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FRANCO CARDIN

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**Existence and uniqueness theorems
for viscous fluids capable of heat conduction
in a relativistic theory
of non stationary thermodynamics**

by

Franco CARDIN (*)

Istituto di Analisi e Meccanica

Università di Padova, Via Belzoni 7, 35100 Padova, Italie

ABSTRACT. — A hyperbolization method for any conservative P. D. E. system, whose solutions must satisfy a conservative inequality (e. g. a Clausius-Duhem generalized inequality), is worked out in General Relativity, in the first part of this paper, on the basis of an early work of K. O. Friedrichs and P. D. Lax [13]. More in detail, a hyperbolization method of T. Ruggeri and A. Strumia [17], very useful to study the evolution of continuous bodies in the frame-work of I. Müller's thermodynamics, is generalized in such a way to be useful to the construction of relativistic continuous theories in a thermodynamic frame-work of B. D. Coleman and W. Noll's type. In [17] basic theorems of existence, uniqueness, and continuous dependence on data are proved to hold for the evolution system of a relativistic simple fluid \mathcal{F} incapable of heat conduction; and certain bounds on the velocities of shock waves travelling in \mathcal{F} are determined. In the second part of this paper the tools worked out in the first one are applied to A. Bressan's relativistic thermodynamics, based on a certain law of heat conduction—see [8]—; and the analogues of the above results for \mathcal{F} are proved here to hold, within A. Bressan's theory, for heat conducting fluids, possibly in the presence of viscosity (with relaxation terms).

(*) Istituto di Analisi e Meccanica dell'Università di Padova, via Belzoni n° 7, 35100 Padova. This paper has been performed in the sphere of activity of research group n. 3 of the C. N. R. (Consiglio Nazionale delle Ricerche).

RÉSUMÉ. — Dans la première partie de cet article, une méthode d'hyperbolisation pour un système d'EDP arbitraire dont les solutions satisfont une inégalité de conservation (par exemple une inégalité de Clausius-Duhem généralisée) est mise en œuvre en Relativité Générale, sur la base d'un travail de K. O. Friedrichs et P. D. Lax [13]. Plus précisément, une méthode d'hyperbolisation récente de T. Ruggieri et A. Strumia [17], très utile pour étudier l'évolution de corps continus dans le cadre de la Thermodynamique de I. Müller, est généralisée de façon à être utile pour construire des théories relativistes continues dans le cadre thermodynamique du type de B. D. Coleman et W. Noll. Dans [17] sont démontrés des théorèmes d'existence, d'unicité et de continuité par rapport aux données pour l'évolution d'un fluide simple relativiste \mathcal{F} ne conduisant pas la chaleur, et des bornes sont obtenues sur la vitesse des ondes de choc dans \mathcal{F} . Dans la seconde partie de cet article, les outils construits dans la première sont appliqués à la thermodynamique relativiste de A. Bressan, basée sur une certaine loi de conduction de la chaleur, voir [8], et les analogues des résultats précédents sont démontrés, dans le cadre de la théorie de Bressan, pour des fluides conducteurs de la chaleur, éventuellement en présence de viscosité (avec des termes de relaxation).

1. Introduction.

In 1971 K. O. Friedrichs and P. D. Lax [13] studied a first order quasi-linear system of P. D. E. s in *conservative* form

$$(1.1) \quad \partial_t U^j + \partial_L f^{jL} = 0, \quad f^{jL} = f^{jL}(U), \quad j = 1, \dots, N, \quad L = 1, 2, 3,$$

which *implies* a further conservation equation

$$(1.2) \quad \partial_t S + \partial_L F^L = 0, \quad S = S(U), \quad F^L = F^L(U),$$

in case S is *convex*, i. e. $\|(\partial^2 S / \partial U^i \partial U^j)(U)\|$ is (strictly) positive definite. In more detail, every solution $U^i(t, x^L)$ of (1.1) is supposed to satisfy (1.2). The above authors deduced some restriction relations among f^{jL} , S , F^L , and their derivatives, and they proved that the convexity of S is sufficient to state that (1.1) is equivalent to a *symmetric* and *hyperbolic* system (in K. O. Friedrich's sense). Nowadays it is well known that Cauchy's problem is *well posed* for these systems see [10], [11].

This approach to hyperbolicity appeared at once important for continuum mechanics: e. g. the evolution equations for non-viscous isentropic fluids constitute a special case of (1.1, 2).

Afterwards G. Boillat—see [3]—, by improving a result of S. K. Godu-

nov—see [14]—, showed that, substantially under the same afore-mentioned hypotheses on S , (1.1) is equivalent to a *symmetric hyperbolic and conservative* system (in a certain new unknown functions); and together with T. Ruggeri—see [4], [5]—he showed that especially in connection with shock wave propagation the last property is very useful.

A different framework for overdeterminate conservative systems was proposed by K. O. Friedrichs in [12]. He considered a more general conservative system (than (1.1))

$$(1.3) \quad \partial_\alpha f^{\Gamma\alpha} = h^\Gamma, \quad \alpha = 0, 1, 2, 3, \quad \Gamma = 1, \dots, \mathcal{N},$$

where $f^{\Gamma\alpha} = f^{\Gamma\alpha}(U)$, $h^\Gamma = h^\Gamma(U)$, $U = (U^i)$, $i = 1, \dots, N$, with ⁽¹⁾: $N < \mathcal{N}$. In order to remove this overdeterminacy, he assumed that there exists a set of functions of U , $\Lambda_\Gamma(U)$, such that the relation

$$(1.4) \quad \Lambda_\Gamma \partial_\alpha f^{\Gamma\alpha} = \Lambda_\Gamma h^\Gamma$$

holds for arbitrary functions $U^i(x^\alpha)$. By a suitable analogue of the convexity hypothesis on S made in [13], K. O. Friedrichs obtained some hyperbolicity results similar to those in [13].

In 1981 this theory of K. O. Friedrichs, together with some previous results of G. Boillat and T. Ruggeri, was set in a relativistic covariant form by T. Ruggeri and A. Strumia—see [17]—; in that paper an application to non-viscous fluids, incapable of heat conduction, was carried out. Furthermore the authors observed that their approach is quite similar to the analytical structure of the thermodynamics of I. Müller—see, e. g., [16], [1].

Now, if one slightly generalizes K. O. Friedrichs and P. D. Lax's theory by replacing (1.2) with

$$(1.5) \quad \partial_t S + \partial_L F^L \geq 0,$$

one can note by analogous arguments, that in this version, their theory is quite similar to the analytical structure of the thermodynamics of B. D. Coleman and W. Noll [9]. In this comparison (1.5) can be identified with the well known Clausius-Duhem inequality. The thermodynamics of B. D. Coleman and W. Noll's type is a theory widely accepted in continuum mechanics.

The aim of the first part of this paper (NN. 2, ..., 5) is to write directly a relativistic version of K. O. Friedrichs and P. D. Lax's theory, i. e. to treat a general conservative system, $A^{i\alpha}{}_{/\alpha} = h^i$, in case it implies a conservative inequality $S^\alpha{}_{/\alpha} \geq 0$. This is done with the following aims:

i) to study, in General Relativity, the behaviour of continuous bodies whose evolutions are governed by conservative systems,

⁽¹⁾ For simplicity, put: $\mathcal{N} = N + 1$.

ii) to draw, in relativistic theories of thermodynamics of the kind B. D. Coleman and W. Noll's classical thermodynamics, some conclusions on the possible hyperbolic character of the evolution equations of the above bodies (NN. 2, 3), and

iii) to study discontinuity waves (NN. 4, 5).

In more detail, in NN. 2, 3 the equivalence of a conservative system to a symmetric hyperbolic one is shown by using a suitable Legendre transformation of the G. Boillat's type. In [4] G. Boillat and T. Ruggeri showed that shock velocities have certain bounds given by characteristic velocities, treated here in N. 4. An analogue of this result for the present relativistic theory is proved in N. 5.

In the second part of this paper the theory developed in the first one is applied to the relativistic non-stationary thermodynamics of A. Bressan — see [7], [8] —, which agrees with B. D. Coleman and W. Noll's as far as the second principle is concerned. Among other things, within A. Bressan's theory the *dynamical law for heat conduction* — see (6.16,17) — is treated in a natural way; that is, unlike other authors on non stationary relativistic thermodynamics — e. g. see [15] —, this law appears as a natural and simple mathematical consequence of the friction interaction between two sub-fluids \mathcal{F}' and \mathcal{F}'' composing a fluid \mathcal{F} ; furthermore A. Bressan states the above law in terms of macroscopic magnitudes (used in the theory of continuous media) ⁽²⁾. In N. 7 constitutive equations and relativistic Clausius-Duhem inequality $S^\alpha_{;\alpha} \geq 0$ are concerned. Furthermore, in N. 8, I state an equivalent conservative version of the system of P. D. E.s for the original thermodynamic theory, that is compatible with the tools presented in the first part, i. e. of the kind: $A^{i\alpha}_{;\alpha} = h^i$.

Incidentally the way in which the afore-mentioned friction interaction, a spatial condition, is put into a conservative form (by introducing suitable additional unknown functions) can be applied to various other spatial conditions.

In N. 9 a class of fluids, whose constitutive equations render symmetric hyperbolic conservative (modulo some transformations) the system, is considered, and in N. 10 the afore-mentioned results are extended to the case where viscosity (with relaxation terms) is present. This result is reached by introducing suitable additional unknown functions connected with the relaxation terms.

⁽²⁾ In [7] (and also in [8]) it is remarked that the (nonstationary) law of heat conduction proposed by A. Bressan slightly differs from C. Eckart's (stationary) law also in the stationary case; this is an advantage in that a certain discrepancy concerning the Coriolis' force on heat flux has been noted there between the dynamic equations and the law of heat conduction belonging to C. Eckart's theory.

PART ONE

**A RELATIVISTIC VERSION OF FRIEDRICHS
AND LAX'S THEORY OF SYSTEMS CONSERVATIVE,
SYMMETRIC, AND HYPERBOLIC**

**2. A generalization of the Friedrichs and Lax's theory, founded
in a relativistic form.**

Let (\mathcal{S}_4, g) be a 4-dimensional Riemannian space-time manifold of General Relativity, where $g = (g_{\alpha\beta})$ is the metric tensor field.

I consider the following first order quasi-linear P. D. E. system, of conservative kind, in the unknown functions $U^1(x^0, \dots, x^3), \dots, U^N(x^0, \dots, x^3)$, or briefly $U(x)$,

$$(2.1) \quad A^{i\alpha}_{/\alpha} = h^i, \quad \alpha = 0, 1, 2, 3, \quad i = 1, \dots, N,$$

where $A^{i\alpha} = A^{i\alpha}(U)$, $h^i = h^i(U)$, in detail ⁽³⁾:

$$(2.1)' \quad A^{i\alpha} U^j_{/\alpha} = h^i, \quad i, j = 1, \dots, N.$$

I suppose that, for some function $S^\alpha = S^\alpha(U)$, system (2.1) implies the inequality of conservative kind

$$(2.2) \quad S^\alpha_{/\alpha} \geq 0, \quad \text{i. e.} \quad S^\alpha_j U^j_{/\alpha} \geq 0,$$

that is, every solution $U(x)$ of (2.1), $x \in \mathcal{S}_4$, satisfies (2.2).

Let ζ_α be a *time-like* unit vector field, $\zeta_\alpha \zeta^\alpha = -1$; one can regard the field ζ_α as a system of *observers* or a *reference frame* in \mathcal{S}_4 . Under suitable hypotheses which connect the functions $A^{i\alpha}$ and S^α to the field ζ_α (that generalize some known hypotheses in K. O. Friedrichs and P. D. Lax [13]). I shall deduce some restrictions relations on $A^{i\alpha}$, h^i and S^α ; afterwards the system (2.1) will be shown to be equivalent to a *symmetric hyperbolic* one. For these systems the Cauchy's problem is *well posed*—cf. [10], [11]—in suitable Sobolev spaces.

The *time* and *space* projectors related to the field ζ_α are $-\zeta_\alpha \zeta^\beta$ and

⁽³⁾ After [13] I set $A^{i\alpha}_j = \frac{\partial}{\partial U^j} A^{i\alpha}$, $S^\alpha_j = \frac{\partial}{\partial U^j} S^\alpha$. Since $g_{\alpha\beta;\gamma} \equiv 0$ holds, for the sake of simplicity I don't consider here the more general case: $A^{i\alpha}(g, U)_{/\alpha} = h^i(g, U)$, $S^\alpha(g, U)_{/\alpha} \geq 0$; in fact we have e. g. : $A^{i\alpha}_{/\alpha} = \frac{\partial A^{i\alpha}}{\partial g_{\rho\sigma}} g_{\rho\sigma;/\alpha} + A^{i\alpha}_j U^j_{/\alpha} = A^{i\alpha}_j U^j_{/\alpha}$.

$\frac{\partial}{\partial g_\alpha^\beta} = g_\alpha^\beta + \zeta_\alpha \zeta^\beta$ respectively — see [6], N. 17—. From (2.1)' and (2.2)₂ we have

$$(2.3) \quad \begin{cases} A^{i\alpha}_j (-\zeta_\alpha \zeta^\beta + \frac{\partial}{\partial g_\alpha^\beta}) U^j_{/\beta} = h^i, \\ S^{\alpha}_j (-\zeta_\alpha \zeta^\beta + \frac{\partial}{\partial g_\alpha^\beta}) U^j_{/\beta} \geq 0. \end{cases}$$

By the definitions

$$(2.4) \quad {}^\zeta A^i_j = -\zeta_\alpha A^{i\alpha}_j; \quad {}^\zeta S_j = -\zeta_\alpha S^{\alpha}_j,$$

from (2.3) one obtains

$$(2.5) \quad \begin{cases} {}^\zeta A^i_j \frac{DU^j}{D\tau} + A^{i\frac{\partial}{\partial g}}_j U^j_{/\frac{\partial}{\partial g}} = h^i, \\ {}^\zeta S_j \frac{DU^j}{D\tau} + S^{\frac{\partial}{\partial g}}_j U^j_{/\frac{\partial}{\partial g}} \geq 0, \end{cases}$$

where, according to definition (17.9) in [6],

$$(2.6) \quad \frac{DU^j}{D\tau} = U^j_{/\alpha} \zeta^\alpha, \quad A^{i\frac{\partial}{\partial g}}_j = A^{i\beta}_j \frac{\partial}{\partial g_\beta^\alpha}, \quad S^{\frac{\partial}{\partial g}}_j = S^{\beta}_j \frac{\partial}{\partial g_\beta^\alpha}.$$

Now let us suppose that

$$(2.7) \quad \det \| {}^\zeta A^i_j \| \neq 0.$$

Then it is meaningful to compose (2.5)₁ with ${}^\zeta S_\kappa ({}^\zeta A^{-1})^{\kappa}_i$:

$$(2.8) \quad {}^\zeta S_j \frac{DU^j}{D\tau} + {}^\zeta S_j ({}^\zeta A^{-1})^j_\kappa A^{\kappa\frac{\partial}{\partial g}}_i U^l_{/\frac{\partial}{\partial g}} = {}^\zeta S_j ({}^\zeta A^{-1})^j_l h^l.$$

By (2.8) (2.5)₂ becomes

$$(2.9) \quad [S^{\frac{\partial}{\partial g}}_i - {}^\zeta S_j ({}^\zeta A^{-1})^j_\kappa A^{\kappa\frac{\partial}{\partial g}}_i] U^l_{/\frac{\partial}{\partial g}} + {}^\zeta S_j ({}^\zeta A^{-1})^j_l h^l \geq 0.$$

Since (2.1) implies (2.2), (2.9) must be satisfied by arbitrary values of U^i and $U^i_{/\frac{\partial}{\partial g}}$. Then one obtains the following *restriction relations* (related to the field ζ_α):

$$(2.10) \quad \begin{cases} S^{\frac{\partial}{\partial g}}_i = {}^\zeta S_j ({}^\zeta A^{-1})^j_\kappa A^{\kappa\frac{\partial}{\partial g}}_i, \\ {}^\zeta S_j ({}^\zeta A^{-1})^j_l h^l \geq 0. \end{cases}$$

Now, in order to show that (2.1) is equivalent to a symmetric hyperbolic system, let us derive ⁽⁴⁾ (2.10)₁ with respect to U^m ,

$$(2.11) \quad S^{\frac{\partial}{\partial g}}_{lm} = [{}^\zeta S_j ({}^\zeta A^{-1})^j_\kappa]_m A^{\kappa\frac{\partial}{\partial g}}_l + [{}^\zeta S_j ({}^\zeta A^{-1})^j_\kappa] A^{\kappa\frac{\partial}{\partial g}}_{lm}.$$

(4) I set: $S^{\frac{\partial}{\partial g}}_{lm} = \frac{\partial^2 S^{\frac{\partial}{\partial g}}}{\partial U^m \partial U^l}, \dots$

Since the left hand side and the second term of the right hand side are symmetric in (l, m) , also the matrices

$$(2.12) \quad a_{ml}^{\frac{1}{2}} = [\zeta S_j (\zeta A^{-1})^j]_{\kappa m} A_{i \kappa}^{\frac{1}{2}}$$

are symmetric in (l, m) : $a_{[ml]}^{\frac{1}{2}} = 0$ ⁽⁵⁾.

It is easy to see that also

$$(2.13) \quad a_{mi} = [\zeta S_j (\zeta A^{-1})^j]_{\kappa m} \zeta A_{i \kappa} = a_{im},$$

in fact

$$a_{ml} = \zeta S_{lm} - \zeta S_j (\zeta A^{-1})^j \zeta A_{lm}.$$

Multiplication of $(2.5)_1$ by the matrix

$$(2.14) \quad b_{mi} = [\zeta S_j (\zeta A^{-1})^j]_{i m},$$

yields

$$(2.15) \quad a_{mj} \frac{DU^j}{D\tau} + a_{mj}^{\frac{1}{2}} U^j /_{\frac{1}{2}} = b_{mi} h^i.$$

This symmetric system is *symmetric hyperbolic* if the matrix a_{mi} , defined in (2.13), is *positive definite* (for all event points and for all U^i):

$$(2.16) \quad a_{mi} \zeta^m \zeta^l > 0, \quad \forall \zeta^m \neq 0.$$

This last hypothesis, together with the hypothesis (2.7), assures us *a posteriori* that the matrix b_{mi} is non-singular (it is : $b = a \zeta A^{-1}$); hence *the conservative system (2.1), which implies the inequality (2.2), under hypotheses (2.7) and (2.16) is equivalent to system (2.15), symmetric hyperbolic with respect to field (observer) ζ_α .*

3. A relativistic version of the Boillat's transformation.

By generalizing an argument of G. Boillat [3] I can show, under some hypotheses stronger than (2.7) and (2.16), that the conservative system (2.1) is equivalent to a symmetric hyperbolic conservative system—see (3.7).

In the sequel I suppose that *both matrices a_{im} and b_{im} are positive definite*, that is (2.16) and

$$(3.1) \quad b_{mi} \zeta^m \zeta^l > 0, \quad \forall \zeta^m \neq 0,$$

hold.

I define the transformation (from a *convex* $\mathcal{D} \subseteq \mathbb{R}^N$ into \mathbb{R}^N)

$$(3.2) \quad V_i = V_i(U) = \zeta S_j (\zeta A^{-1})^j_i (= \hat{V}_i(\zeta, U)).$$

⁽⁵⁾ $2\bar{\mathcal{C}}_{[lm]} = \bar{\mathcal{C}}_{lm} - \bar{\mathcal{C}}_{ml}$, $2\bar{\mathcal{C}}_{(l,m)} = \bar{\mathcal{C}}_{lm} + \bar{\mathcal{C}}_{ml}$.

The (symmetric part of the) Jacobian matrix of function (3.2), i. e. $\partial V_j / \partial U^m = b_{mi}$, is positive definite. Then—see [2], p. 137—(3.2) is *injective*. Hence, given for every event point $x \in \mathcal{S}_4$ the value of the field ζ_α , there exists a one—to—one map between the elements U of \mathcal{D} and V of $V(\mathcal{D})$.

I define now (Legendre transformation):

$$(3.3) \quad \mathcal{F}^\alpha(V) = V_i A^{i\alpha}(U(V)) - S^\alpha(U(V)).$$

We have

$$(3.4) \quad \frac{\partial \mathcal{F}^{\frac{1}{\alpha}}}{\partial V_j} = A^{j\frac{1}{\alpha}} + (V_i A^{i\frac{1}{\alpha}}_\kappa - S_\kappa^{\frac{1}{\alpha}}) \frac{\partial U^\kappa}{\partial V_j} = A^{j\alpha},$$

where (3.4)₂ holds by (3.2) and (2.10)₁.

Furthermore

$$(3.5) \quad \frac{\partial}{\partial V_j} (-\zeta_\alpha \mathcal{F}^\alpha) = -\zeta_\alpha A^{j\alpha} + (V_i \zeta_\alpha A^i_\kappa - \zeta_\alpha S_\kappa) \frac{\partial U^\kappa}{\partial V_j} = -\zeta_\alpha A^{j\alpha},$$

where (3.5)₂ holds by (3.2) and (2.4). In addition (3.4)₂ and (3.5)₂ are equivalent to

$$(3.6) \quad \frac{\partial \mathcal{F}^\alpha}{\partial V_i}(V) = A^{i\alpha}(U(V)).$$

Hence the conservative system (2.1), in the new unknowns V(x), becomes a symmetric hyperbolic conservative system ⁽⁶⁾:

$$(3.7) \quad \frac{\partial^2 \mathcal{F}^\alpha}{\partial V_j \partial V_i} V_{j/\alpha} = h^i.$$

The hyperbolicity of (3.7) is clear when one introduces the time and spatial projectors with respect to ζ_α :

$$(3.8) \quad \left(-\zeta_\alpha \frac{\partial^2 \mathcal{F}^\alpha}{\partial V_j \partial V_i} \right) \frac{DV_j}{D\tau} + \frac{\partial^2 \mathcal{F}^{\frac{1}{\alpha}}}{\partial V_j \partial V_i} V_{j/\frac{1}{\alpha}} = h^i,$$

or

$$(3.9) \quad \zeta_\alpha A^i_\kappa \frac{\partial U^\kappa}{\partial V_j} \frac{DV_j}{D\tau} + A^{i\frac{1}{\alpha}}_\kappa \frac{\partial U^\kappa}{\partial V_j} V_{j/\frac{1}{\alpha}} = h^i.$$

⁽⁶⁾ The term h^i in the right hand side of (3.7) means:

$$h^i(\hat{U}(\zeta, V)) - \frac{\partial \mathcal{A}^{i\alpha}}{\partial U^j}(\hat{U}(\zeta, V)) \frac{\partial \hat{U}^j}{\partial \zeta_\rho}(\zeta, V) \zeta_{\rho/\alpha},$$

where $\hat{U}^j(\zeta, V)$ is the inverse function of (3.2)₃.

From (2.14) and (3.2) we have that $\partial U^k / \partial V_j = (b^{-1})^{jk}$; by remembering definitions (2.12, 13) one sees that $a_{ml}^{\frac{\alpha}{2}} = b_{mi} A_l^{\frac{\alpha}{2}}$, $a_{ml} = b_{mi}^{\zeta} A_l^i$; lastly system (3.9) reads (7)

$$(3.10) \quad (b^{-1} a^T b^{-1})^{ij} \frac{DV_j}{D\tau} + (b^{-1} a^{\frac{\alpha}{2}})^T b^{-1})^{ij} V_{j/\frac{\alpha}} = h^i.$$

Now $(b^{-1} a^T b^{-1})$ is positive definite because a is positive definite and b is positive definite, hence non-singular.

4. Discontinuity waves and characteristic velocities.

Let $V = V(x)$ be a solution of (3.7) and let \mathcal{W} be the support of V in \mathcal{S}_4 , $\mathcal{W} = \{x \in \mathcal{S}_4 : V(x) \neq 0\}$. Let $\Sigma_3 \subset \mathcal{W}$ be a 3-dimensional surface, $f(x) = 0$, of discontinuity for the first derivatives of V , that is $[[V_{i/\alpha}]] \neq 0$, but $[[V_i]] = 0$. The unit normal vector of Σ_3 is

$$(4.1) \quad N_\alpha = \frac{f_{,\alpha}}{|f_{,\rho} f_{,\sigma} g^{\rho\sigma}|^{1/2}},$$

and its natural decomposition with respect to ζ_α reads

$$(4.2) \quad N_\alpha = \sigma \zeta_\alpha + N_{\frac{\alpha}{\sigma}}, \quad \sigma = -N_\beta \zeta^\beta,$$

thus σ denotes the propagation velocity of the discontinuity with respect to (observer) ζ_α .

The Hugoniot-Hadamard's conditions, $[[V_{i/\alpha}]] = \lambda_i N_\alpha$, become, on using (4.2),

$$(4.3) \quad \left[\left[\frac{DV_i}{D\tau} \right] \right] = -\lambda_i \sigma, \quad [[V_{i/\alpha}]] = \lambda_i N_{\frac{\alpha}{\sigma}}.$$

By (4.3) and (3.10) one obtains

$$(4.4) \quad [- (b^{-1} a^T b^{-1})^{ij} \sigma + (b^{-1} a^{\frac{\alpha}{2}})^T b^{-1})^j N_{\frac{\alpha}{\sigma}}] \lambda_j = 0.$$

There exist non-trivial solutions for λ_j if and only if

$$(4.5) \quad \det \| a^{-1} a^{\frac{\alpha}{2}} N_{\frac{\alpha}{\sigma}} - \sigma \mathbb{1} \| = 0,$$

that is $\sigma (= \tilde{\sigma}(x, N, V)) \in \text{Spec} (a^{-1} a^{\frac{\alpha}{2}} N_{\frac{\alpha}{\sigma}}) \subset \mathbb{R}$, where the last inclusion

(7) ${}^T b_{lm} = b_{ml}$.

holds because the product of a positive definite matrix (a^{-1}) with a symmetric matrix ($a^{\frac{1}{2}}N_{\alpha}$) is *similar* to a symmetric matrix.

It is standard to call the elements $\tilde{\sigma}(x, N, V)$ of $\text{Spec}(a^{-1}a^{\frac{1}{2}}N_{\alpha})$ *characteristic velocities*.

5. Shock waves.

Assume that $V(x)$ is a solution of (3.7) and that $\Sigma_3, f(x) = 0$, is a 3-dimensional discontinuity surface for V , $[[V_i]] = V_i - V_i^{(0)} \neq 0$; and let N_{α} be the unit normal vector of Σ_3 , as well in (4.1, 2).

By a well known integral balance relation for (3.7), one obtains the *Rankine-Hugoniot's* equations:

$$(5.1) \quad [[A^{i\alpha}N_{\alpha}]] = 0.$$

Let \mathcal{D} be any convex subset of \mathbb{R}^N and set

$$(5.2) \quad F^i(V) = N_{\alpha}A^{i\alpha}(V) \quad (V \in \mathcal{D}).$$

Then (5.1) becomes

$$(5.3) \quad F^i(V) = F^i(V^{(0)}).$$

By a theorem mentioned above (see [2], p. 137), if $(\partial F^i/\partial V_j)(V)$ is positive [negative] definite $\forall V \in \mathcal{D}$, then $F^i: \mathcal{D} \subseteq \mathbb{R}^N \rightarrow \mathbb{R}^N$ is *injective*. By the remark below (3.9) we have

$$\frac{\partial F^i}{\partial V_j} = N_{\alpha}A^{i\alpha}_{\kappa} \frac{\partial U^{\kappa}}{\partial V_j} = [-b^{-1}a^{-1}b^{-1}\sigma + b^{-1}a^{\frac{1}{2}}b^{-1}N_{\alpha}]^{ij};$$

hence

$$(5.4) \quad \frac{\partial F^i}{\partial V_j} = [b^{-1}(a^{\frac{1}{2}}N_{\alpha} - \sigma a)^{-1}b^{-1}]^{ij}.$$

Since b is non-singular, $(\partial F^i/\partial V_j)$ is positive [negative] definite if and only if $(a^{\frac{1}{2}}N_{\alpha} - \sigma a)$ is such. By standard algebraic arguments one obtains that $(\partial F^i/\partial V_j)$ is positive [negative] definite if

$$(5.5) \quad \sigma < m = \inf_{V \in \mathcal{D}} \text{Spec}(a^{-1}a^{\frac{1}{2}}N_{\alpha}) \quad [\sigma > M = \sup_{V \in \mathcal{D}} \text{Spec}(a^{-1}a^{\frac{1}{2}}N_{\alpha})].$$

In this case F^i is injective and (5.3) implies $V_i = V_i^{(0)}$; hence (non-trivial) shocks can exist only for some choices of the propagation velocities σ such that

$$(5.6) \quad m \leq \sigma \leq M.$$

I conclude by remarking (as in [4] and [17]) that *systems which admit discontinuities of the first derivatives with causal propagations, i. e. with time or light-like characteristic velocities, admit, at most, shocks with causal propagation.*

PART TWO

**APPLICATION OF THE PRECEDING RELATIVIZATION
OF FRIEDRICHS
AND LAX'S THEORY TO VISCOUS FLUIDS CAPABLE
OF HEAT CONDUCTION ACCORDING
TO BRESSAN'S RELATIVISTIC
NON STATIONARY HEAT CONDUCTION LAW**

**6. Summary of A. Bressan's relativistic
non stationary thermodynamics.**

In this presentation of the theory in [8] and in the following application, for the sake of simplicity I only consider the typical fluid \mathcal{F} (possibly viscous, see N. 10 below) that is capable of heat conduction and has the energy tensor

$$(6.1) \quad \mathcal{U}^{\alpha\beta} = \rho u^\alpha u^\beta + X^{\alpha\beta} + 2u^{(\alpha} q^{\beta)};$$

where ρ is the gravitational energy density; u^α is the 4-velocity, $u^\alpha u_\alpha = -1$; $X^{\alpha\beta}$ is the Cauchy spatial stress tensor, $X^{[\alpha\beta]} = 0 = u_\alpha X^{\alpha\beta}$; and q^α is the spatial energy flux, $u_\alpha q^\alpha = 0$.

On the basis of the mass-energy equivalence principle, \mathcal{F} is regarded —as well as in [15]—as divided in two parts, \mathcal{F}' and \mathcal{F}'' , where the motion of \mathcal{F}' represents the energy flux of \mathcal{F} . The u^α -gravitational energy densities of \mathcal{F}' and \mathcal{F}'' are ρ' and ρ'' respectively, and the 4-velocity of \mathcal{F}' is v^α , hence $v^\alpha v_\alpha = -1$. The above assumption about \mathcal{F} (that is: $\mathcal{F} = \mathcal{F}_{(\text{heat})} + \mathcal{F}_{(\text{particles})}''$) implies

$$(6.2) \quad \rho = \rho' + \rho''.$$

I use the conventional mass density κ of \mathcal{F} —see e. g. N. 21 in [6]—, which satisfies the continuity equation

$$(6.3) \quad (\kappa u^\alpha)_{;\alpha} = 0,$$

and the u^α -internal ⁽⁸⁾ energy densities w' and w'' for \mathcal{F}' and \mathcal{F}'' , which are defined by

$$(6.4) \quad \rho' = \kappa w', \quad \rho'' = \kappa(c^2 + w'').$$

⁽⁸⁾ After [8] the prefix $u^\alpha[v^\alpha]$ expresses that the observer $u^\alpha[v^\alpha]$ is being referred to (to evaluate e. g. the mentioned energy densities).

Hence $w = w' + w''$, where w is the u^α -internal energy density for \mathcal{F} , such that $\rho = \kappa(c^2 + w)$.

The following two energy tensors are assumed for \mathcal{F}' and \mathcal{F}'' respectively:

$$(6.5) \quad \mathcal{U}'^{\alpha\beta} = \rho'_0 v^\alpha v^\beta + \mathcal{P} \gamma^{\alpha\beta}, \quad \gamma^{\alpha\beta} = g^{\alpha\beta} + v^\alpha v^\beta,$$

$$(6.6) \quad \mathcal{U}''^{\alpha\beta} = \rho'' u^\alpha u^\beta + X''^{\alpha\beta},$$

where $X''^{\alpha\beta} u_\beta = 0 = X''^{[\alpha\beta]}$, ρ'_0 is the v^α -gravitational energy density of \mathcal{F}' and \mathcal{P} is a heat pressure that A. Bressan (unlike other authors on non stationary thermodynamics, see e. g. [15]) introduces in [8]. Of course the energy tensor of $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$ is

$$(6.7) \quad \mathcal{U}^{\alpha\beta} = \mathcal{U}'^{\alpha\beta} + \mathcal{U}''^{\alpha\beta}.$$

Furthermore I set—cf. [8] and [6]—

$$(6.8) \quad \mathcal{V} = -u_\alpha v^\alpha, \quad \overset{\perp}{g}^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta, \quad \overset{\perp}{\mathcal{C}}^{\alpha\dots} = \overset{\perp}{g}^{\alpha\dots} \overset{\perp}{\mathcal{C}}^{\beta\dots}.$$

By (6.7), (6.1), and (6.6) one obtains

$$(6.9) \quad \mathcal{U}'^{\alpha\beta} = \mathcal{U}^{\alpha\beta} - \mathcal{U}''^{\alpha\beta} = \rho' u^\alpha u^\beta + (X^{\alpha\beta} - X''^{\alpha\beta}) + 2u^{(\alpha} q^{\beta)},$$

hence the following relations are true:

$$(6.10) \quad \rho' = u_\alpha u_\beta \mathcal{U}'^{\alpha\beta} = \rho'_0 \mathcal{V}^2 + (\mathcal{V}^2 - 1)\mathcal{P},$$

$$(6.11) \quad X'^{\alpha\beta} = \overset{\perp}{\mathcal{U}}'^{\alpha\beta} = (\rho'_0 + \mathcal{P}) \overset{\perp}{v}^\alpha \overset{\perp}{v}^\beta + \mathcal{P} \overset{\perp}{g}^{\alpha\beta},$$

$$(6.12) \quad q^\alpha = -u_\rho \mathcal{U}'^{\rho\sigma} \overset{\perp}{g}_\sigma^\alpha = (\rho'_0 + \mathcal{P}) \mathcal{V} \overset{\perp}{v}^\alpha.$$

It is easy to see that v^α can be expressed by a (universal) function of ρ'_0 , \mathcal{P} , q^ρ , and u^σ , i. e.

$$(6.13) \quad v^\alpha = \tilde{v}^\alpha(\rho'_0, \mathcal{P}, q^\rho, u^\sigma) = \overset{\perp}{v}^\alpha + \mathcal{V} u^\alpha = \frac{q^\alpha}{(\rho'_0 + \mathcal{P}) \mathcal{V}} + \mathcal{V} u^\alpha,$$

where \mathcal{V} is given by

$$(6.14) \quad \mathcal{V} = \sqrt{\frac{1 + \sqrt{1 + 4 \frac{q^\rho q_\rho}{(\rho'_0 + \mathcal{P})^2}}}{2}}.$$

Let the field $g_{\alpha\beta}$ be assigned. The following *dynamical* equations (6.15, 16) governing the fluid \mathcal{F} are postulated in [8]: the well known consequences of the Einstein equations

$$(6.15) \quad \mathcal{U}^{\alpha\beta}{}_{|\beta} = 0,$$

and the *interaction law* between the two sub-fluids \mathcal{F}' and \mathcal{F}''

$$(6.16) \quad \mathcal{U}'^{\alpha\beta}{}_{|\beta} = -\mathbb{K}^{\alpha\gamma} \overset{\perp}{v}^\gamma + \mathbf{J} u^\alpha,$$

where $\mathbb{K}^\alpha{}_\beta$ is an u^α -spatial, $\mathbb{K}^\alpha{}_\beta = \overset{\perp}{\mathbb{K}}^\alpha{}_\beta$, and positive definite tensor, such

that in its motion within \mathcal{F}'' (or \mathcal{F}) \mathcal{F}' meets a resistance $-\mathbb{K}^\alpha_\gamma \frac{1}{v^\gamma}$ per unit of u^α -volume, which vanishes for $v^\alpha = u^\alpha$; the scalar J , called *energy influx* from \mathcal{F}'' into \mathcal{F}' per unit of u^α -volume, is defined by: $J = -u_\alpha \mathcal{U}'^{\alpha\beta}{}_{|\beta}$, hence only the spatial components of (6.16), i. e.

$$(6.17) \quad \mathring{g}^\alpha_\rho \mathcal{U}'^{\rho\beta}{}_{|\beta} = -\mathbb{K}^\alpha_\gamma \frac{1}{v^\gamma},$$

are dynamically meaningful.

By (6.12, 13) it is clear that (6.17) is to regarded as a dynamical equation for the heat flux, and not as a *constitutive* equation, as is usually done in other theories. A detailed physical interpretation of (6.16), or (6.17), (and its consequences) is made in [8] (see [7] too); there, the reader can see that \mathbb{K}^α_γ is essentially (up to scalar factor) a Fourier conductivity tensor.

7. Constitutive equations for \mathcal{F} and relativistic Clausius-Duhem inequality.

Now I list a set of constitutive equations for the non-viscous fluid \mathcal{F} (viscosity is included in N. 10) and, in compliance with the *principle of equipresence*—see [19], p. 703, I regard w, \dots, \mathcal{P} as functions of κ, T , and q^ρ , where T is the absolute temperature:

$$(7.1) \quad \begin{aligned} w &= \tilde{w}(\kappa, T, q^\rho), & X^{\alpha\beta} &= \tilde{X}^{\alpha\beta}(\kappa, T, q^\rho), \\ \mathbb{K}^\alpha_\beta &= \tilde{\mathbb{K}}^\alpha_\beta(\kappa, T, q^\rho), & \rho'_0 &= \tilde{\rho}'_0(\kappa, T, q^\rho), \\ \mathcal{P} &= \tilde{\mathcal{P}}(\kappa, T, q^\rho). \end{aligned}$$

Together with equations (7.1) I consider a constitutive equation for entropy density

$$(7.2) \quad \eta = \tilde{\eta}(\kappa, T, q^\rho),$$

and the following relativistic Clausius-Duhem inequality as a version of the « second principle of thermodynamics ».

DEFINITION 7.1. — *A set of fields $\{\kappa(x), T(x), q^\rho(x), u^\sigma(x)\}$, $x \in \mathcal{S}_4$, that—under (6.1), (6.5), (6.13), (6.14), and (7.1)—solve (6.3), (6.15), (6.17), $q^\alpha u_\alpha = 0$, and $u^\alpha u_\alpha = -1$, is a thermodynamically admissible process for \mathcal{F} if and only if the following inequality (7.3)₁ is satisfied*

$$(7.3) \quad S^\alpha_{|\alpha} \geq 0, \quad \text{where} \quad S^\alpha = \kappa \eta u^\alpha + \frac{q^\alpha}{T},$$

and η is given by (7.2).

POST. 7.1. — *Every solution $\{\kappa(x), T(x), q^\rho(x), u^\sigma(x)\}$ solving the equations mentioned in Def. 7.1 is thermodynamically admissible for \mathcal{F} .*

Often S^α is called the 4-entropy flux vector. The requirement that every ⁽⁹⁾ solution $\{\kappa(x), \dots, u^\sigma(x)\}$ be a thermodynamically admissible process for \mathcal{F} is equivalent to the requirement that the system of P. D. E.s for \mathcal{F} implies inequality (7.3)₁ in the sense of N. 2. The same requirement is accepted in the classical thermodynamics according to e. g. [9]. Now, for the full applicability of the theory developed in NN. 2, . . . , 5, I must show that the system of P. D. E. s for \mathcal{F} can be posed in a conservative form (see the next sections).

The choice (7.1) of constitutive functions is slightly different from the one in [8]; there A. Bressan uses the quantity $\varepsilon_1 = (\rho'_0 + \mathcal{P})/T$ instead of ρ'_0 : obviously the two choices are equivalent.

Some constitutive hypotheses less general than (7.1), but perhaps more natural ⁽¹⁰⁾, are e. g.:

$$(7.4) \quad w = \hat{w}(\kappa, T), \quad X^{\alpha\beta} = \hat{X}^{\alpha\beta}(\kappa, T),$$

and $\mathbb{K}^\alpha_\beta, \rho'_0$, and \mathcal{P} as in (7.1)_{3,4,5}.

A further restriction on the possible choices of constitutive equations for a fluid \mathcal{F} consists in A. Bressan's Assumption 5.2 in [8]—based on certain arguments belonging to the relativistic equilibrium of the black body radiation—which asserts the relation

$$(7.5) \quad \mathcal{P}_{/x} = \varepsilon_1 T_{/x} \quad \left(\varepsilon_1 = \frac{\rho'_0 + \mathcal{P}}{T} \right)$$

holds in every physical situation. Now, one has to look at (7.5) as at a constitutive restriction on $\tilde{\rho}'_0$ and $\tilde{\mathcal{P}}$, and one easily sees that (7.5) is equivalent to

$$(7.6) \quad \rho'_0 = \tilde{\rho}'_0(T), \quad \mathcal{P} = \tilde{\mathcal{P}}(T) = T \left(\int_{T_0}^T \frac{\tilde{\rho}'_0(\mu)}{\mu^2} d\mu + \frac{\mathcal{P}_0}{T_0} \right).$$

8. A conservative version of the dynamical system for the fluid \mathcal{F} .

As we have seen in Def. 7.1, the dynamical equations for \mathcal{F} —under (6.1), (6.5), (6.13), (6.14), and (7.1)—are (6.3), (6.15), (6.17), $q^\alpha u_\alpha = 0$, and $u^\alpha u_\alpha = -1$. While (6.3) and (6.15) are already in a conservative form, equations (6.17) and the conditions $q^\alpha u_\alpha = 0$ and $u^\alpha u_\alpha = -1$ must be replaced with conservative equations that are equivalent to them. Thus the results in NN. 2, . . . , 5 can be applied to the fluid \mathcal{F} .

⁽⁹⁾ Suitably smooth, here I don't enter into regularity matter.

⁽¹⁰⁾ That is, more similar to the usual thermodynamic relations.

It is easy to see that

$$(8.1) \quad [\kappa(q^\rho u_\rho)u^\alpha]_{/\alpha} = 0, \quad [\kappa(u^\rho u_\rho)u^\alpha]_{/\alpha} = 0$$

are the substitutes sought for. In fact, under the validity of (6.3), equations (8.1) read

$$(8.2) \quad \kappa \frac{D}{Ds} (q^\rho u_\rho) = 0, \quad \kappa \frac{D}{Ds} (u^\rho u_\rho) = 0;$$

if on a Cauchy surface Ω one chooses $q^\rho u_\rho|_\Omega = 0$ and $u^\rho u_\rho|_\Omega = -1$, then $q^\rho u_\rho = 0$ and $u^\rho u_\rho = -1$ hold in the world-tube of \mathcal{F} (where $\kappa \neq 0$).

In order to treat (6.17), I consider the following additional unknowns, $\Gamma = 1, 2, 3$:

$$(8.3) \quad \begin{cases} e_\alpha^\Gamma & (12 \text{ scalar components}), \\ E_{\alpha\beta}^\Gamma & (48 \text{ scalar components}), \end{cases}$$

and I add the following differential equations in conservative form to the evolution system for \mathcal{F} :

$$(8.4) \quad \begin{cases} [\kappa(e_\alpha^\Gamma u^\alpha)u^\beta]_{/\beta} = 0 & (3 \text{ scalar equations}), \\ [\kappa(e_\alpha^\Gamma \Delta_\beta^\alpha g^{\alpha\beta})u^\gamma]_{/\gamma} = 0 & (6 \text{ scalar equations}), \\ (e_\alpha^\Gamma g_\beta^\gamma)_{/\gamma} = E_{\alpha\beta}^\Gamma & (48 \text{ scalar equations}), \\ [\kappa(E_{\alpha\beta}^\Gamma \mathcal{U}'^{\alpha\beta})u^\gamma]_{/\gamma} = 0 & (3 \text{ scalar equations}). \end{cases}$$

While (8.4)₃ denotes simply that $E_{\alpha\beta}^\Gamma = e_{\alpha/\beta}^\Gamma$, under the validity of (6.3) equations (8.4)_{1,2,4} say that, if on a Cauchy surface Ω we choose $e_\alpha^\Gamma u^\alpha|_\Omega = 0$, $e_\alpha^\Gamma e_\beta^\Delta g^{\alpha\beta}|_\Omega = \delta^{\Gamma\Delta}$, and $E_{\alpha\beta}^\Gamma \mathcal{U}'^{\alpha\beta}|_\Omega = 0$, then $\{e_\alpha^\Gamma\}_{\Gamma=1,2,3}$ is an orthonormal triple of u^α -spatial vectors, and

$$(8.5) \quad e_{\alpha/\beta}^\Gamma \mathcal{U}'^{\alpha\beta} = 0$$

holds in the world-tube of \mathcal{F} . Now it can be shown that the dynamical equations for the heat flux (6.17) are *equivalent* to the conservative ones

$$(8.6) \quad (e_\alpha^\Gamma \mathcal{U}'^{\alpha\beta})_{/\beta} = -e_\alpha^\Gamma \mathbb{K}^\alpha_\beta v^\beta.$$

In fact, if (6.17) holds, then, since $e_\alpha^\Gamma u^\alpha = 0$ and (8.5) holds, (8.6) follows. Conversely, let (8.6) hold and choose a locally pseudo-euclidean and proper frame at the typical event point $x \in \mathcal{S}_4$ ($g_{rs} = \delta_{rs}$, $g_{0r} = 0$, $g_{00} = -1$, and $u^\alpha = \delta_0^\alpha$, hence e. g. $g_{rs}^\perp = \delta_{rs}$, and $g_{0\alpha}^\perp = 0$). Then, by (8.5), (8.6) can be written in the form

$$(8.7) \quad e_r^\Gamma B^r = 0, \quad \text{where} \quad B^r = \mathcal{U}'^{r\beta}{}_{/\beta} + \mathbb{K}^r_\beta v^\beta.$$

Since $\det \|e_r^\Gamma\| = 1$, (8.7) implies $B^r = 0$ at x , which is equivalent to equations (6.17), i. e. $g_{\rho}^\alpha \mathcal{U}'^{\rho\sigma}{}_{/\sigma} = -\mathbb{K}^\alpha_\beta v^\beta$, in the same pseudo-euclidean and proper frame at x .

9. Hyperbolicity.

We have seen that an equivalent conservative version of the system of P. D. E. s for \mathcal{F} is given by (6.3), (6.15), (8.1), (8.4), and (8.6) in the seventy scalar unknown fields U^i , $i=1, \dots, 70$:

$$(9.1) \quad U^1 = \kappa, \quad U^2 = T, \quad U^{3+\alpha} = q^\alpha, \quad U^{7+\alpha} = u^\alpha, \\ \{U^{10+l_1}\}_{l_1=1, \dots, 12} = \{e_\alpha^\Gamma\}_{\Gamma=1,2,3, \alpha=0, \dots, 3}, \quad \{U^{22+l_2}\}_{l_2=1, \dots, 48} = \{E_{\alpha\beta}^\Gamma\}_{\Gamma=1,2,3, \alpha, \beta=0, \dots, 3}.$$

By using the same symbols and notations as in NN. 2, ..., 5, I define $A^{i\beta}(U)$, $i=1, \dots, 70$:

$$(9.2) \quad A^{(1+\alpha)\beta} = \mathcal{U}^{\alpha\beta}, \quad A^{5\beta} = \kappa u^\beta, \quad A^{6\beta} = \kappa(u^\rho u_\rho)u^\beta, \\ A^{7\beta} = \kappa(q^\rho u_\rho)u^\beta, \quad A^{(7+\Gamma)\beta} = \kappa(e_\alpha^\Gamma u^\alpha)u^\beta, \\ \{A^{(10+l_3)\beta}\}_{l_3=1, \dots, 6} = \{\kappa(e_\rho^\Gamma e_\sigma^\Delta g^{\rho\sigma})u^\beta\}_{\Gamma, \Delta=1,2,3}, \\ A^{(16+\Gamma)\beta} = \kappa(E_{\rho\sigma}^\Gamma \mathcal{U}'^{\rho\sigma})u^\beta, \\ \{A^{(19+l_4)\beta}\}_{l_4=1, \dots, 48} = \{e_\alpha^\Gamma g_\gamma^\beta\}_{\Gamma=1,2,3, \alpha, \gamma=0, \dots, 3}, \\ A^{(67+\Gamma)\beta} = e_\alpha^\Gamma \mathcal{U}'^{\alpha\beta},$$

and

$$(9.3) \quad h^{l_5} = 0 \quad l_5 = 1, \dots, 19, \\ \{h^{19+l_6}\}_{l_6=1, \dots, 48} = \{E_{\alpha\beta}^\Gamma\}_{\Gamma=1,2,3, \alpha, \beta=0, \dots, 3}, \\ h^{67+\Gamma} = -e_\alpha^\Gamma \mathbb{K}^{\alpha\beta} v^\beta.$$

It is clear that the dependence of $A^{i\beta}$ and h^i on U^i is obtained by using (6.1), (6.5), (6.13), (6.14), and the constitutive relations (7.1). Lastly, $S^\alpha(U)$ is given by (7.3)₂ and (7.2). Since $g_{\alpha\beta/\gamma} \equiv 0$, I don't express the dependence of $A^{i\alpha}$, h^i , and S^α on $g_{\rho\sigma}$.

Under the above definitions the evolution system for and the Clausius-Duhem inequality read

$$(9.4) \quad A^{i\alpha}(U)_{/x} = h^i(U), \quad S^\alpha(U)_{/x} \geq 0.$$

At this point the theory developed in the 1st part can be used.

Let ζ_α be a typical time-like unit vector field in \mathcal{S}_4 , i. e. a reference frame (an observer). I say that the fluid \mathcal{F} , defined by (7.1) and (7.2), is a ζ_α -admissible fluid if, for every U belonging to some convex subset of \mathbb{R}^{70} and for every U belonging to some convex subset of \mathbb{R}^{70} and for every $x \in \mathcal{S}_4$, the matrices a_{lm} and b_{lm} ($l, m = 1, \dots, 70$)—cf. (2.13) and (2.14)—are positive definite. In this case:

i) the conservative system (9.4)₁, which implies (9.4)₂, is equivalent to a symmetric hyperbolic conservative system—see N. 3—, hence, by a theo-

rem of A. E. Fisher and J. E. Marsden presented in [10], the Cauchy problem is well posed,

ii) acceleration (or characteristic) waves travel with finite speed,

iii) any (possible) shock wave travels with a speed $\sigma < c$ (speed of light) if the same occurs for acceleration waves.

10. Non stationary conservative viscosity.

In this section I consider a possible simple way for including linear viscosity in the preceding non stationary conservative treatment of relativistic fluids.

I recall that in classical fluid mechanics we have $X^{ij} = p\delta^{ij} + X^{ij}_{(irr.)}$, where for example—see [18]—

$$(10.1) \quad X^{ij}_{(irr.)} = -\lambda\delta^{ij} \operatorname{div} \mathbf{v} - 2\mu v^{(i,j)}.$$

Now I define, for some $\xi, \lambda, \mu \in \mathbb{R}$,

$$(10.2) \quad \begin{cases} \mathbb{X}^{\alpha\beta\gamma} = \xi\kappa Z^{\alpha\beta}_{\rho\sigma} X^{\rho\sigma}_{(irr.)} u^\gamma - \lambda g^{\alpha\beta} u^\gamma - 2\mu g^{(\alpha\gamma} u^{\beta)}, \\ \mathbb{Y}^{\alpha\beta} = \kappa u_\rho X^{\rho\sigma}_{(irr.)} u^\beta, \end{cases}$$

where $Z^{\alpha\beta\gamma\delta}$ is a relaxation tensor field, to be considered as an additional unknown together with $X^{\alpha\beta}_{(irr.)}$; furthermore I require on it that

$$(10.3) \quad Z^{[\alpha\beta]\gamma\delta} = 0 = Z^{\alpha\beta[\gamma\delta]}.$$

Let us add the equations (9.4)₁ with the following ones:

$$(10.4) \quad \begin{cases} \mathbb{X}^{\alpha\beta\gamma}_{|\gamma} = X^{\alpha\beta}_{(irr.)}, \\ \mathbb{Y}^{\alpha\beta}_{|\beta} = 0 \end{cases}$$

and in (7.1)₂ let us perform the replacement:

$$\tilde{X}^{\alpha\beta} \rightarrow \tilde{X}^{\alpha\beta} + X^{\alpha\beta}_{(irr.)}.$$

Since $(\kappa u^\alpha)_{|\alpha} = 0$, (10.4) become

$$(10.5) \quad \begin{cases} \xi\kappa \frac{D}{Ds} (Z^{\alpha\beta}_{\gamma\delta} X^{\gamma\delta}_{(irr.)}) - \lambda g^{\alpha\beta} u^\gamma_{|\gamma} - 2\mu u^{(\alpha\beta)} = X^{\alpha\beta}_{(irr.)}, \\ \kappa \frac{D}{Ds} (u_\rho X^{\rho\alpha}_{(irr.)}) = 0. \end{cases}$$

By (10.3) and (10.5)₂, $X^{\alpha\beta}_{(irr.)}$ is a symmetric and u^α -spatial tensor (whenever on a Cauchy surface $\Omega: u_\rho X^{\rho\alpha}_{(irr.)}|_\Omega = 0$). Furthermore the spatial projection of (10.5)₁ shows that, for $\xi \rightarrow 0$, we just obtain the relativistic analogues of the linear classical viscosity relations

$$(10.6) \quad X^{\alpha\beta}_{(irr.)} = -\lambda g^\perp{}^{\alpha\beta} u^\gamma_{|\gamma} - 2\mu u^{(\alpha\beta)\perp} + O(\xi).$$

The equations-unknowns balance is satisfied if there are *four* independent scalar components of $Z^{\alpha\beta\gamma\delta}$; in fact, in this case we have:

$$\begin{aligned} \text{number of new equations} &= 10 \text{ for } (10.5)_1 \text{ plus } 4 \text{ for } (10.5)_2, \\ \text{number of new unknowns} &= 10 \text{ for } X_{(irr.)}^{\alpha\beta} \text{ plus } 4 \text{ for } Z^{\alpha\beta\gamma\delta}. \end{aligned}$$

This last requisite on $Z^{\alpha\beta\gamma\delta}$ is satisfied when $Z^{\alpha\beta\gamma\delta}$ is expressed by a function of a 4-vector τ^α , relaxation vector, For example, the choice

$$(10.7) \quad Z^{\alpha\beta\gamma\delta} = \tau^\alpha \tau^\beta \tau^\gamma \tau^\delta$$

is (in accord with a natural requisite of isotropy and it is) such that relations (10.3) are identically satisfied.

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