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Conservation of axiomatic field theory properties by the G-convolution associated with simple graphs and the Φ_4^4 equations of motion

by

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ABSTRACT. — We prove that simple graphs with two generalized vertices and an arbitrary number of internal lines, can be associated with generalized G-convolution products which conserve all algebraic and analytic properties resulting from the linear Wightman axioms. The method used is an extension in the renormalized case of the Bros-Lassalle iterative procedure.

RÉSUMÉ. — On démontre qu'à des graphes simples à deux vertex généralisés et un nombre arbitraire de lignes internes, on peut associer des produits de G-convolution généralisés qui conservent toutes les propriétés algébriques et analytiques résultant des axiomes de Wightman linéaires. La méthode utilisée est une extension au cas renormalisé de la méthode itérative de Bros et Lassalle.

INTRODUCTION

In [1] a renormalized normal product in the r-dimensional ($1 \leqslant r \leqslant 4$) Euclidean space has been introduced to study the equations of motion for the Schwinger functions of a Φ^4 model in the Euclidean Axiomatic field theory framework. The essential tool for this purpose was the renormalized G-convolution product defined in [2] in the Euclidean case. In the special case of two dimensions it has been proved that these equations of motion conserve:

I) All analytic and algebraic properties in 2-momenta complex Minkowski space, implied by the linear program of a Wightman field theory, and which characterize the structure of a general N-point function (see def. of section 1 and [3]). These conservation laws have been established by using the iterative integration method introduced by Bros and Lassalle in [3] and [4].

II) Properties of asymptotic behaviour at infinity in 2 momenta Euclidean space by application of the main theorem of [5]. The extension of the above results in the more interesting case of four dimensional complex Minkowski space can be obtained only by an improvement of the method used. More precisely it is necessary to establish new topological statements concerning the graphical representation of the renormalized G-convolution product (R. G. P.), H_G^{ren}, together with the corresponding analytic and algebraic properties (of the renormalized integrand R_G) which are appropriate to the graphical iterative features of the renormalization. procedure. This is what exactly we have realized in the present work.

We have to notice at this point that the local character of each term in the perturbation theory framework, for the Feynman integrals, has been established by Epstein and Glaser in [8]. These authors have proved rigourously that at each order of renormalized perturbation theory, a causal system of time ordered products can be defined and that there exists an equivalence between the existence of this system and that of a set of analytic n-point functions which can be identified with sums of Feynman integrals in the corresponding order of perturbative theory.

The method of renormalized G-convolution product used in the present work is non perturbative and has the advantage that by the generalization of Feynman amplitude in complex Minkowski momentum space yields directly the analyticity of all integrals associated with general graphs and in particular with Feynman graphs.

For technical reasons we restrict ourselves to a class of the so called « simple graphs » (see def. (4.1.1). This class is large enough to contain all typical graphs of the equations of motion of a Φ^4 theory (see [1]) and so sufficient for our purposes.

In section 1 we recall all definitions and results of the references [3], [4] and [5] which will be used in the forthcoming sections. In section 2 we associate a set of simply connected graphs (trees) to a general graph G and to all its subgraphs γ in the « forest » formalism of renormalization as defined in [6], [7] and extended in [2]. We thus establish the extension $\tilde{R}_G$ of the renormalized integrand R_G of G. We call this technical tool a « blowing up procedure ».

In section 3 we establish some mathematical results concerning the algebraic and analytic properties of the so called n-point functions of « tree type ». The results of the above two previous sections concern a graph G of general structure, and the larger part of the corresponding proofs are found in [10].

In section 4 we restrict ourselves to the « simple type » graphs which contain only two generalized vertices linked by an arbitrary number of internal lines. We apply the statements of section 2 and 3 to this case and show the general n-point analytic and algebraic properties of the extended renormalized integrand $\tilde{R}_G$. We then apply the analog of the iterative integration of [3] and [4] to prove theorems 4.2 and 4.3. These theorems ensure the conservation of the character general n-point function by the renormalized G-convolution products H_G^{ren} of simple type involved, in particular, by the Φ_4^4 equations of motion.

SECTION 1

1.1. ANALYTIC PROPERTIES OF GENERAL n-POINT FUNCTIONS (cf. [3] [4])

Let us consider the set $X = \{1, 2, \ldots, n\}$; we call $\mathscr{P}^*(X)$ the set of all proper subsets of X, and (X, X\I) every proper partition of X.

DEFINITION 1*a*. — A function $H^{(n)}$ defined in:

$$\mathbb{C}_X^{4(n-1)} = \left\{ (K_1, \ldots, K_n) : K_j = P_j + iQ_j ; P_j, Q_j \in \mathbb{R}^4 ; \sum_{j \in X} K_j = 0 \right\}$$

is called a general n-point function if it satisfies the following properties:

i) analyticity in the union of a certain family of tubes:

$$\{ \mathscr{T}_\lambda = \mathbb{R}^{4(n-1)} + i\mathscr{C}_\lambda ; \lambda \in \Lambda^{(n)} \}$$

and slow increase near the real; $\mathscr{C}_\lambda$ is the cone of $\mathbb{R}^{4(n-1)}$ defined by:

$$\mathscr{C}_\lambda = \{ Q \in \mathbb{R}^{4(n-1)} : \lambda(I) Q_I \in V^+, \forall I \in \mathscr{P}^*(X) \}$$

where λ is a function defined on $\mathscr{P}^*(X)$ with values in $\{ -1, 1 \}$ such that $\mathscr{C}_\lambda$ is not empty [1]; we call $\Lambda^{(n)}$ the set of such functions;

the real boundary values $\{ H_\lambda^{(n)}(P) ; \lambda \in \Lambda^{(n)} \}$ are distributions.

ii) the boundary values $H_{\lambda_1}^{(n)}(P)$ and $H_{\lambda_2}^{(n)}(P)$ associated with two adjacent tubes $\mathscr{T}_{\lambda_1}$ and $\mathscr{T}_{\lambda_2}$ separated by the partition $(I, X\backslash I)$ coincide, in the distributional sense, in the « edge of the wedge » region $\mathscr{R}_I$ defined by:

$$\mathscr{R}_I = \{ P \in \mathbb{R}^{4(n-1)} : P_I^2 < M_I^2, P_I^2 \neq m_I \}$$

where: $P_I = \sum_{i \in I} P_i$, M_I is a suitable mass-threshold in the channel $(I, X\backslash I)$, and m_I is the discrete mass-spectrum of the theory.

We say that *i*) and *ii*) characterize the primitive domain of analyticity of a general *n*-point function.

iii) The boundary values $\{ H_\lambda^{(n)}(P) ; \lambda \in \Lambda^{(n)} \}$ satisfy general « Steinmann relations ».

More precisely, for a quartet of adjacent tubes $\{ \mathscr{T}_{\lambda_i} : 1 \leq i \leq 4 \}$ corresponding to a pair of transverse partitions [2] $(I, X\backslash I)$, $(J, X\backslash J)$, say: $\mathscr{T}_{\lambda^{++}}$, $\mathscr{T}_{\lambda^{-+}}$, $\mathscr{T}_{\lambda^{--}}$, $\mathscr{T}_{\lambda^{+-}}$, the four corresponding boundary values satisfy:

$$H_{\lambda^{++}}^{(n)}(P) + H_{\lambda^{--}}^{(n)}(P) = H_{\lambda^{+-}}^{(n)}(P) + H_{\lambda^{-+}}^{(n)}(P)$$

iv) The τ-boundary value of $H^{(n)}$ denoted by $H_\tau^{(n)}(P)$ is defined by the « Ruelle prescription ».

$$H_\tau^{(n)}(P) = \lim_{\substack{K \to P \\ P \in \Omega_\lambda \\ Q \in \mathscr{C}_\lambda}} H^{(n)}(K)$$

The open set Ω_λ are defined by:

$$\Omega_\lambda = \{ P \in \mathbb{R}^{4(n-1)} : \lambda(I) P_I \in \complement \overline{V}_I^-, \forall I \in \mathscr{P}^*(X) \}$$

where:

$$\overline{V}^+ = -V^- = \{ P \in \mathbb{R}^4 : P^0 \geqslant ((\vec{P})^2 + M_I^2)^{1/2} \} \cup \{ P \in \mathbb{R}^4 : P^0 = ((\vec{P})^2 + m_I^2)^{1/2} \}$$

When $H^{(n)}(K)$ is the physical *n*-point function, then $H_\tau^{(n)}$ is the connected time-ordered product $\tau_c(P)$.

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(1) V^+ is the forward light cone defined by $V^+ = \{ x \mid x^2 > 0, x^0 > 0 \}$.

(2) Transverse partitions $(I, X\backslash I)$, $(J, X\backslash J)$ satisfy: $I \cap J \neq \emptyset$, $I \cap (X\backslash J) \neq \emptyset$, $(X\backslash J) \cap J \neq \emptyset$, $(X\backslash I) \cap (X\backslash J) \neq \emptyset$.

1.2. ASYMPTOTIC BEHAVIOUR OF *n*-POINT FUNCTIONS IN EUCLIDEAN REGIONS (see [2] and [5])

a) Class of Weinberg $A_N^{(\alpha)}$ (3).

Let E an N-dimensional vector space on $\mathbb{R}$.

Let α a bounded real-valued function on the set of all the linear subspaces $S \neq \{0\}$.

DEFINITION 1*b*. — A complex-valued function f on E belongs to the Weinberg class $A_N^{(\alpha)}$ if, for every set of $m \leq N$ independent vectors $L_1, \ldots, L_m$ and every bounded region W on E, there exists a set of numbers $b_1, b_2, \ldots, b_m \geqslant 1$ and a constant $M > 0$ (depending on $L_1, L_2, \ldots, L_m$ and W) such that:

$$\left| f\left(\sum_{j=1}^{m} L_j y_j \ldots y_m + C \right) \right| \leq M \prod_{j=1}^{m} y_j^{\alpha(\overline{\{L_1, \ldots L_j\}})}$$

where the real variables y_j $(j = 1, \ldots, m)$ belong to the region $\{ y_j \geqslant b_j \}$ and where $C \in W$, and $\overline{\{ L_1, \ldots, L_j \}}$ denotes the linear closure of the set of vectors $\{ L_1, \ldots, L_j \}$.

b) Class Σ_n^μ.

Let $\mathscr{E}$ an n-dimensional vector space with a norm $\|\cdot\|$.

Let $P_\nu(D)$ any homogeneous polynomial of degree ν in the derivatives with respect to the coordinates in $\mathscr{E}$. In an intrinsic way, $P_\nu(D)$ is a linear operator on $C^\infty(\mathscr{E})$, associated with an element P_ν of the symetrized tensor product $\mathscr{E}^{\bigotimes_s^\nu}$ and is defined as follows; let $f \in C^\infty(\mathscr{E})$; if $f^{(\nu)}(K)$ denotes the derivative application of order ν of f at K, which is a linear form on $\mathscr{E}^{\bigotimes_s^\nu}$, one has:

$$\forall K \in \mathscr{E} \qquad (P_\nu(D)f)(K) = f^{(\nu)}(K)(P_\nu).$$

We define then the following class of symbols:

DEFINITION 1*c*. — Let μ an arbitrary real number. A function f on $\mathscr{E}$

(3) Note that the original Weinberg classes involve (cf [*11*]) a logarithmic index β. However, in all the paper, we shall denote by the abusive notation A_N^α the genuine class $A_N^{\alpha,\beta}$ with $\beta = 0$. This abuse has no influence upon the integrability criterium (for a detailed study of this fact, see [*12*]).

is said to belong to the class Σ_n^μ if it belongs to $C^\infty(\mathscr{E})$ and if, for every integer $v \geqslant 0$ and every homogeneous polynomial $P_v(D)$ in derivative with respect to K, there is a constant C_v such that:

$$|(P_v(D)f)(K)| \leq C_v \||P_v\|| \cdot (1 + \|K\|)^{\mu - v}.$$

Here $\||\cdot\||$ is a certain norm on the symetrized tensor product $\mathscr{E}^{\otimes_s v}$.

c) Weinberg's theorem (Power Counting Theorem).

THEOREM 1.1. — *Let* $f(K, k)$ *belong to the Weinberg class* $A_{rN}^{(\alpha)}$ *on the vector space* $\mathscr{E}_{(K)}^{r(n-1)} \times \mathscr{E}_{(k)}^{rL}$ *for* $1 \leq r \leq 4$.

If: $\sup_{S \subset E_{(k)}^{rL}} (\alpha(S) + \dim S) < 0$ (Weinberg's Criterium).

Then:

i) The integral $I(K) = \int_{E_{(k)}^{rL}} f(K, k) d^{rL}k$ *converges absolutely.*

ii) $I(K)$ *belongs to a class* $A_{r(n-1)}^{\alpha_I}$ *of Weinberg functions with the following asymptotic indicatrix* α_I*:*

$$\forall S \subset \mathscr{E}_{(K)}^{r(n-1)} : \alpha_I(S) = \max_{\chi(S')=S} \{ \alpha(S') + \dim S' - \dim S \}$$

where χ *is the canonical projection of:* $\mathscr{E}_{(K,k)}^{rN}$ *on* $\mathscr{E}_{(K)}^{r(n-1)}$.

SECTION 2

2.1. SETS OF INDEPENDENT INTERNAL LINES OF G AND OF SUBGRAPHS γ OF G

DEFINITION 2.*a*.

2*a*.1. — On the set $\tilde{\mathscr{L}}$ of internal lines of a given connected graph G, we decide once for all to put an order Ω which we denote by the sign >, so that:

$$\tilde{\mathscr{L}} = \{ l_1, l_2, \ldots, l_\lambda \} \tag{2.1}$$

with $\lambda = |\tilde{\mathscr{L}}|$ (the number of elements of $\tilde{\mathscr{L}}$) and:

$$l_1 > l_2 \ldots l_\lambda .$$

2*a*.2. — The order Ω will allow us to associate with G a well defined tree graph G_0 as follows.

We first define the following subset K_G of internal lines of G (as a subset of $\tilde{\mathscr{L}}$, it is necessarily ordered by Ω):

$$\mathscr{K}_G = \{ \bar{l}_1 > \bar{l}_2 \ldots \bar{l}_N \} \quad \bar{l}_j \in \tilde{\mathscr{L}} \quad 1 \leq j \leq L \tag{2.2}$$

satisfying the properties:

a) $G \backslash \bar{l}_1$ connected and $\forall \bar{l}_k > \bar{l}_1$, $G \backslash \bar{l}_k$ disconnected.

b) $\forall j \leq L$, $G \backslash \bar{l}_1 \ldots \backslash \bar{l}_j$ connected and
$\forall l_k : \bar{l}_{j-1} > l_k > l_j$, $G \backslash \bar{l}_1 \backslash \ldots \backslash \bar{l}_{j-1} \backslash \bar{l}_k$ disconnected

c) $\forall l_k : \bar{l}_L > l_k$, $G \backslash \bar{l}_1 \backslash \ldots \bar{l}_L \backslash l_k$ disconnected.

We call $\mathscr{K}_G$ « the set of independent lines of G » induced by the order Ω. Since the order Ω induces an order Ω_γ on the set of internal lines of every connected subgraph $\gamma \subset G$, we can define in a analogous way (by properties *a*) *b*) *c*)) the set of independent internal lines of γ:

$$\mathscr{K}_\gamma = \{ \bar{l}_{1_\gamma}, \ldots, \bar{l}_{L_\gamma} \} \tag{2.3}$$

and the set of independent internal lines of $G \backslash \mathscr{K}_\gamma$, denoted by $\mathscr{K}_\gamma^G$. Moreover, inside a forest U of G we consider the set $\{ \gamma_a \; a = 1, \ldots, c \}$ of maximal disjoint subgraphs of γ in U(γ) and the corresponding union of the sets $\bigcup_a \mathscr{K}_{\gamma_a}$. We then define the set of independent internal lines of $\gamma \backslash \bigcup_a \mathscr{K}_{\gamma_a}$ induced by Ω as the set of ordered lines:

$$\mathscr{K}^\gamma_{\bigcup_a \gamma_a} = \{ \bar{l}_1 > \bar{l}_2 \ldots > \bar{l}_n \}, \; n = L - \sum_a L_a \tag{2.4}$$

with L_a the number of independent loops of γ_a.

We now present without proof a useful statement concerning the above defined sets. The corresponding proofs of all these properties are given in detailed form in [*10*].

PROPOSITION 2.1. — *i*) The subgraphs $G \backslash \mathscr{K}_G$ of G, $\gamma \backslash \mathscr{K}_\gamma$ and $G \backslash \mathscr{K}_\gamma \backslash \mathscr{K}_\gamma^G$ are tree graphs depending on the order Ω.

ii) *a*) For a given connected graph G with the order Ω on its set of internal lines, and for every couple of connected subgraphs $\mu \subset \gamma$ of G, the corresponding sets of independent internal lines satisfy:

$$\begin{aligned} \mathscr{K}_G &= \mathscr{K}_\gamma \cup \mathscr{K}_\gamma^G \\ \mathscr{K}_\gamma &= \mathscr{K}_\mu \cup \mathscr{K}_\mu^\gamma \end{aligned} \tag{2.5}$$

b) Given a Forest U of G and an arbitrary $\gamma \in U$, the following generalized decomposition formula holds:

$$\mathscr{K}_G = \left(\bigcup_a \mathscr{K}_{\gamma_a} \right) \cup \mathscr{K}^G_{\bigcup_a \gamma_a} \tag{2.6}$$

$$\mathscr{K}_\gamma = \left(\bigcup_\tau \mathscr{K}_{\gamma_\tau} \right) \cup \mathscr{K}^\gamma_{\bigcup_\tau \gamma_\tau} \tag{2.7}$$

Here, γ_a, $a = 1, \ldots, c$ are the maximal disjoint subgraphs of U(G), and γ_τ, $\tau = 1, \ldots, d$ are the maximal disjoint subgraphs of U(γ).

2.2. AN ADMISSIBLE SET OF BASIC INTERNAL MOMENTA: THE Ω-ASSIGNMENT

With a fixed order Ω on the internal lines of a graph G, we are going to associate a special « admissible » ([4]) set of basic internal momenta for G and for all connected subgraphs γ of G; this special choice will be called an « Ω-assignment » of internal momenta. We choose as a particular solution of (2.10) in [*6*] the one corresponding to the choice:

$$p_{ab\sigma} = 0 \quad \text{for every line} \quad l_{ab\sigma} \in \mathscr{K}_G . \tag{2.8}$$

The set of equations (2.10) in [*6*] then reduces to a system of $4(N_v - 1)$ equations for the $N_v - 1$ momenta $p_{ab\sigma}$ associated with the lines $l_{ab\sigma} \in G \backslash \mathscr{K}_G$ (the latter being a tree graph). Furthermore, with every line $l_{ab\sigma} \in \mathscr{K}_G$ we associate one of the L-independent variables (whose set is denoted by k).

$$k_{ab\sigma} = k_j, \quad 1 \leq j \leq L, \qquad \text{for every} \quad l_{ab\sigma} \in \mathscr{K}_G \tag{2.9}$$

So in the following the lines of $\mathscr{K}_G$ will be in one-to-one correspondence with the L independent variables of integration. Similarly, for every connected subgraph γ of G, we choose as a particular solution of (2.27) in [*6*] the one corresponding to the choice:

$$p^{\gamma}_{ab\sigma} = 0 \qquad \forall l^{\gamma}_{ab\sigma} \in \mathscr{K}_{\gamma} \tag{2.10}$$

The set $\{ p^{\gamma}_{ab\sigma}, \forall l^{\gamma}_{ab\sigma} \in \gamma \backslash \mathscr{K}_{\gamma} \}$ being then determined as the solution of a system of $4(N_v - 1)$ equations. We also choose the set k^{γ} of independent variables of integration in γ as the set of four momenta $k^{\gamma}_{ab\sigma}$ associated with all the lines $l^{\gamma}_{ab\sigma}$ in $\mathscr{K}_{\gamma}$. Taking into account Proposition 2.1 we can then put:

$$l^{\gamma}_{ab\sigma} \in \mathscr{K}_{\gamma} : k^{\gamma}_{ab\sigma} = k_{ab\sigma} \qquad \text{where} \qquad k_{ab\sigma} \in k \tag{2.16}$$

That means the set of independent variables k^{γ} is a subset of the set k defined in (2.14). Taking into account the preceding results and definitions we have shown in [*10*] the following.

PROPOSITION 2.2. — *For a given graph* G *the above defined Ω-assignment of internal momenta is an admissible one. That means it satisfies for every subgraph* $\gamma \subset G$:

i)
$$\tilde{l}_{ab\sigma}(k, p) = \tilde{l}^{\gamma}_{ab\sigma}(k^{\gamma}, p^{\gamma}) \tag{2.12}$$

ii)
$$p^{\gamma}_{ab\sigma} = p^{\gamma}_{ab\sigma}(K, k) \tag{2.13}$$

$$k^{\gamma}_{ab\sigma} = k^{\gamma}_{ab\sigma}(k) \tag{2.14}$$

([4]) In the sense of Zimmermann [*6*] as already explained in [*1*].

(p^γ(resp. p) *denotes the set of basic internal momenta of* γ (resp. *of* G)).

iii) For every couple of subgraphs μ, γ *of* G *such that* $\mu \subset \gamma$*:*

$$p^{\mu}_{ab\sigma} = p^{\mu}_{ab\sigma}(p^\gamma, k^\gamma) \tag{2.15}$$

$$p^{\mu}_{ab\sigma} = k^{\mu}_{ab\sigma}(k^\gamma) \tag{2.16}$$

2.3. THE « BLOWING UP PROCEDURE » ASSOCIATED WITH Ω

DEFINITION 2c.1. — For an arbitrary graph $\gamma \subseteq G$ we consider now the following procedure. We « cut » the lines of the set $\mathscr{K}_\gamma$ so that from each internal line $\bar{l} \in \mathscr{K}_\gamma$ we obtain a pair of external lines $(\bar{l}, \bar{l}')$. We then define for G a set:

$$\mathscr{K}'_G = \{ \bar{l}'_1 > \dots > \bar{l}'_L \}$$

and for all $\gamma \subseteq G$ we define the corresponding sets:

$$\mathscr{K}'_\gamma = \{ \bar{l}'_i \quad \text{with} \quad \bar{l}_i \in \mathscr{K}_\gamma \quad \text{and} \quad \bar{l}'_i \in \mathscr{K}'_G, \forall i = 1, \dots, L \} ;$$

so that the following conditions are satisfied: $\mathscr{K}'_\gamma \subset \mathscr{K}'_G$ and for every couple γ, μ with $\gamma \subset \mu \subset G$:

$$\mathscr{K}'_\gamma \subset \mathscr{K}'_\mu \subset \mathscr{K}'_G \tag{2.17}$$

Proposition 2.1 then allows us to state the following:

LEMMA 2.1. — *Given a graph* G *with n external lines and given an order* Ω *on* $\tilde{\mathscr{L}}$, *the « cutting » of all lines* $l_j \in \mathscr{K}_G(\Omega)$ *inside every forest* U *yields simultaneously:*

From G *a tree graph* T_G *having* $n + 2|\mathscr{K}_G|$ *external lines and from every subgraph* γ *of* U(G) *a tree graph* T_γ *with* $n_\gamma + 2|\mathscr{K}_\gamma|$ *external lines. We call this operation a « blowing up » procedure for the forest* U *of* G.

After this « blowing up » procedure we define the new renormalized integrand $\tilde{R}_G$, in a way analogous to [*2*].

DEFINITIONS 2.*c*.2. — With every pair of cut lines $(\bar{l}_j, \bar{l}'_j) \in \{ \mathscr{K}_G \cup \mathscr{K}'_G \}$ (resp. $(l^\gamma_i, l^{\gamma'}_i) \in \{ \mathscr{K}_\gamma \cup \mathscr{K}'_\gamma \}$) we associate momenta $(k_j, k'_j) \in \mathbb{C}^8$ (resp. $(k^\gamma_j, k^{\gamma'}_j) \in \mathbb{C}^8$). We use the notation (K, k, k') (resp. (K^γ, k^γ, $k^{\gamma'}$)) to denote the set of external and internal momenta of T_G (resp. T_γ). These momenta satisfy:

$$\sum_{1 \leq i \leq n} K_i + \sum_{1 \leq j \leq N} (k_j + k'_j) = 0 \quad \left(\text{resp.} \sum_{1 \leq i \leq n_\gamma} K^\gamma_i + \sum_{1 \leq j \leq L_\gamma} (k^\gamma_j + k^{\gamma'}_j) = 0 \right) \tag{2.18}$$

2.c.3. — In a way analogous to [1] (5) we define the mappings:

$$\begin{aligned} &\forall v \in \mathcal{N} : (\mathrm{K}, k, k') \overset{\lambda'_v}{\to} \mathrm{K}^v(\mathrm{K}, k, k') \qquad && (\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}) \overset{\lambda_v^{\gamma\prime}}{\to} \mathrm{K}^v(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}) \\ & \qquad\qquad\qquad \text{and resp.} && \\ &\forall l_i \in \tilde{\mathcal{L}} : (\mathrm{K}, k, k') \overset{\lambda'_i}{\to} \tilde{l}_i(\mathrm{K}, k, k') && (\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}) \overset{\lambda_i^{\gamma\prime}}{\to} \tilde{l}_i(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}) \end{aligned} \tag{2.19}$$

and we associate with every vertex V of $\mathrm{T_G}$ (resp. with every line l_i) a completely amputated general n_v-point function $\hat{\mathrm{H}}^{n_v}$ (resp. $\tilde{\mathrm{H}}_i^{(2)}$) satisfying:

$$\begin{aligned} \lambda'_v{}^*\tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v) &= \tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}, k, k')) \\ (\text{resp. } \lambda_v^{\gamma\prime *}\tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}\text{-}) &= \tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}))) \\ \lambda'_i{}^*\tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i) &= \tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i(\mathrm{K}, k, k')) \\ (\text{resp. } \lambda_i^{\gamma\prime *}\tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i) &= \tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}))) \end{aligned} \tag{2.20}$$

We shall suppose that:

$$\tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}, k, k'))\Big|^{(6)}_{k+k'=0} = \hat{\mathrm{H}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}, k)) \tag{2.21}$$

$$\tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i(\mathrm{K}, k, k'))\big|_{k+k'=0} = \mathrm{H}_i^{(2)}(\tilde{l}_i(\mathrm{K}, k)) \tag{2.22}$$

$$\text{resp.} \begin{cases} \tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}))\big|_{k^\gamma+k^{\gamma\prime}=0} = \hat{\mathrm{H}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}^\gamma, k^\gamma)) & (2.23) \\ \tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime}))\big|_{k^\gamma+k^{\gamma\prime}=0} = \mathrm{H}_i^{(2)}(\tilde{l}_i(\mathrm{K}^\gamma, k^\gamma)) & (2.24) \end{cases}$$

We then define the « integrand » corresponding to $\mathrm{T_G}$ (resp. T_γ) by:

$$\tilde{\mathrm{I}}_{\mathrm{G}}(\mathrm{K}, k, k') = \prod_{v \in \mathcal{N}} \tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}, k, k')) \prod_{l_i \in \mathcal{L}_{\mathrm{T_G}}} \tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i(\mathrm{K}, k, k')) \tag{2.25}$$

Here $\mathcal{L}_{\mathrm{T_G}}$ denotes the set of internal and external lines of $\mathrm{T_G}$. Respectively:

$$\tilde{\mathrm{I}}_\gamma = \prod_{v \in \mathcal{N}_\gamma} \tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime})) \prod_{l_i \in \mathcal{L}_{\mathrm{T}_\gamma}} \tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime})) \tag{2.26}$$

For the « integrand » corresponding to a tree graph $\mathrm{T}_{\bar{\gamma}}$ which results from a « blown up » reduced subgraph $\bar{\gamma} = \gamma/\gamma_1\gamma_2 \ldots \gamma_c$ with γ_a, $a = 1, \ldots, c$ disjoint maximal subgraphs of $\mathrm{U}(\gamma)$ we have the analog of formula (2.40) of [6].

$$\tilde{\mathrm{I}}_{\bar{\gamma}} = \prod_{\substack{v \in \mathcal{N}_\gamma \\ v \notin \mathcal{N}_{\gamma_a} \\ a=1,\ldots,c}} \tilde{\hat{\mathrm{H}}}^{(n_v)}(\mathrm{K}^v(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime})) \cdot \prod_{\substack{l_i \in \mathcal{L}_{\mathrm{T}_\gamma} \\ l_i \notin \mathcal{L}_{\mathrm{T}_{\gamma_a}} \\ a=1,\ldots,c}} \tilde{\mathrm{H}}_i^{(2)}(\tilde{l}_i(\mathrm{K}^\gamma, k^\gamma, k^{\gamma\prime})) \tag{2.27}$$

---

(5) See in section 2 of [1] all exact definitions given there.

(6) This is a shorthand notation to indicate that we take the restriction of the function $\mathrm{H}^{(n_v)}(\mathrm{K}, k, k')$ on the submanifold $k_i = -k'_i$, $1 \leqslant i \leqslant |\mathcal{K}_{\mathrm{G}}|$.

We can show the following:

LEMME 2.2.

$$\tilde{I}_{\bar{G}}(K, k, k')|_{k+k'=0} = I_{\bar{G}}(K, k) \cdot \prod_{\substack{l_i \in \mathscr{K} \bigcup_\tau^G \gamma_\tau \\ l_i \notin \mathscr{L}_{\gamma_\tau} \\ \gamma_\tau \in U(G)}} H_i^{(2)}(k_i) \tag{2.28}$$

$$\text{(resp. } \tilde{I}_{\bar{\gamma}}(K^\gamma, k^\gamma, k^{\gamma'})|_{k^\gamma+k^{\gamma'}=0} = I_{\bar{\gamma}}(K^\gamma, k^\gamma) \cdot \prod_{\substack{l_i^\gamma \in \mathscr{K} \bigcup_a^\gamma \gamma_a \\ l_i^\gamma \notin \mathscr{L}_{\gamma_a} \\ \gamma_a \in U(\gamma)}} H_i^{(2)}(k_i^\gamma) \tag{2.29}$$

2.c.4. — Taking into account the order Ω_γ on the internal and coupled blown up external lines of T_γ we select the first $2|\mathscr{K}_\gamma| - 1$ momenta:

$$\{ (k_i^\gamma, k_i^{\gamma'}; i = 1, \ldots, |\mathscr{K}_\gamma| - 1), k_{|\mathscr{K}_\gamma|}^\gamma \}$$

corresponding to the first ordered « cut » lines: $\{ \bar{l}_j, \bar{l}_j' \} \in \mathscr{K}_\gamma \cup (\mathscr{K}_\gamma' \backslash \bar{l}_{|\mathscr{K}_\gamma|}')$.

We define on this way a set of independent internal momenta of γ and we denote it by:

$$\underset{\sim}{k}^\gamma = \{ k_j^\gamma, k_j^{\gamma'}; j = 1, \ldots, |\mathscr{K}_\gamma| - 1, \text{ and } k_{|\mathscr{K}_\gamma|}^\gamma \} \tag{2.30}$$

From Lemma 2.5 and the momentum Ω-assignment (Proposition 2.2), we then obtain:

LEMMA 2.3. — *For any two graphs γ_1, γ_2 with $\gamma_2 \subset \gamma_1$, the corresponding sets $\underset{\sim}{k}^{\gamma_a}$, $a = 1, 2$ of independent internal momenta satisfy:*

$$\underset{\sim}{k}^{\gamma_1} \subset \underset{\sim}{k}^{\gamma_2}. \tag{2.31}$$

2.c.5. — Given two graphs $\gamma_2 \subset \gamma_1$ we consider a function $g(\underset{\sim}{k}^{\gamma_2}, p^{\gamma_2}(K^{\gamma_2}))$ of the internal and external momenta of γ_2. The inclusion property (2.39) allows us to define the functions $\underset{\sim}{k}^{\gamma_2}(\underset{\sim}{k}^{\gamma_1})$ by:

$$k_j^{\gamma_2} = k_j^{\gamma_1} \quad 1 \leqslant j \leqslant |\mathscr{K}_{\gamma_2}|; \; k_i^{\gamma_2'} = k_i^{\gamma_1'}, \; 1 \leqslant i \leqslant |\mathscr{K}_{\gamma_2}| - 1 \tag{2.32}$$

The basic internal momenta denoted by $p_{ab\sigma}^{\gamma_2}(K^{\gamma_2})$ (for an internal line $l_{ab\sigma}^{\gamma_2}$ of T_{γ_2}) are defined as in sec. 2 of [6] through the momentum conservation equations at all the vertices of the tree graphs $T_{\gamma_2}, T_{\gamma_1}$ by (see prop. 2.1):

$$p_{ab\sigma}^{\gamma_2} = p_{ab\sigma}^{\gamma_2}(\underset{\sim}{k}^{\gamma_1}, p^{\gamma_1}). \tag{2.33}$$

Finally the substitution operator $\tilde{S}_{\gamma_2}^{\gamma_1 *}$ (see [6]) acting on the functions of the variables of any graph $\gamma_2 \subset \gamma_1$ is defined as follows:

$$\tilde{S}_{\gamma_2}^{\gamma_1 *} g(\underset{\sim}{k}^{\gamma_2}, p^{\gamma_2}) = g(\underset{\sim}{k}^{\gamma_2}(\underset{\sim}{k}^{\gamma_1}), p^{\gamma_2}(\underset{\sim}{k}^{\gamma_1}, p^{\gamma_1})) \tag{2.34}$$

and satisfies the following properties:

If:

$$\gamma_3 \subset \gamma_2 \subset \gamma_1 : \begin{cases} \tilde{S}^{\gamma_1 *}_{\gamma_3} = \tilde{S}^{\gamma_1 *}_{\gamma_2} \circ \tilde{S}^{\gamma_2 *}_{\gamma_3} & a) \\ (\tilde{S}^{\gamma_1}_{\gamma_2} g) \,|_{k^{\gamma_1} = -k^{\gamma'_1}} = \tilde{S}^{\gamma_1}_{\gamma_2}(g) \,|_{k^{\gamma_2} = -k^{\gamma'_2}} & b) \end{cases} \tag{2.35}$$

2.*c*.6. — The renormalized integrand $\tilde{R}_G$ (analog of eq. (2.43) in [*2*] after the « blowing up » procedure is defined by:

$$\tilde{R}_G = \sum_U \tilde{F}_U \cdot \prod_{\substack{l_i \in \mathscr{K}_G \setminus \mathscr{L}_\gamma \setminus \mathscr{K}_\gamma \\ \gamma \in U(G)}} [H_i^{(2)}(k_i)]^{-1} \tag{2.36}$$

with

$$\tilde{F}_U = I_{T_G}(-t^{d(G)})\tilde{Y}_G \quad \text{for a full forest} \tag{2.37}$$

$$\tilde{F}_U = \tilde{Y}_G \quad \text{for a normal forest}. \tag{2.38}$$

Here the functions $\tilde{Y}_\gamma$ are defined recursively by:

$$\tilde{Y}_\gamma = \tilde{I}_{\bar{\gamma}} \cdot \prod_a \tilde{S}^{\gamma *}_a(-t^{d(\gamma_a)})\tilde{Y}_{\gamma_a} \tag{2.39}$$

and

$$I_{T_G} = \prod_{i=1}^{n} H^{(2)^{-1}}(\tilde{l}_i(K)), \tag{2.40}$$

the product in (2.40) is taken over all external lines of G. We state the following.

PROPOSITION 2.2. — *For a given general graph* G *with* $|\mathscr{K}_G|$ *independent loops. The G. R. I. Z.* $R_G(K, k)$ *(eq.* (2.43)) *of* [*2*]) *is related to* $\tilde{R}_G$ *(eq.* (2.36)) *by*:

$$R_G(K, k) = \tilde{R}_G(K, k, k')\,|_{k+k'=0}. \tag{2.41}$$

Proof. — We use the following recursion hypothesis. We suppose that for each γ_a maximal subgraph of $U(\gamma)$ the function $\tilde{Y}_{\gamma_a}$ satisfies:

$$\tilde{Y}_{\gamma_a}(K^{\gamma_a}, k^{\gamma_a}, k^{\gamma'_a})\,|_{k^{\gamma_a}+k^{\gamma'_a}=0} = Y_{\gamma_a} \prod_{\substack{l_j \in \mathscr{K}_\mu \\ \mu \in U(\gamma_a)}} H_j^{(2)}(k_j^{\gamma_a}) \tag{2.42}$$

We then prove that a similar relation holds between Y_γ and $\tilde{Y}_\gamma$. The operator $-t^{d(\gamma_a)}$ acts only on the external variables and S^*_a satisfies property (2.35); therefore taking into account the recursion hypothesis (2.42) we can write (in view of (2.9))

$$(\tilde{S}^*_\alpha(-t^{d(\gamma_a)})\tilde{Y}_{\gamma_a})\,|_{k^\gamma+k^{\gamma'}=0} = \prod_{\substack{l_j \in \mathscr{K}_\mu \\ \mu \in U(\gamma_a)}} H_j^{(2)}(k_j^\gamma) S^*_a(-t^{d(\gamma_a)})Y_{\gamma_a} \tag{2.43}$$

We combine Lemma (2.2) with (2.43) inside the formula (2.39) to obtain:

$$\tilde{Y}_\gamma |_{k^\gamma + k^{\gamma'} = 0} = \prod_{\substack{l_i \in \mathscr{K}_{\gamma_a} \\ \gamma_a \in U(\gamma)}} H_i^{(2)}(k_i^\gamma) Y_\gamma(K^\gamma, k^\gamma) . \qquad (2.44)$$

We then apply the result (2.44) to $\gamma = G$ and obtain:

$$\tilde{F}_U |_{k+k'=0} \cdot \prod_{\substack{l_i \in \mathscr{K}_\gamma \\ \gamma \in U(G)}} [H_i^{(2)}(k_i)]^{-1} = F_U . \qquad (2.45)$$

The sum over all forests U yields (2.41).

The above statement allows us to study all properties of $\tilde{R}_G(K, k, k')$ (this will be the aim of the two next sections) and then to try a procedure (to be specified in section 4) of successive integrations of $\tilde{R}_G |_{k+k'=0}$ over all internal four momenta k_j, $1 \leqslant j \leqslant |\mathscr{K}_G|$, in order to obtain the extension in the complex space of the finite part H_G^{ren}, and prove all its analytic and algebraic properties.

SECTION 3

3. SOME OPERATIONS ON GENERAL n-POINT FUNCTIONS

We shall prove some useful properties of the general n-point functions.

LEMME 3.1. — *Let* $H^{(n)}(K)$, $K = \{K_i\} \in \mathbb{C}^{4(n-1)}$, $i \in X = \{1, \ldots, n\}$, *be a general n-point function. The partial derivatives with respect to arbitrary directions:* $\dfrac{\partial^r}{\partial K_{i_1} \partial K_{i_2} \ldots \partial K_{i_r}} H^{(n)}(K)$ (*for any choice of* $4(n-1)$ *independent components of the vectors* K_i) *are general n-point functions of the same set of variables* $K \in \mathbb{C}_X^{4(n-1)}$.

Proof. — The analyticity inside the tubes of every derivative of the analytic function $H^{(n)}$ is evident. The Steinmann and coincidence relations follow from the linearity of the derivation operator together with the fact that the real boundary value of any derivative of the analytic function $H^{(n)}$ is equal to the corresponding derivative (in the sense of distributions on real space $\mathbb{R}_X^{4(n-1)}$) of the boundary value of $H^{(n)}$.

LEMMA 3.2. — *a) Let* $H^{(n)}(K)$ *be as in Lemma* 3.1. *The product of* $H^{(n)}(K)$ *by a polynomial* B(K) *in the same variables is a general n-point function.*

b) The sum $\sum_i H_i^{(n)}(K)$ *of general n-point functions is a general n-point functions.*

The proof is trivial.

LEMME 3.3. — *Let* $H^{(n)}(K)$ *be as in Lemma* 3.1. *For every* $I \subseteq X$ *the restriction:*

$$H^{(n)}(K)\,|_{\{K_i=0,\, i\in I\}}$$

is a general $n - |I|$ *point function.*

Proof. — It has been proved in [3] that a general n-point function is analytic at all the boundary points of the tubes $\mathcal{T}_\lambda^{(n)}$ which do not belong to the union of the family of sets $\{\Delta_J, J \in \mathcal{P}^*(X)\}$:

$$\Delta_J = \Delta_{X\setminus J} = \{K \in \mathbb{C}_X^{4(n-1)} : K_J^2 = m_J^2\} \cup \{K \in \mathbb{C}_X^{4(n-1)} : K_J^2 = M_J^2 + \rho, \forall \rho \geq 0\}.$$

It is then easy to check that in the linear manifold:

$$\mathbb{C}_{X\setminus I}^{4(n-|I|-1)} = \mathbb{C}_X^{4(n-1)} \cap \{(K_1, \ldots, K_n); K_i = 0, \forall i \in I\}$$

every tube $\mathcal{T}_\nu^{n-|I|}(\nu \in \Lambda^{n-|I|})$ belongs to the intersection of the boundaries of several tubes $\mathcal{T}_{\lambda_\alpha(\nu)}^n$ in $\mathbb{C}_X^{4(n-1)}$ and has no point in any set Δ_J defined above. Moreover the coincidence relations for $H^{(n-|I|)}$ between adjacent tubes $(\mathcal{T}_\nu^{n-|I|}, \mathcal{T}_{\nu'}^{n-|I|})$ are the restrictions to $\mathbb{C}_{X\setminus I}^{4(n-|I|-1)}$ of the coincidence relations for $H^{(n)}$ between the corresponding sets of tubes $(\mathcal{T}_{\lambda_\alpha(\nu)}^n, \mathcal{T}_{\lambda_{\alpha'}(\nu')}^n)$; Steinmann relations for $H^{(n-|I|)}$ are also restrictions of corresponding Steinamnn relations for $H^{(n)}$.

DEFINITION 3.*a*. — A partition $(I, X\setminus I)$ is called trivial for a general n-point function if the corresponding discontinuity function is identically zero; more precisely if for every pair of adjacent tubes $\mathcal{T}_\lambda$, $\mathcal{T}_{\lambda'}$ separated by the face $q_I = q_{X\setminus I} = 0$, one has $H_\lambda^{(n)} = H_{\lambda'}^{(n)}$ in the whole space $\mathbb{R}_X^{4(n-1)}$.

DEFINITION 3.*b*. — Let ϖ_m be a partition $(I, \ldots. I_m)$ of X: $X = \bigcup_{j\in M} I_j$ with $M = \{1, \ldots, m\}$. We define the following linear mapping i_{ϖ_m} of $\mathbb{C}_X^{4(n-1)} = \left\{(K_1, \ldots, K_n) = K, \sum_{i\in X} K_i = 0\right\}$, onto

$$\mathbb{C}_M^{4(m-1)} = \left\{k = (k_j = p_j + iq_j, j \in M), \sum_{j\in M} k_j = 0\right\}:$$

$$i_{\varpi_m}(K) = \left\{k_j = \sum_{i\in I_j} K_i ; \forall j \in M\right\} = i_{\varpi_m}(P) + i i_{\varpi_m}(Q) \qquad (3.1)$$

With every proper subset N of M, we associate in X the subset:

$$\mathrm{I(N)} = \bigcup_{j \in \mathrm{N}} \mathrm{I}_j, \qquad \mathrm{I(N)} \subset \mathrm{X} \tag{3.2}$$

We thus have: $p_{\mathrm{N}} \equiv \sum_{j \in \mathrm{N}} p_j = \sum_{i \in \mathrm{I(N)}} p_i \equiv p_{\mathrm{I(N)}}$.

LEMMA 3.4. — *For every general m-point function* $\mathrm{H}^{(m)}(k)$ *the inverse image of* $\mathrm{H}^{(m)}$ *by* i_{ϖ_m}*:*

$$\mathrm{H}^{(n)}(\mathrm{K}) = \mathrm{H}^{(m)}(i_{\varpi_m}(\mathrm{K}))$$

is a general n-point function.

Moreover the following specifications hold:

a) for every partition (N, M\N) *of* M *which is trivial* (*resp. non trivial*) *for* $\mathrm{H}^{(m)}(k)$, *the corresponding partition* (I(N), X\I(N)) *of* X *is trivial* (*resp. non trivial*) *for* $\mathrm{H}^{(n)}(\mathrm{K})$.

b) For a partition (I, X\I) *of* X *to be non-trivial for* $\mathrm{H}^{(n)}(\mathrm{K})$ *the following condition is* necessary.

There exists a subset $\mathrm{N} \in \mathscr{P}^*(\mathrm{M})$ *such that* I = I(N) (*in the sense of* (3.2)).

The proof of this statement is given in [*10*].

DEFINITION 3.*c*. — In the following, we shall consider a tree-graph T whose external lines are labelled by the elements of the set X, and carry the respective (complex) four-momenta $\mathrm{K}_i \left(\sum_{i \in \mathrm{X}} \mathrm{K}_i = 0 \right)$.

DEFINITION 3.*c*.1. — We say that a general *n*-point function $\mathrm{H}_{\mathrm{T}}^{(n)}(\mathrm{K})$ $(\mathrm{K} \in \mathbb{C}_{\mathrm{X}}^{4(n-1)})$ is of « tree type » if it is a tree-graph convolution in the sense of [*4*], which is associated with a tree T with *n* external lines.

We recall the definition of such a function $\mathrm{H}_{\mathrm{T}}^{(n)}$:

a) With every vertex v of T a general n_v-point function $\mathrm{H}^{(n_v)}(k^{(v)})$ $(k^{(v)} \in \mathbb{C}_{\mathrm{X}_v}^{4(n_v-1)})$ is associated; here X_v denotes the set of indices which label the lines incident to the vertex v; n_v is the number of these lines. Each line $l_j (j \in \mathrm{X}_v)$ carries four momentum $k_j^{(v)}$, and $k^{(v)}$ is a set $\{ k_j^{(v)} ; j \in \mathrm{X}_v \}$ satisfying $\sum_{j \in \mathrm{X}_v} k_j^{(v)} = 0$.

b) Let $\varpi^{(v)}$ be the partition of X whose elements are all non-empty subsets $\mathrm{I}(j, v)$ of X for j varying in X_v. (Each element $i \in \mathrm{X}$ belongs to one and only one subset $\mathrm{I}(j, v)$, since the external line l_i of T is connected to v through a unique path which contains a unique line l_j with $j \in \mathrm{X}_v$).

Let then $i_{\varpi^{(v)}}$ be the linear mapping from $\mathbb{C}_{\mathrm{X}}^{4(n-1)}$ onto $\mathbb{C}_{\mathrm{X}_v}^{4(n_v-1)}$ associated with the partition $\varpi^{(v)}$ of X (see formula (3.1)). In view of Lemma 3.4, $H^{(n)}(i_{\varpi^{(v)}}(K))$ is a general n-point function, and we then define a tree convolution product $H_T^{(n)}$ as follows,

$$H_T^{(n)}(K) = \prod_{v\in\mathcal{N}} H^{(n_v)}(i_{\varpi^{(v)}}(K)) \cdot \prod_{l_i\in\mathcal{L}} [H^{(2)}(K_I)]^{-1}. \tag{3.3}$$

The first product is taken over the set $\mathcal{N}$ of vertices of T. The second one is taken over all internal lines $l_i(I, X\backslash I)$ denotes the line partition of T which is associated with the line l_i); the inverse of the general two point function $H^{(2)}$ associated with the internal line l_i, is used in order to produce a simple pole at $P_I^2 = m^2$ in the corresponding channel (I, X\I).

The following property has been proved in [*4*].

LEMMA 3.5. — $H_T^{(n)}(K)$ *defined by* (3.3) *is a general n-point function.*

DEFINITION 3.*d*.1. — Let us consider a function $H_T^{(n)}$ of tree type as given by def. 3.*c*.1. A vertex v_0 of T is said to be a virtual vertex for $H_T^{(n)}(K)$ if every pure vertex partition connected with v_0 is a trivial partition for $H_T^{(n)}(K)$.

DEFINITION 3.*d*.2. — A general n-point function $H_T^{(n)}$ of tree type is said to be of « special tree type » if at least one vertex v of T is virtual for $H_T^{(n)}$, in the sense of Definition 3.*d*.1.

DEFINITION 3.*e*.

3.*e*.1. — Let X be a set of indices with N elements: $|X| = N$. We consider a family F of subsets of X: $F = \{ X_a \subset X, a \in \mathcal{A} \}$ such that $\forall a, b \in \mathcal{A}$ one has either $X_b \subset X_a$ or $X_a \subset X_b$; $\forall a \in \mathcal{A}$ we put $|X_a| = n_a$.

3.*e*.2. — We define an order Ω over the set X of indices and we denote by $K \in \mathbb{C}_X^{4(n-1)}$ the following ordered set of variables:

$$K = \left\{ K_i \in \mathbb{C}^4 \quad i \in X; \quad \sum_{i\in X} K_i = 0 \right\}. \tag{3.4}$$

For every $a \in \mathcal{A}$ we denote by Ω_a the order induced by Ω on the set X_a of indices and we denote by $K^a \in \mathbb{C}_{X_a}^{4(n_a-1)}$ the set:

$$K^a = \left\{ K_i^a \quad i \in X_a; \quad \sum_{i\in X_a} K_i^a = 0 \right\}. \tag{3.5}$$

Let n_a denote the maximal (following Ω_a) index of X_a. We define the following linear mapping of $\mathbb{C}_X^{4(N-1)}$ onto $\mathbb{C}_{X_a}^{4(n_a-1)}$

$$i_a(K) = \left\{ K_j^a = K_j \quad \forall j \in X_a \backslash \{ n_a \}, \quad K_{n_a}^a = K_{n_a} + \sum_{i \in X \backslash X_a} K_i \right\}. \quad (3.6)$$

3.*e*.3. — With every set $X_a \in F$ we associate a general n_a-point function $H^{n_a}(K^a)$ in $\mathbb{C}_{X_a}^{4(n_a-1)}$. We then consider the inverse image of such a function:

$$H_a^N(K) = H^{n_a}(i_a(K)) . \quad (3.7)$$

By application of Lemma 3.4 we obtain the following:

LEMMA 3.6. — $H_a^N(K)$ *defined by* (3.14) *is a general* N-*point function.*

Concluding remark.

Applying all the preceding lemmas to the « blown-up » renormalized integrand $\tilde{R}_G$, we would like to obtain the n-point character for it. Unfortunately, after Bros-Lassalle ([*3*] [*4*]) we know that the corner stone of their construction which allows the conservation of the primitive analyticity domains of n-point functions by G-convolution is the set of Steinman relations; in the renormalized context, the procedure used to work on these relations does not allow us to prove them for a general graph, giving only a partial result ([*10*]). This fact may be seen as a crucial obstruction due to renormalization, and forces us to a strong technical restriction analyzed in the last section, useful in the ϕ^4 and Bethe-Salpeter contexts.

SECTION 4

4.1. PROPERTIES OF THE « BLOWN-UP » RENORMALIZED INTEGRAND

4.1.1. Definition of simple graphs.

In the following sections, we restrict ourselves to a certain kind of graphs G called « simple graphs » which is sufficient for our present purpose (in view of [*1*]).

These graphs have the following structure (see Fig. 1).

There exists only two generalized vertices V_i, $i = 1, 2$ and $L + 1$ internal lines linking them.

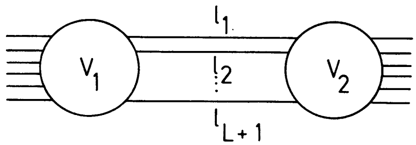

FIG. 1.

To each vertex V_i (resp. to each line l_i) is associated a general n_{V_i}-point function (resp. a general two-point function).

We are going to prove that for such graphs, the corresponding renormalized G-convolution product H_G^{ren} (see definition 4.13 below) is a general n-point function.

For this purpose we use the technique of Lassalle [*4*], starting with tree graphs and then going on by a recursive integration associated with the blowing-up procedure of Section 2.

4.1.2. Analytic properties of $\tilde{R}_G$.

In a given G-forest U we consider the function $\tilde{Y}_\gamma$ corresponding to a given $\gamma \in U(G)$ (after the « blowing up » operation of sec. 2) following the recursion formula:

$$\tilde{Y}_\gamma = \tilde{I}_{\bar{\gamma}} \cdot S_a^{\gamma *}(- t^{d(\gamma_a)})\tilde{Y}_{\gamma_a} \tag{2.39}$$

with γ_a the maximal element of $U(\gamma)$. Notice that now, for every $\gamma \subset G$, there is at most one maximal subgraph.

In what follows we shall apply all general results of sec. 3 on the formula (2.39) when $\gamma = G$, which yields recursively the contribution of every forest term in eq. (2.36) of $\tilde{R}_G$.

DEFINITION 4.*a*.

4.*a*.1. — Let $X_1^{(\gamma)} = \{1, \ldots, n_\gamma\}$ (resp. $X_1 = \{1, \ldots, n\}$) denote the set of indices of external momenta K^γ of T_γ (resp. K of T_G). Let $X_2^{(\gamma)} \cup \bar{X}_2^{(\gamma)}$ (resp. $X_2 \cup \bar{X}_2$) denote the set of indices of internal independent momenta $(k^\gamma, k^{\gamma'})$ of T_γ (resp. k, k' of T_G) as given by the Ω-assignment (eq. (2.12)-(2.16)); more precisely: with the ordered (following Ω_γ) set of lines $\mathscr{K}_\gamma \cup \mathscr{K}'_\gamma$ (resp. $\mathscr{K}_G \cup \mathscr{K}'_G$) of T_γ (of T_G) we associate the ordered set of indices $X_2^{(\gamma)} \cup \bar{X}_2^{(\gamma)}$ (resp. $X_2 \cup \bar{X}_2$), so we write:

$$X^{(\gamma)} = X_1^{(\gamma)} \cup X_2^{(\gamma)} \cup \bar{X}_2^{(\gamma)}, \ |X_1^{(\gamma)}| = n_\gamma, \ |X_2^{(\gamma)}| = |\bar{X}_2^{(\gamma)}| = L_\gamma \tag{4.1}$$

$$(\text{resp. } X = X_1 \cup X_2 \cup \bar{X}_2, \ X = n, \ |X_2| = |\bar{X}_2| = L) \tag{4.1'}$$

In what follows we shall use the followings sets of variables:

$$(K^\gamma, k^\gamma, k'^\gamma) = \Big\{ K_i \quad i \in X_1^\gamma, k_j^{(\gamma)} \quad j \in X_2^{(\gamma)}, k_{j'}^{(\gamma)} \quad j' \in \overline{X}_2^{(\gamma)} \quad \text{with:} \sum_{i \in X_1^{(\gamma)}} K_i^\gamma + \sum_{\substack{j \in X_2^{(\gamma)} \\ j' \in \overline{X}_2^{(\gamma)}}} k_j^{(\gamma)} + k_{j'}^{(\gamma)} = 0 \Big\}$$

$$(k^\gamma, k^{\gamma'}) = \Big\{ k_j^\gamma, j \in X_2^{(\gamma)} \quad k_{j'}^{(\gamma)}) j' \in \overline{X}_2^{(\gamma)} \quad \text{with:} \sum_{\substack{j \in X_2^{(\gamma)} \\ j' \in \overline{X}_2^{(\gamma)}}} k_j^\gamma + k_{j'}^\gamma = 0 \Big\} \tag{4.2}$$

Analogous definitions hold for the sets (K, k, k'), (k, k') when $\gamma = G$.

4.*a*.2. — Let $\{ L'_\gamma \}$ (resp. L') denote the maximal following Ω_γ (resp. Ω_G) index in $\overline{X}_2^{(\gamma)}$ (resp. in $\overline{X}_2$). We define the analogs of linear application (3.6) for every $\gamma \in U(G)$ and G:

$$i_\gamma(K, k, k') = \Big\{ k_i^{(\gamma)} = k_i \quad i \in X_2^{(\gamma)}, k_{i'}^{(\gamma)} = k_{i'} \quad i' \in \overline{X}_2^{(\gamma)} \setminus \{ L'_\gamma \} ;$$
$$k_{L'_\gamma}^{(\gamma)} = k_{L'_\gamma} + \sum_{i \in X_1} K_i + \sum_{j \in X_2 \setminus X_2^{(\gamma)}} k_j + \sum_{j' \in \overline{X}_2 \setminus \overline{X}_2^{(\gamma)}} k_{j'} \Big\} \tag{4.3}$$

$$i_G(K, k, k') = \Big\{ k_i^G = k_i \; i \in X_2, k_{i'}^G = k_{i'} \; i' \in \overline{X}_2 \setminus \{ L \} ; k_{L'}^G = k_{L'} + \sum_{i \in X_1} K_i \Big\} \tag{4.4}$$

For a given forest U we then state the following:

PROPOSITION 4.1. — *The function* F_U *in formula* (2.37) *is a general* $n + 2L$ *point function in the space* $\mathbb{C}^{(4+2L-1)}$ *of the set* (K, k, k').

For the proof of this property let us first show two useful statements:

LEMMA 4.1. — *For every subgraph* $\gamma \in U(G)$ *the corresponding function* $\tilde{I}_{\bar\gamma}(K^\gamma, k^\gamma, k^{\gamma'})$ *is a general* $(n_\gamma + 2L_\gamma)$*-point function of special tree type.*

Proof. — From Lemma 2.1 of the « blowing up » operation and Definitions 2.27 (for $\tilde{I}_{\bar\gamma}$) and 3.*c*.1 of tree type functions, it follows that $\tilde{I}_{\bar\gamma}(K^\gamma, k^\gamma, k^{\gamma'})$ is a general $n_\gamma + 2L_\gamma$-point function of tree type in the space $\mathbb{C}^{4(n_\gamma + 2L_\gamma - 1)}$ of the variables $(K^\gamma, k^\gamma, k^{\gamma'})$. Notice that as we can verify from formula (2.27), the corresponding tree graph $T_{\bar\gamma}$ (with $n_\gamma + 2L_\gamma$ external lines) has the following special property: the general n_v-point functions (resp. the general 2-point functions) which correspond to all vertices $v \in \mathcal{N}_\gamma$ such that $v \in \mathcal{N}_{\gamma_a}$ (resp. $l_i^\gamma \in \mathcal{L}(\gamma_a)$) with γ_a maximal subgraph in $U(\gamma)$, are equal to the constant 1.

LEMMA 4.2. — *For each subgraph* $\gamma \in U(G)$ *the corresponding function* $\tilde{Y}_\gamma$

is a finite sum of products of $\tilde{I}_{\bar\gamma}$ *by general* $2L_{\gamma_a}$*-point functions corresponding to* γ_a*, with* γ_a *maximal subgraph in* $U(\gamma)$*. Precisely:*

$$\tilde{Y}_\gamma = \tilde{I}_{\bar\gamma}(K^\gamma, k^\gamma, k^{\gamma'}) \cdot \sum_j P^{(j)}(K^\gamma, k^\gamma, k^{\gamma'}) \cdot \tilde{S}_a^{\gamma *} H_{T_a}(k^{\gamma_a}, k^{\gamma'_a}) \tag{4.6}$$

here $P^{(j)}$ *are polynomials and* H_{T_a} *are general* $2L_{\gamma_a}$ *point functions of special tree type.*

Proof. — For the proof we use the following recurrence hypothesis. We suppose that the statement holds for the maximal element γ_a of $U(\gamma)$; we then show that it is true for γ.

a) Action of $(-t^{d(\gamma_a)})$:

From the recurrence hypothesis (4.6) and the property of Taylor operator to act only on the external variables K^{γ_a} we obtain:

$$(-t^{d(\gamma_a)})\tilde{Y}_{\gamma_a} = \sum_{j_a} \tilde{S}_\tau^{a*} H_{T_\tau}(k^{\gamma_a}, k^{\gamma'_a}) \sum_{\lambda_a} K^{(\gamma_a)\lambda_a} \frac{1}{\lambda_a!} \frac{\partial^{|\lambda_a|}}{\partial K^{(\gamma_a)\lambda_a}} \left\{ \tilde{I}_{\bar\gamma_a}(K^{\gamma_a}, k^{\gamma_a}, k^{\gamma_a})' P^{(j_a)}(K^{\gamma_a}, k^{\gamma_a}, k^{\gamma'_a}) \right\}_{K^{\gamma_a}=0} \tag{4.7}$$

where τ is the maximal subgraph of $U(\gamma_a)$.

From Lemmas 3.1 and 3.3 the derivations and the restriction to $K^{\gamma_a} = 0$ yield products of polynomials and general $2L_{\gamma_a}$ point functions $H_{T_a}(k^{\gamma_a}, k^{\gamma'_a})$; more precisely H_{T_a} are of (special) tree type because the linearity of the derivative operators conserves the (special) tree type character of $\tilde{I}_{\bar\gamma_a}$ (Lemma 4.1) given by formula (2.27); the corresponding tree graph T_a results from T_{γ_a} when all the external lines of the latter are eliminated, so:

$$\frac{\partial^{|\lambda_a|}}{\partial K^{(\gamma_a)\lambda_a}} (\tilde{I}_{\gamma_a} \cdot P^{(j_a)}) = \sum_{l_a} \tilde{P}^{(l_a)} H_{T_a}^{(l_a)}(k^{\gamma_a}, k^{\gamma'_a}). \tag{4.8}$$

b) Action of $\tilde{S}_a^{\gamma *}$.

Following definition (2.34) and property 2.35 *a*) we obtain:

$$\begin{aligned} \tilde{S}_a^{\gamma *}(\tilde{S}_\tau^{a*} H_{T_\tau}(k^{\gamma_\tau}, k^{\gamma'_\tau})) &= \tilde{S}_\tau^{\gamma *} H_{T_\tau}(k^{\gamma_\tau}, k^{\gamma'_\tau}) \\ \tilde{S}_a^{\gamma *}(K^{(\gamma_a)\lambda_a}) &= P^{(\lambda_a)}(K^\gamma, k^\gamma, k^{\gamma'}) \end{aligned} \tag{4.9}$$

where $P^{(\lambda_a)}$ is a polynomial of degree λ_a in $(K^\gamma, k^\gamma, k^{\gamma'})$.

Finally, by combining Lemma 4.1 with properties (4.9) we obtain formula (4.6).

Proof of Proposition 4.1. — We apply Lemmas 4.1 and 4.2 to $\gamma = G$ to obtain:

$$\tilde{Y}_G(K, k, k') = \tilde{I}_{\bar G}(K, k, k') \cdot \sum_l P^{(l)}(K, k, k') \cdot S_\gamma^{G*} H_{T_\gamma}^{(l)}(k^\gamma, k^{\gamma'}) \tag{4.10}$$

where $\tilde{I}_{\bar{G}}$ is a general $n + 2L$ point function in $\mathbb{C}^{4(n+2L-1)}_{(K,k,k')}$ and $H_{T_\gamma}(k^\gamma, k^{\gamma'})$ is a general $2L_\gamma$ point function in $\mathbb{C}^{4(2L_\gamma-1)}_{(k^\gamma,k^{\gamma'})}$. We take the inverse images of H_{T_γ} by the mapping (4.3) to obtain a sum of products of polynomials by general $n + 2L$ point functions in $\mathbb{C}^{4(n+2L-1)}_{(K,k,k')}$.

$$\tilde{Y}_G = \tilde{I}_{\bar{G}}(K, k, k') \cdot \sum_l P^{(l)} \cdot \tilde{H}_{T_\gamma}(i_\gamma(K, k, k')) \qquad (4.11)$$

with $\tilde{Y}_G$, we associate the following tree graph (Fig. 2).

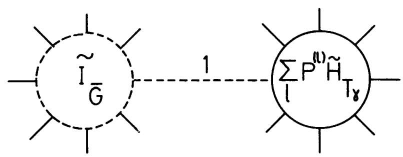

FIG. 2.

We then apply Lemma 3.5 of tree type functions and Lemma 3.2 for the sum to obtain that $\tilde{Y}_G(K, k, k')$ or F_U satisfies Proposition 4.1.

On the other hand, application of the proof of Lemmas 4.1, 4.2.*a*) for I_{G_T} and $(-t^{d(G)})Y_G$ yields that:

$$F_U = \tilde{I}_{G_T}(K, k, k') \cdot \sum_j P^{(j)}(K, k, k') \cdot H^{(j)}_{T_G}(k^G, k^{G'}) \cdot S^{G*}_\gamma H^{(l)}_{T_\gamma}(k^\gamma, k^{\gamma'}) \qquad (4.12)$$

By taking the inverse images of H_{T_G}, H_{T_γ} by def. (4.3), (4.4) respectively and by applying Proposition 4.1 and Lemma 3.2 we obtain the announced result for a full forest.

We note now that the inverse 2-point functions $[H^{(2)}(k_i)]^{-1}$ in formula (2.37) are analytic functions which change nothing in the previous results, so we apply Proposition 4.1 inside the formula (2.37) for every forest U; by Lemma 3.2 *b* then we obtain finally for the sum:

THEOREM 4.1. — *For a graph* G *of simple type the renormalized integrand* $\tilde{R}_G(K, k, k')$ *defined after the « blowing up » procedure of* G, *is a general* $n + 2L$ *point function in the space* $\mathbb{C}^{4(n+2L-1)}$ *of the set* (K, k, k').

4.2. ALGEBRAIC AND ANALYTIC PROPERTIES OF THE RENORMALIZED G-CONVOLUTION PRODUCT H^{ren}_G

4.2.1. Definition of H^{ren}_G.

We have seen in section 2 the alternate « blown-up » definition of the euclidean renormalized integrand R_G, which is:

for
$$R_G(K, k) = \tilde{R}_G(K, k, k')\,|_{k+k'=0}$$
$$(K, k) \in \mathscr{E}_{(K,k)} = \mathscr{E}_{(K)} \times E^{rL}_{(k)}, \quad r \leq 4 .$$

Due to theorem 4.1, we have defined the extension of $R_G(K, k, k')$ in complex Minkowski space.

This definition leads to a recursive definition of the euclidean G-convolution product H_G^{ren}, and is well-suited to obtain all algebraic and analytic properties of H_G^{ren} in complex Minkowski space, by means of successive integrations in the appropriate analyticity domain, using Bros-Lassalle's procedure [*4*].

Let us consider then the formal expression:

$$H_G^{ren}(K) = \int_{\Gamma(K)} \tilde{R}_G(K, k, k')\,|_{k+k'=0}\,dk \tag{4.13}$$

where $\Gamma(K)$ is a rL-dimensional complex cycle to be precised.

In view of the blown-up procedure, the announced recursion is the following:

Suppose that, for a simple graph with $l < L$ independent loops, we can define a general n-point function, then the same can be done for a graph with L independent loops.

More precisely, we consider a simple graph G (fig. 3), with N external lines and $|\mathscr{K}_G| = L$ independent lines $l_1, \ldots, l_L$. Suppose we have « closed » (integrated over) m « blowing-up » lines $l_1 \ldots l_m$, $m < L$. Then we define for $m \leqslant L$:

$$H_G^{(m)}(K) = \int H_G^{(m-1)}(K, k_{m+1}, k'_{m+1})\,|_{k_{m+1} = -k'_{m+1} = t}\,dt \tag{4.14}$$

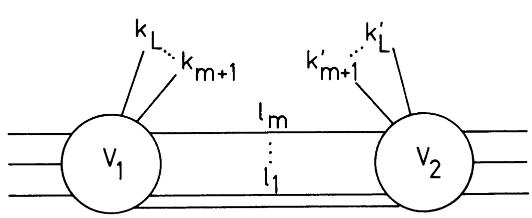

FIG. 3.

The definition of H_G is completely specified by (4.14), and:

Indeed, we have:
$$H_G^0(K) = \tilde{R}_G(K).$$
$$H_G^L(K) = H_G^{ren}(K). \tag{4.15}$$

In these expressions, the symbolic notation K denotes all external momenta, including « blown-up » ones.

We assume now the recursion hypothesis:
— $H_G^{(m-1)}$ is a general $(N + \nu)$-point function, with:

$$\nu = 2(L - (m - 1)) \tag{4.16}$$

— $H_G^{(m-1)}(K, k)$ belongs to the Weinberg class $A_{r\nu}^{\alpha_{m-1}}$ for every fixed value of K in $\mathscr{E}_{(K)}^{r(N-1)}$, with:

$$\sup_{\substack{S' \neq \{0\} \\ S' \in E_{(k)}^{r\nu}}} (\alpha_{m-1}(S') + \dim S') = -1 . \tag{4.17}$$

Notice first that, by theorem 4.1, $\tilde{R}_G$ is a general $(N + 2L)$-point function, and, by theorem 4.1 *a*) of [2], $\tilde{R}_G$ belongs to a precise Weinberg class $A_{rL}^{\alpha_0}$, with:

$$\sup_{\substack{S' \neq \{0\} \\ S' \subset E_{(k)}^{rL}}} (\alpha_0(S') + \dim S') = -1 .$$

So, the recursion hypothesis being true for $H_G^0 = \tilde{R}_G$, it ramains to prove it for $H_G^{(m)}$.

4.2.2. **Algebraic and analytic properties of H_G^{ren}.**

The strategy we use to prove algebraic and analytic properties of H_G, has been introduced by Bros [3] and Lassalle [4] for convergent graphs (which need no renormalization). Modulo several adaptations, we follow closely their iterative procedure, emphasizing only the differences between the convergent case of [3] [4] and our renormalized one. For details, we refer to [4].

In (4.14) we investigate first the G-convolution on the analytic submanifold defined by:

$$k_{m+1} + k'_{m+1} = 0 . \tag{4.18}$$

In order to make the notations more symmetric, we slightly modify them, in term of a new graph $\tilde{G}$ obtained from G by cutting the line l_m, we also adopt a sequential notation of blown-up momenta k and k'. Then (4.14) becomes:

$$H_G(k) = \int H_{\tilde{G}}(k, k_{n+1}, k_{n+2})\,|_{k_{n+1} = -k_{n+2} = t} dt \tag{4.14 bis}$$

where H_G is a n-point function, with $n = N + 2(L - m)$, and $H_{\tilde{G}}$ the corresponding $(n + 2)$-function associated with $\tilde{G}$. So, we study the G-convolution on the submanifold of $\mathbb{C}^{r(n+1)}$ defined by:

$$k_{n+1} + k_{n+2} = 0 . \tag{4.18 bis}$$

We deduce global analyticity by successive local analytic continuations.

a) *Choice of a time direction.*

Let $\hat{e} \in V^+$ a time direction, and the related coordinate system in $\mathbb{R}^r$. We note:

$$\forall i = 1, \ldots, n : \quad k_i = p_i + iq_i = (\vec{k}_i, k_i^0) = (\vec{p}_i + i\vec{q}_i, p_i^0 + iq_i^0)$$
$$t = u + iv = (\vec{t}, t^0) = (\vec{u} + i\vec{v}, u^0 + iv^0)$$

and

$$V_{\hat{e}} = \{ k \in \mathbb{C}^{r(n-1)} : \vec{q}_i = 0 \quad i = 1, \ldots, n \}.$$

Considerations of holomorphy domain lead to the introduction of the flat tubes:

$$S_\mu = \{ k \in \mathbb{C}^{r(n-1)} : \text{Im } \vec{k} = 0, \text{Im } k^0 \in \sigma_\mu \}$$

associated with the so-called « pseudocell » σ_μ.

b) *Analyticity of the integrand.*

We shall use the following notation:

$$\pi = \{ (k, k_{n+1}, k_{n+2}) \in \mathbb{C}^{r(n+1)} : k_{n+1} + k_{n+2} = 0 \}$$

and $H_{\tilde{G},\pi}$ is the restriction of $H_{\tilde{G}}$ to this analytic hyperplane π. Let $(k, t) \in \mathbb{C}^{rn}$ be the current point of π, with $t = k_{n+1} = -k_{n+2}$, then $H_{\tilde{G},\pi}$ becomes: $H_{\tilde{G},\pi}(k, t)$.

We use the following notations:

$$\Xi_I = \{ (k, t) \in \pi, k_I^2 = m_I^2 \} \cup \{ (k, t) \in \pi, k_I^2 = M_I^2 + \rho, \rho \geqslant 0 \}$$
$$\forall I \in \mathscr{P}^*(X)$$
$$\Sigma_I = \{ (k, t) \in \pi, (k_I - t)^2 = m_I'^2 \} \cup \{ (k, t) \in \pi, (k_I - t)^2 = M_I'^2 + \rho, \rho \geqslant 0 \}$$
$$\forall I \in \mathscr{P}^*(X)$$
$$\Sigma_0 = \{ (k, t) \in \pi, t^2 = m^2 \} \cup \{ (k, t) \in \pi, t^2 = M^2 + \rho, \rho \geqslant 0 \}$$
$$\Sigma = \Xi_I \cup \Sigma_I \cup \Sigma_0$$
$$W_{\hat{e}} = \{ (k, t) \in \pi, \vec{q}_i = 0, i = 1, \ldots, n; \vec{v} = 0 \}$$

where X stands for the set $\{ 1, \ldots, n \}$ of indices labelling the external lines of G.

Then we have (prop. 8 of [*4*]):

PROPOSITION 4.2. — $H_{\tilde{G},\pi}$ *is analytic in the variables* $\{ t^0, k_i^0 ; 1 \leq i \leq n \}$ *in the region:* $W_{\hat{e}} \backslash \Sigma$.

We must prove then, integrability of $H_{\tilde{G},\pi}$ at the infinity, in the euclidean directions.

By the recursion hypothesis:

$$H_{\tilde{G}} \in A_{r(n+2)}^{\alpha_{n+2}}$$

with

$$\sup_{\substack{S' \neq \{0\} \\ S' \subset E^{r(n+2)}}} (\alpha_{n+2}(S') + \dim S') < 0 .$$

Then:

$$H_{\tilde{G},\pi} \in A^{\tilde{\alpha}_{n+1}}_{r(n+1)}$$

where $\tilde{\alpha} = \alpha|_\pi$ is the restriction of α to π.

Then, Weinberg's theorem holds and:

$$H_{\tilde{G},\pi} \in L^1(E^{r(n+2)}_{(k,kn+1,kn+2)} \cap \pi). \tag{4.19}$$

We have now the description of the analyticity domain of $H_{\tilde{G},\pi}$, by its sections in t_0 when k is fixed outside the family $\{ \Xi_I, I \in \mathscr{P}^*(X) \}$, and $\vec{t}$ is fixed, arbitrary:

PROPOSITION 4.3 [*4*]. — $H_{\tilde{G},\pi}$ *is analytic at all points of the complex plane* t^0 *which do not belong to the union* $\hat{\Sigma}$ *of the cuts:*

$$\hat{\Sigma}_0 = \{ t^0 \in \mathbb{C} : (t^0)^2 = (\vec{t})^2 + m^2 \} \cup \{ t^0 \in \mathbb{C} : (t^0)^2 = (\vec{t})^2 + M^2 + \rho, \rho \geqslant 0 \}$$

$$\hat{\Sigma}_I = \{ t^0 \in \mathbb{C} : (t^0 - k_I^0)^2 = (\vec{t} - \vec{p}_I)^2 + m_I'^2 \} \cup \{ t^0 \in \mathbb{C} : (t^0 - k_I^0)^2 = (\vec{t} - \vec{p}_I)^2 + M_I'^2 + \rho, \rho \geqslant 0 \} \quad \forall I \in \mathscr{P}^*(X).$$

So, any line $\mathscr{L} \in \pi$ passing through the origin and each point $\{ k_I^0, I \in \mathscr{P}^*(X) \}$ with a slope different from zero, lies inside the t^0-section of the domain of analyticity of $H_{\tilde{G},\pi}$ if:

i) $\quad \forall I \in \mathscr{P}^*(X) \quad \hat{\Sigma}_0 \neq \hat{\Sigma}_I \quad$ (i. e. : $q_I^0 \neq 0$)

ii) $\quad \forall I, J \in \mathscr{P}^*(X) \quad \hat{\Sigma}_I \neq \hat{\Sigma}_J \quad$ (i. e. : $q_I^0 \neq q_J^0$).

c) *Analytic continuations.*

. The analogous of proposition 11 of [*4*] holds:

PROPOSITION 4.4. — *With any point k inside a flat tube of the family* $\{ S_\mu, \mu \in \Theta^{(n)} \}$ *it is possible to associate a four dimensional real region* $\Gamma_{\hat{e},k} \subset \mathbb{C}^4$ *such that:*

$$H_{\tilde{G},\hat{e},\mu}(k) = \int_{\Gamma_{\hat{e},k}} H_{G,\pi}(k, t) dt$$

is analytic inside a neighbourhood of S_μ.

$\Gamma_{\hat{e},k} = \mathbb{R}^3 \times \mathscr{L}_k$ with $\mathscr{L}_k$ passing inside the analytic domain of $H_{\tilde{G},\pi}$ in t_0, through the origin and the points $\{ k_I^0, I \in \mathscr{P}^*(X) \}$, such that its infinite parts be parallel with the imaginary axis, to ensure euclidean convergence.

We can therefore move $\mathscr{L}_k$ so that it does not intersect the cuts, so we have the analyticity inside a neighbourhood of S_μ.

. We must verify the integrability of $H_{G,\hat{e},\mu}$ in the euclidean region. We claim this later property holds, thank to the Weinberg's criterium (eq. (4.19)) without further proof.

PROPOSITION 4.5. — $H_{G,\hat{e},\mu}$ *is integrable in euclidean regions.*

. We consider now a given cell γ_λ and the family $\{ \sigma_\mu, \mu \in M_\lambda \}$ of the pseudocells in γ_λ. In any flat tube S_μ of $\{ \sigma_\mu, \mu \in M_\lambda \}$, a function $H_{G,\hat{e},\mu}$ analytic inside a neighbourhood of S_μ has been introduced. We have then:

PROPOSITION 4.6. — *The* $\{ H_{G,\hat{e},\mu} ; \mu \in M_\lambda \}$ *are pieces of a unique function* $H_{G,\hat{e},\lambda}$ *which is integrable in euclidean directions and analytic inside a neighbourhood* $\mathcal{D}_{\hat{e},\lambda}$ *of the flat tube* $\mathcal{T}_\lambda \cap V_{\hat{e}}$. $H_{G,\hat{e},\lambda}$ *is increasing slowly near the real in* $\mathcal{D}_{\hat{e},\lambda}$ *and has therefore a boundary value in the distributional sense.*

Remark. — We notice here the crucial necessity of « complete Steinmann relations » in order to avoid the « pinched singularities », and recover analyticity of $H_{G,\hat{e},\mu}$. This is the reason which forced us to restrict general graphs to simple ones: The most general cases of graphs need more sophisticated technical tools to ensure the Steinmann relations for all overlapping subgraphs.

We find then that proposition 14 of [*4*] is still valid: we can move the time direction $\hat{e}$ in the light cone, and the various $H_{G,\hat{e},\lambda}$ are pieces of the same analytic function $H_{G,\lambda}$. Then:

PROPOSITION 4.7. — *The* $\{ H_{G,\lambda} ; \lambda \in \Lambda^{(n)} \}$ *are pieces of a unique function* H_G *which is a general n-point function.*

. Finally, the proof of the independence of the « antecedent » by the blowing-up procedure, still holds in our case and, taking into account the initial point of the recursion (tree graphs), we get the final result:

THEOREM 4.2. — *The convolution product* $H_G^{\text{ren}}(K)$ *associated with a simple graph with n external lines is a general n-point function.*

4.2.3. Application to Φ^4 coupling.

The relevant graphs to be considered in this theory are the following, corresponding to $(n + 1)$-point functions symbolized by « Bubble-vertices »:

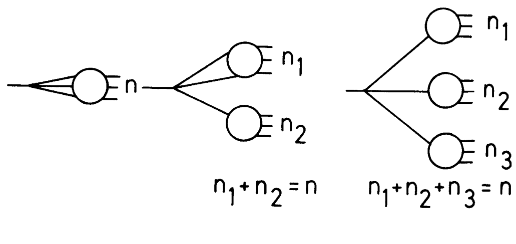

FIG. 4.

These graphs correspond to the three principal terms of the equations of motion for Schwinger functions of Φ_4^4.

The graph 4.*a* is a special case of a simple graph.

The graph 4.*c* is a graph of tree type. In 4.*b*, the only one particle-irreducible renormalization part is:

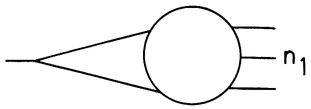

FIG. 5.

So, by a trivial extension of case of simple graphs, we are in position to apply theorem 5.1, and to get our main result.

THEOREM 4.3. — *The renormalized* G-*convolution conserves all linear algebraic and analytic properties of axiomatic field theory for all constitutive graphs of the* Φ_4^4 *theory.*

REFERENCES

[1] M. MANOLESSOU-GRAMMATICOU, *Renormalized normal product and equations of motion of* Φ^4*-coupling*. Preprint, Bielefeld, 1982.

[2] J. BROS, M. MANOLESSOU-GRAMMATICOU, *Commun. Math. Phys.*, t. **72**, 1980, p. 175-205; *Commun. Math. Phys.*, t. **72**, 1980, p. 207-237.

[3] J. BROS, Thèse. In *Analytic Methods in Mathematical Physics*, Newton Ed., Gordon and Breach, 1970.

[4] M. LASSALLE, *Commun. Math. Phys.*, t. **36**, 1974, p. 185-226.

[5] M. MANOLESSOU-GRAMMATICOU, *Ann. Phys.*, t. **122**, 1979, p. 297-320.

[6] W. ZIMMERMANN, *Commun. Math. Phys.*, t. **15**, 1969, p. 208-234.

[7] W. ZIMMERMANN, *Commun. Math. Phys.*, t. **11**, 1968, p. 1-8.

[8] H. EPSTEIN, V. GLASER, Rep. TH1400 *CERN*, 1971.

[9] H. EPSTEIN, V. GLASER, *Ann. I. H. P.*, t. **19**, 1973, p. 211-295.

[10] M. MANOLESSOU-GRAMMATICOU, *Thesis*, Orsay, 1977.

[11] S. WEINBERG, *Phys. Rev.*, t. **118**, 1960, p. 838-849.

[12] B. DUCOMET, *Logarithmic asymptotic behaviour of the renormalized* G-*convolution product in* 4-*dimensional euclidean space*. (To appear).

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