4.21) we find

$$f^*\hat{D}\Phi = \underset{\sim}{\psi} \tag{4.23}$$

so that the first n components $\hat{D}\Phi^i$ of $\hat{D}\Phi$ are linearly independent due to the condition (1.5). For a local cross section σ of $\hat{P}$, $\sigma^*\hat{D}\Phi^i$ is therefore a « tetrad field ».

The part of the curvature form of $\hat{\omega}$ which corresponds to the linear connection defines a curvature tensor in the same way as in sections 2 and 3. The field equations for the prerelativistic theory of gravitation [*6*]-[*13*] can now be stated as follows:

$$\hat{\Theta} = \hat{D}^2\Phi = 0 \tag{4.24}$$

$$R_{ij} = 4\pi G \rho \Psi_i \Psi_j \tag{4.25}$$

$$\hat{\nabla}_j T^{ij} = 0 \tag{4.26}$$

with the gravitational constant G and

$$\rho = T^{ij}\Psi_i\Psi_j . \tag{4.27}$$

T^{ij} is a mass-momentum 0-form on P which transforms only under the homogeneous Galilei group (like $\hat{D}\Phi^i$).

To incorporate the energy law we have to pass to a mass-momentum-energy 0-form which transforms according to a tensor product of the $(n + 1)$-dimensional representation (4.6) (cf. [9]-[10]).

CONCLUSION

We have demonstrated how a gauge theory of a certain class of Lie groups can lead to space-time structures. One of the crucial points in this construction is the introduction of a Goldstone field.

Using this additional field we have formulated theories of gravitation as pure gauge theories of the Yang-Mills type. In particular, our presentation shows the equivalence of various approaches to the Poincaré gauge theory (cf. e. g. [2]-[3]-[5]).

Dealing with the Galilei group and its central extension we have obtained some more insight into the geometric structure of the prerelativistic theory of gravitation and especially the significance of the class of Newtonian connections.

ACKNOWLEDGMENT

The author is grateful to A. Dimakis and H. Goenner for stimulating comments.

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(Manuscrit reçu le 4 novembre 1982)