ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 42, nº 4 (1985), p. 341-361
<http://www.numdam.org/item?id=AIHPA_1985__42_4_341_0>

*Ann. Inst. Henri Poincaré,*
Vol. 42, n° 4, 1985, p. 341-361. *Physique théorique*

# Pointwise bounds on the asymptotics of spherically averaged $L^2$-solutions of one-body Schrödinger equations

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ABSTRACT. — Let $(-\Delta + V - E)\psi = 0$ in $\Omega_R = \{ x \in \mathbb{R}^n : |x| > R \}$, $\psi \in L^2(\Omega_R)$ where $V = V_1(|x|) + V_2(x)$ and $E < 0$, and let

$$(-\Delta + V_1(|x|) - E)v(|x|) = 0 \quad \text{for} \quad |x| \geq R' \geq R,$$

R' sufficiently large, with $v > 0$. We shall suppose that V tends to zero in some sense as $|x| \to \infty$. We give conditions on V so that for $r = |x|$ large

$$0 < c_- \leq \left( \int_{S^{n-1}} |\psi(r\omega)|^2 d\omega \right)^{1/2} / v(r) \leq c_+ < \infty$$

where $d\omega$ denotes integration over the unit sphere. Our conditions on $V(x)$ include e. g. Hamiltonians describing one electron in the field of fixed

(*) Supported by « Fonds zur Förderung der wissenschaftlichen Forschung in Österreich », Project nr. P4925.

(**) Supported by « Fonds zur Förderung der wissenschaftlichen Forschung in Österreich », Project nr. P4829.

(***) Research in partial fulfilment of the requirements of a Ph. D. degree at the University of Vienna.

nuclei or the case of short range potentials, i. e. $|V(x)| \leq Cr^{-1-\varepsilon}$ for $|x|$ large, $\varepsilon > 0$.

RÉSUMÉ. — Soit $(-\Delta + V - E)\psi = 0$ dans $\Omega_R = \{ x \in \mathbb{R}^n : |x| > R \}$, $\psi \in L^2(\Omega_R)$ où $V = V_1(|x|) + V_2(x)$ et $E < 0$, et soit

$$(-\Delta + V_1(|x|) - E)v(|x|) = 0 \quad \text{pour} \quad |x| \geq R' \geq R,$$

R′ assez grand, avec $v > 0$. On suppose que V tend vers zéro en un sens convenable quand $|x| \to \infty$. On donne des conditions sur V telles que pour $r = |x|$ grand,

$$0 < c_- \leqslant \left( \int_{S^{n-1}} |\psi(r\omega)|^2 d\omega \right)^{1/2} / v(r) \leqslant c_+ < \infty$$

où $d\omega$ est l'intégration sur la sphère unité. Les conditions sur $V(x)$ couvrent les cas de Hamiltoniens décrivant un électron dans le champ de noyaux fixes ou le cas de potentiels à courte portée, à savoir $|V(x)| \leq Cr^{-1-\varepsilon}$ pour $|x|$ grand, $\varepsilon > 0$.

## INTRODUCTION

In this paper we investigate asymptotic properties of $L^2$-solutions of one-body Schrödinger equations

$$(-\Delta + V - E)\psi = 0 \quad \text{on} \quad \Omega_R, \qquad E < 0, \tag{1.1}$$

where $\Omega_R = \{ x \in \mathbb{R}^n : |x| > R > 0 \}$ and where $V(x)$ is multiplication by a real valued function. Without loss we assume $\psi$ real valued. V will be assumed to tend to zero for $|x| \to \infty$ in a sense to be specified below.

The asymptotic properties of $L^2$-solutions to (1.1) and its generalizations for the manybody case have been studied quite intensively by various authors (see [*1*]-[*5*] for recent results and references to earlier work). If in (1.1) $\psi$ cannot be taken to be positive a natural quantity to consider is

$$\psi_{av} = \left( \int_{S^{n-1}} \psi(r\omega)^2 d\omega \right)^{1/2} \tag{1.2}$$

where $r = |x|$ and $d\omega$ denotes integration over the unit sphere.

If V satisfies some very mild conditions a remark given in [*5*] tells us that $\psi_{av}(r) > 0$ for $r \geq \overline{R} \geq R$ provided

$$\inf_{\substack{\phi \in C_0^\infty(\Omega_{\overline{R}}) \\ \|\phi\| = 1}} (\phi, (-\Delta + V - E)\phi) > 0. \tag{1.3}$$

For suppose $\psi_{av}(r) = 0$ for some $r \geq \overline{R}$ then this contradicts (1.3).

Under suitable conditions on V ( $|V| \leq r^{-1/2-\varepsilon}$ for $r$ large will do for instance) it is known by the work of Bardos and Merigot [6] and of Froese and Herbst [7] together with the upper bounds obtainable via maximum principle arguments [8]-[10] that for $r$ large

$$c_\delta \exp\left[-(\sqrt{|E|}+\delta)r\right] \le \psi_{av} \le c_\varepsilon \exp\left[(-\sqrt{|E|}+\varepsilon)r\right] \tag{1.4}$$

for all positive $\varepsilon$, $\delta$.

In this paper we shall improve upon (1.4). More specifically we shall take a perturbation theoretic point of view. We shall compare the asymptotics of $\psi_{av}$ with the asymptotics of a radially symmetric positive function $v(r)$ which is the $L^2$-solution of the radially symmetric problem.

$$(-\Delta + V_1(r) - E)v(r) = 0 \quad \text{in} \quad \Omega_{R'} \tag{1.5}$$

for some $R' \ge R$ and we shall give conditions on V in (1.1) so that

$$0 < c_- \le \psi_{av}(r)/v(r) \le c_+ < \infty \tag{1.6}$$

for $r$ large.

Previous results which in some sense are sharper than (1.4) are those of Ahlrichs *et al.* [11] on ground states of two-electron systems and those of Froese *et al.* [12] who also study one-body systems and obtain « $L^2$-lower bounds ». For instance they show that if in (1.1) $|V(x)| = o(r^{-1-\delta})$ then for $\gamma$ sufficiently large $r^\gamma \exp(\sqrt{-E}r)\psi \notin L^2(\Omega_R)$. Our methods here are partly motivated by this work, where also the case $E \in \sigma_{ess}(-\Delta + V)$ is considered (see also [7] [13] for recent results on decay properties of eigenfunctions of Schrödinger operators where the corresponding energies are imbedded in the essential spectrum). We do not treat $E \in \sigma_{ess}(-\Delta + V)$ here since then $\psi_{av}$ is not necessarily positive any more for $r$ sufficiently large as for instance the Wigner-von Neumann example shows [14]. We also do not consider the many body case. There the situation is a lot more complicated and bounds like (1.4) are usually not available. See [7] for recent results.

In Section 2 we shall state our $n$-dimensional result and some applications to quantum mechanical problems. One dimensional results and the proof of the upper bound are given in section 3. The lower bound is proven in section 4.

As will be seen from the following the proof of the lower bound to $\psi_{av}$ is much more involved than that of the upper bound (this is in accordance for instance with [6]). The reason is the following: In the 1-dimensional case any $L^2$-solution $\psi$ of (1.1) can be chosen positive in $\Omega_R$, provided (1.3) holds. Hence by standard comparison arguments (using the maximum principle) upper and lower bounds to $\psi$ are easily obtained (see e. g. [6]). In the $n$-dimensional case ($n \ge 2$) the upper bound to $|\psi|$ (and similarly to $\psi_{av}$) is still available via comparison arguments since (due to Kato's inequality) $(-\Delta + V - E)|\psi| \le 0$ (see e. g. [2]). However, for the lower bound this method is no longer applicable because of the nodes of $\psi$. But

$\psi_{av} > 0$ in $\Omega_{\bar{R}}$. So one would have to show that $(-\Delta + W - E)\psi_{av} \geq 0$ in $\Omega_{\bar{R}}$ with some suitable $W(x)$ in order to derive a lower bound to $\psi_{av}$ by comparison theorems. However, such an inequality is not available so far.

## 2. STATEMENTS OF THE RESULTS

In order to allow for locally unbounded potentials we need the following

DEFINITION. — Define the differential operator

$$A = x.\nabla + n/2 \tag{2.1}$$

($\nabla$ denoting the $n$-dimensional gradient) on the class of functions $\{ \phi \}$ which can be written as $\phi = \tilde{\phi}\chi$ with $\tilde{\phi} \in H^{1,2}_{loc}(\Omega_R)$ and $\chi \in C_0^\infty(\Omega_R)$ (therefore $\| \nabla\phi \| + \| \phi \| < \infty$). We say that a function $V \in L^2_{loc}(\Omega_R)$ is A-bounded with bound $a$, if for all such $\phi$

$$\| V\phi \|^2 \leq a \| A\phi \|^2 + b \| \phi \|^2 \quad \text{for some } b \in \mathbb{R}\,. \tag{2.2}$$

REMARK 2.1. — This condition on potentials $V(x)$ was introduced in [*12*]. A sufficient condition for V to be A-bounded is for instance [*12*] that $V = V^{(1)} + V^{(2)}$ with $V^{(1)} \in L^\infty(\Omega_R)$ and

$$r^{-1/2} \sup_{\omega \in S^{n-1}} | V^{(2)}(r\omega) | \in L^2((R, \infty), dr)\,. \tag{2.3}$$

Since we want to exclude the possibility that a solution of (1.1) has compact support we assume in the following that $-\Delta + V$ has the « unique continuation property »: For instance $V \in L^p_{loc}$ with $p = 1$ for $n = 1$, [*2*], $p > n/2$ for $2 \leq n \leq 4$ [*15*] and $p \geq (4n - 2)/7$ for $n \geq 5$ [*16*] imply that if a distributional solution of (1.1) vanishes near some point it must vanish identically. Actually it is widely believed that also for $n \geq 5, p > n/2$ should suffice [*2*].

We can state now our main result.

THEOREM 2.1. — Let $n \geq 2$ and let $V(x)$ be a real valued function such that $V \in L^p_{loc}(\Omega_R)$ uniformly, $p=2$ for $n=2$, 3, and for $n>3$, $p$ such that $-\Delta+V$ has the unique continuation property. Assume that $V = V_1 + V_2 + V_3$ and suppose that there is an $\varepsilon > 0$ so that on $\Omega_R$

*i)* $V_1(x) = V_1(r)$, $V_1$ absolutely continuous, $V_1 \to 0$ for $r \to \infty$ and $V_1' \geq -r^{-1-\varepsilon}$.

*ii)* $\left[\left(V_2 + \frac{1}{2}x\nabla V_2\right)_+ r^{1+\varepsilon}\right]^{1/2}$ is A-bounded with bound zero, where $(.)_+$ denotes max $((.), 0)$ and $\nabla$ the distributional gradient.

*iii)* $r^{1+\varepsilon}V_3$ is A-bounded.

*iv)* $\inf_{\omega \in S^{n-1}} (V_2 + V_3)(r\omega) \in L^1((\mathbb{R}, \infty), dr)$.

Suppose $E < 0$ and that there is a distributional solution $\psi \not\equiv 0$, so that

$$(-\Delta + V - E)\psi = 0 \quad \text{on} \quad \Omega_R,\ \psi \in L^2(\Omega_R), \tag{2.4}$$

then there exists for $r \geq R' \geq R$, $R'$ sufficiently large a radially symmetric function $v(r) > 0$ such that

$$(-\Delta + V_1 - E)v = 0 \quad \text{in} \quad \Omega_{R'} \tag{2.5}$$

and

$$c_- v(r) \leq \psi_{av}(r) \leq c_+ v(r) \quad \text{in} \quad \Omega_{R'} \tag{2.6}$$

for some constants $0 < c_- \leq c_+ < \infty$, depending on $R'$.

REMARK 2.2. — A result corresponding to Theorem 2.1 for $n = 1$ is proven in section 3. The $L^p$ conditions on V can be probably relaxed for $n < 4$ somewhat. These conditions imply that any distributional solution $\psi$ of (2.4) is in $H^{2,2}_{loc}(\Omega_R)$ that means for every $\chi \in C_0^\infty(\Omega_R)$, $\| \Delta(\chi\psi) \| + \| \chi\psi \| < \infty$. Furthermore these conditions imply that a solution of (2.4) is continuous [*2*] [*17*].

REMARK 2.3. — Since $V_1(r) \to 0$ for $r \to 0$, we have for $r$ sufficiently large $V_1(r) - E > 0$. In section 3 we will show using this fact that there exists an exponentially decreasing $v(r) > 0$ so that (2.5) is satisfied.

REMARK 2.4. — The proof of Theorem 2.1 (see section 3 and 4) naturally consists of two parts, namely the proof of the upper and the lower bound to $\psi_{av}$. For the upper bound we shall need only the conditions (*i*) and (*iv*) where as for the lower bound in (2.6) all conditions on V will be needed.

Let us now state some immediate consequences of Theorem 2.1. Suppose $H = -\Delta + V$ on $\mathbb{R}^n$ and H is selfadjoint with V satisfying the conditions of Theorem 2.1 in $\Omega_R$. Suppose $E < 0$ and E is an $m$-fold degenerate eigenvalue such that $(-\Delta + V - E)\psi_i = 0$, $i = 1, \ldots, m$, then the spherical averages of the $\psi_i$ have the same asymptotics. This is for instance illustrated by hydrogenic wave functions. To end this chapter we give two explicit examples of physical relevance.

COROLLARY 2.2. — *i)* Suppose that $(-\Delta + V - E)\psi = 0$, $\psi \in L^2(\mathbb{R}^3)$, $E < 0$ and that

$$V(x) = -\sum_{i=1}^{k} \frac{Z_i}{|X_i - x|}$$

where the $Z_i$ are real numbers and the $X_i$ are fixed points in $\mathbb{R}^3$, then for $r$ sufficiently large

$$c_- r^{Z/\sqrt{|E|}-1} e^{-\sqrt{|E|}r} \leq \psi_{av} \leq c_+ r^{Z/\sqrt{|E|}-1} e^{-\sqrt{|E|}r} \tag{2.7}$$

where $Z = \sum_{i=1}^{k} z_i$.

*ii*) Suppose that $(-\Delta + V - E)\psi = 0$, $\psi \in L^2(\Omega_R)$, $E < 0$ and that in $\Omega_R$ $|V| \leq c(r+1)^{-1-\delta}$, $\delta > 0$. Then for $r$ sufficiently large

$$c_- r^{-\frac{n-1}{2}} e^{-\sqrt{|E|}r} \leq \psi_{av} \leq c_+ r^{-\frac{n-1}{2}} e^{-\sqrt{|E|}r}. \tag{2.8}$$

Note that by (*i*) for instance an electron in the field of fixed nuclei is described if the $Z_i > 0$.

*Proof.* — (*i*) Take in Theorem 2.1, $V_1(r) = -Z/r$, then $|V - V_1| \leq Cr^{-2}$ for $r$ large. Hence it suffices to investigate the asymptotic behaviour of an $L^2$-solution of

$$\left(-\Delta - \frac{Z}{r} - E\right)v = 0 \quad \text{in} \quad \Omega_R \tag{2.9}$$

(2.9) can be solved explicitly by $r^{-1}W$, W a Whittaker function, whose asymptotics is well-known [*18*].

(*ii*) Pick $V = V_3$.

## 3. ONE DIMENSIONAL RESULTS AND THE PROOF OF THE UPPER BOUND

We start by showing that the statements of Theorem 2.1 make sense, namely that one can find a function $v(r)$ satisfying (2.5) and that $\psi_{av} > 0$.

LEMMA 3.1. — Let $W(r)$ be bounded and $W(r) - E \geq \alpha^2 > 0$ for $r \geq R$. Suppose $U(r) = U_+ - U_-$ with $U_+ = \max(U, 0)$ and that $\int_R^\infty U_- dr < 2\alpha$, $U \in L^1_{loc}(R, \infty)$. Then there is a function $v > 0$, $v \in L^2(R, \infty)$, $v, v'$ absolutely continuous so that for $r \geq R$

$$-v'' + (W + U - E)v = 0. \tag{3.1}$$

*Proof.* — We first show that

$$\inf_{\substack{\phi \in C_0^\infty(R,\infty) \\ \|\phi\| = 1}} \left(\phi, \left(-\frac{d^2}{dr^2} + W + U - E\right)\phi\right) > 0. \tag{3.2}$$

We have for $\phi \in C_0^\infty(R, \infty)$

$$\left(\phi,\left(-\frac{d^2}{dr^2}+W+U-E\right)\phi\right) \geq \int|\phi'|^2dr+\alpha^2\int|\phi|^2dr-2\alpha\sup|\phi|^2. \tag{3.3}$$

By an inequality of Block [*19*] see also [*20*] we have

$$\sup|\phi|^2 \leq \frac{1}{2}\inf_{\kappa>0}\left(\kappa\int|\phi|^2dr+\frac{1}{\kappa}\int|\phi'|^2dr\right). \tag{3.4}$$

If we take $\kappa = \alpha$ in (3.4) and insert this estimate into (3.3), (3.2) follows. Next assume $a > 0$ and consider for some $c > 0$

$$\inf_{\substack{\phi\in C_0^\infty(R-a,\infty)\\ \|\phi\|=1}}\left(\phi,\left(-\frac{d^2}{dr^2}+W+U-E\right)\phi\right),\ \text{with}\ W+U-E=-c\ \text{for}\ r<R. \tag{3.5}$$

According to the variational principle (see e. g. [*14*]) there will be some a so that this infimum is actually zero and attained. We note that $C_0^\infty(\Omega_{R-a})$ is a core of the quadratic form of $H = -\frac{d^2}{dr^2} + W + U - E$ with Dirichlet conditions at $R-a$. It follows that H has a distributional eigenfunction $v$ so that $Hv = 0$, $v(R-a) = 0$, $v$, $v'$ absolutely continuous and $v$ strictly positive for $r > R-a$ [*2*] [*17*] [*21*].

Now we show that $\psi_{av}(r) > 0$ for $r$ large: According to section 1 we just have to show (1.3). First we note that for $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi$ real valued

$$\|\nabla\phi\| \geq \|\nabla\phi_{av}\| \tag{3.7}$$

since

$$\phi_{av}'^2 = \left(\int_{S^{n-1}}\phi\phi'd\omega\right)^2/\phi_{av}^2 \leq \int_{S^{n-1}}\phi'^2d\omega. \tag{3.8}$$

Hence with V obeying the conditions of Theorem 2.1 we have for real valued $\phi \in C_0^\infty(\Omega_R)$

$$\begin{aligned}(\phi,(-\Delta+V_1+V_2+V_3-E)\phi) &\geq \\ \geq \int_R^\infty (r^{(n-1)/2}\phi_{av})'^2dr &+ \int_R^\infty (r^{(n-1)/2}\phi_{av})^2\left(\frac{(n-1)(n-3)}{4r^2}+|E|-\delta(R)\right)dr \\ &+ \int_R^\infty \inf_{\omega\in S^{n-1}}(V_2+V_3)(r\omega)dr\ \sup_{r\geq R}(r^{n-1}\phi_{av}^2)\end{aligned} \tag{3.9}$$

with $\delta(R) \to 0$ for $R \to \infty$. Now we proceed as in the proof of Lemma 3.1 to conclude that the r. h. s. of (3.9) is positive for R sufficiently large. Note that as in the one dimensional case $C_0^\infty(\Omega_R)$ is a form core of $H=-\Delta+V$, for $n \geq 2$ due to our conditions on V also an operator core [*2*] [*17*].

Next we consider a one dimensional version of Theorem 2.1.

THEOREM 3.1. — Suppose $0 < \alpha^2 \leq W(r) \leq \beta^2 < \infty$ for $r \geq R$. Let $U(r) \in L^1(R, \infty)$ and define

$$\gamma_\pm(R) = \frac{1}{2\alpha} \int_R^\infty U_\pm dr. \tag{3.10}$$

Suppose that $\gamma_- < 1$. Let $f, g > 0$ in $[R, \infty)$ and $f, g \to 0$ for $r \to \infty$. If in the distributional sense

$$\begin{aligned} -f'' + Wf &= 0 \\ -g'' + Wg + Ug &= 0 \end{aligned} \quad \text{in} \quad (R, \infty) \tag{3.11}$$

then for $r \geq R$

$$\frac{f(r)}{f(R)} \exp\left[-(1-e^{-2\alpha(r-R)})\gamma_+(R)\right] \leq \frac{g(r)}{g(R)} \leq \frac{f(r)}{f(R)}\left(1 + \frac{\gamma_-(R)}{1-\gamma_-(R)}\right). \tag{3.12}$$

REMARK 3.1. — This Theorem can be extended in various directions, one can consider for instance the case $W \to \infty$ for $r \to \infty$. Note that Lemma 3.1 implies that positive $f, g$, with $f, g, f', g'$ locally absolutely continuous exist so that (3.11) is satisfied. If we let in (3.12) $r$ tend to infinity we obtain an estimate of the rate at which $f(r)/g(r)$ tends to a constant:

$$e^{-\gamma_+(r)} - 1 \leq \frac{f(r)}{g(r)} \lim_{r\to\infty} \frac{g(r)}{f(r)} - 1 \leq \frac{\gamma_-(r)}{1-\gamma_-(r)}.$$

We start the proof of Theorem 3.1 with a simple one dimensional comparison lemma.

LEMMA 3.2. — Let $U_1(x) \leq U_2(x)$ for $x \geq x_0$ and suppose that $U_1$ and $U_2$ obey for $x \geq x_0 + 1$, $\sup_x \int_{|x-x'|\leq 1} |U_i|\, dx' < \infty$, $i = 1, 2$. We assume also that

$$\inf_{\substack{\phi \in C_0^\infty(x_0,\infty) \\ \|\phi\|=1}} \left(\phi, \left(-\frac{d^2}{dx^2} + U_2\right)\phi\right) > 0. \tag{3.13}$$

Let $f, g, f', g' \in L^2(x_0, \infty)$ and locally absolutely continuous. If $f, g > 0$ and

$$\begin{aligned} -f'' + U_1 f &\geq 0 \\ -g'' + U_2 g &\leq 0 \end{aligned} \quad \text{for} \quad x \geq x_0 \tag{3.14}$$

in the distributional sense, then for $x \geq x_0$

$$\frac{f(x)}{f(x_0)} \geq \frac{g(x)}{g(x_0)} \tag{3.15}$$

and

$$\frac{f'(x)}{f(x)} \geq \frac{g'(x)}{g(x)}. \tag{3.16}$$

*Proof.* — We assume $f(x_0) = g(x_0)$. Let $D = \{ x : f < g \}$. D can be written as the union of intervals $(a_i, b_i)$ with $a_i < b_i$, $(g-f)(a_i) = (g-f)(b_i) = 0$. Let $(a, b)$ denote one of these intervals. Then by (3.14)

$$0 \leq \int_a^b (U_2 - U_1) f g dx \leq \int_a^b (-gf'' + fg'') dx = f(b)(g-f)'(b) - f(a)(g-f''(a) \leq 0 \tag{3.17}$$

where we used that $(g-f)'(b) \leq 0$ and $(g-f)'(a) \geq 0$ due to the definition of D. If $U_2 - U_1 > 0$ a. e. in $(a, b)$ we get a contradiction from (3.17). It remains to consider $U_2 = U_1$ in $(a, b)$. But then $-(g-f)'' + U_1(g-f) \leq 0$ in $(a, b)$ and hence

$$\int_a^b (g-f)'^2 dx + \int_a^b U_1(g-f)^2 dx \leq 0$$

which implies that $g = f$ and therefore D is empty for if $g - f > 0$ there we get a contradiction to (3.13). (3.16) is an immediate consequence of (3.15) since for $x > x_1 \geq x_0$

$$\frac{f(x) - f(x_1)}{f(x_1)(x - x_1)} \geq \frac{g(x) - g(x_1)}{g(x_1)(x - x_1)}.$$

A related many dimensional result can be found in [*9*].

Armed with this lemma we proceed to prove Theorem 3.1. We consider the differential equations

$$\begin{aligned} -g_1'' + (W + U_+)g_1 &= 0 \\ -g_2'' + (W - U_-)g_2 &= 0 \end{aligned} \quad \text{in } [R, \infty) \quad g_1(R) = f(R) = g_2(R). \tag{3.18}$$

Lemma 3.1 implies that $g_1, g_2 \in L^2$ exist so that $g_1, g_2 > 0$ in $[R, \infty)$. Harnack type estimates show that $g_1, g_2, g_1', g_2' \to 0$ for $r \to \infty$, see [*2*]. Lemma 3.2 implies that $g_1 \leq g.f(R)/g(R) \leq g_2$. First we prove the lower bound to $g(r)/g(R)$. Differentiation shows that

$$\frac{g_1(r)}{g_1(R)} = \frac{f(r)}{f(R)} \exp\left[ - \int_R^r \left( \frac{1}{fg_1} \int_x^\infty fg_1 U_+ dy \right) dx \right]. \tag{3.19}$$

Since $W \geq \alpha^2$ we can use (3.16) by considering

$$\left( -\frac{d^2}{dx^2} + \alpha^2 \right) e^{-\alpha x} = 0 \tag{3.20}$$

to obtain

$$\frac{(fg_1)'}{fg_1} = \frac{f'}{f} + \frac{g_1'}{g_1} \leq -2\alpha. \tag{3.21}$$

(3.21) leads upon integration to

$$\frac{(fg_1)(y)}{(fg_1)(x)} \leq e^{-2\alpha(y-x)} \qquad \text{for} \qquad y \geq x \geq \mathrm{R}. \tag{3.22}$$

Therefore we can estimate (3.19) by using the following inequalities:

$$\int_{\mathrm{R}}^{r}\left(e^{2\alpha x}\int_{x}^{\infty} e^{-2\alpha y}\mathrm{U}_{+}(y)dy\right)dx = \frac{1}{2\alpha}\left[e^{2\alpha r}\int_{r}^{\infty} e^{-2\alpha x}\mathrm{U}_{+}dx - e^{2\alpha \mathrm{R}}\int_{\mathrm{R}}^{\infty} e^{-2\alpha x}\mathrm{U}_{+}dx + \int_{\mathrm{R}}^{r}\mathrm{U}_{+}dx\right] \leq \frac{1}{2\alpha}\left[(1 - e^{-2\alpha(r-\mathrm{R})})\int_{\mathrm{R}}^{\infty}\mathrm{U}_{+}dx\right]. \tag{3.23}$$

To prove the upper bound we must proceed in a different way since we cannot estimate $g_2'/g_2$. We consider the inhomogeneous equation

$$-h_c'' + \mathrm{W}h_c - c\mathrm{U}_{-}f = 0, \qquad h_c(\mathrm{R}) = f(\mathrm{R}) = g_2(\mathrm{R}), \qquad c \geq 0. \tag{3.24}$$

If we write

$$h_c = \mu_c f + f, \tag{3.25}$$

$\mu_c$ is easily seen to satisfy

$$f\mu_c'' + 2f'\mu_c' + c\mathrm{U}_{-}f = 0 \tag{3.26}$$

with

$$\mu_c = c\int_{\mathrm{R}}^{r}\int_{x}^{\infty}\frac{f^2(y)}{f^2(x)}\mathrm{U}_{-}(y)dydx. \tag{3.27}$$

Now as before $f^2(y)/f^2(x) \leq e^{-2\alpha(y-x)}$ for $y \geq x \geq \mathrm{R}$ and we get an upper bound to $\mu_c$

$$\mu_c \leq c\gamma_{-}(x) \tag{3.28}$$

and hence

$$h_c \leq (1 + c\gamma_{-}(\mathrm{R}))f. \tag{3.29}$$

Thus

$$-h_c'' + \mathrm{W}h_c - \frac{c}{1 + c\gamma_{-}}\mathrm{U}_{-}h_c \geq 0 \tag{3.30}$$

and for $c = \dfrac{1}{1-\gamma_{-}}$ we obtain

$$-h_c'' + \mathrm{W}h_c - \mathrm{U}_{-}h_c \geq 0. \tag{3.31}$$

By Lemma 3.2, $h_c \geq g_2 \geq g$.

We proceed by proving the upper bound to $\psi_{av}$.

LEMMA 3.3. — For $x \in \Omega_{\mathrm{R}}$

$$(-\Delta + \mathrm{V}_1 - \mathrm{E} + \inf_{\omega \in \mathrm{S}^{n-1}}(\mathrm{V}_2 + \mathrm{V}_3)(r\omega))\psi_{av} \leq 0 \tag{3.32}$$

in the distributional sense.

*Proof.* — We just have to show that in $\Omega_R$, $\Delta\psi \in L^1_{loc}(\Omega_R)$ and

$$-\psi_{av}\Delta\psi_{av} \leq -\int_{S^{n-1}} \psi\Delta\psi d\omega . \tag{3.33}$$

This follows from

$$\Delta\psi_{av}^2 = 2\psi_{av}\Delta\psi_{av} + 2\,|\nabla\psi_{av}|^2 = \int_{S^{n-1}} \Delta\psi^2 d\omega$$

$$= 2\int_{S^{n-1}} \psi\Delta\psi d\omega + 2\int_{S^{n-1}} |\nabla\psi|^2 d\omega \tag{3.34}$$

and (3.7). Next we transform (3.32) to the one dimensional differential inequality

$$\left(-\frac{d^2}{dr^2} + \frac{(n-1)(n-3)}{4r^2} + (V_1-E) + \inf_{\omega\in S^{n-1}} (V_2+V_3)(r\omega)\right)\Phi(r) \leq 0 \tag{3.35}$$

with $\Phi = r^{(n-1)/2}\psi_{av}$. We note that $\psi_{av}$ and $\partial\psi_{av}/\partial r$ are locally absolutely continuous in $\Omega_R$ by a standard argument due to Morrey [*22*] since $\nabla\psi \in L^1_{loc}(\Omega_R)$ implies $\psi(r\omega)$ locally absolutely continuous in $r$ for almost every $\omega$ and $\frac{\partial^2}{\partial r^2}\psi \in L^1_{loc}(\Omega_R)$ implies that $\frac{\partial}{\partial r}\psi(r\omega)$ absolutely continuous for a. e. $\omega$ and therefore

$$\frac{\partial\psi_{av}}{\partial r} = \frac{1}{2\psi_{av}}\frac{\partial}{\partial r}\int \psi^2 d\omega .$$

REMARK 3.2. — (3.33) is related to Kato's distributional inequality (see [*23*]). The proof we have given here follows the proof of a related inequality given in [*24*].

Due to Lemma 3.1 there is a $\eta > 0$ such that

$$\left(-\frac{d^2}{dr^2} + V_1 - E + (n-1)(n-3)/(4r^2) + \inf_{\omega\in S^{n-1}} (V_2+V_3)\right)\eta = 0 \text{ for } r \geq R' \geq R.$$

Application of Lemma 3.2 to $\eta$ and $\Phi$ gives

$$\Phi(r) \leq \frac{\Phi(R')}{\eta(R')}\eta(r) \qquad \text{for} \qquad r \geq R' \tag{3.36}$$

and application of Theorem 3.1 to $vr^{(n-1)/2}$ and $\eta$ gives

$$\frac{\eta(r)}{\eta(R')} \leq \frac{v(r)r^{(n-1)/2}}{v(R')R'^{(n-1)/2}}\left(1 + \frac{\gamma_-(R')}{1-\gamma_-(R')}\right) \quad \text{for} \quad r \geq R'. \tag{3.37}$$

Finally combining (3.36) and (3.37) we obtain

$$\psi_{av}(r) \leq \frac{\psi_{av}(\mathrm{R}')}{v(\mathrm{R}')} v(r)c \qquad \text{for} \qquad r \geq \mathrm{R}' \geq \mathrm{R} \tag{3.38}$$

with some $0 < c(\mathrm{R}) < \infty$ not depending on R′, since $\gamma_{-}(\mathrm{R}') < \gamma_{-}(\mathrm{R})$ for $\mathrm{R}' > \mathrm{R}$. This verifies the upper bound to $\psi_{av}$ given in (2.6).

## 4. PROOF OF THE LOWER BOUND

As already mentioned in the introduction the proof of the lower bound to $\psi_{av}$ is cumbersome. In order to gain flexibility we introduce an auxiliary potential $\mathrm{V}_a(r)$ which depends on a positive parameter $a$:

Define

$$\mathrm{V}_a(r) = \mathrm{V}_1(r) + ar^{-1-\varepsilon}, \qquad a > 0 \tag{4.1}$$

and let $v_a(r) > 0$ obey

$$(-\Delta + \mathrm{V}_a - \mathrm{E})v_a(r) = 0 \qquad \text{in} \qquad \Omega_{\mathrm{R}}. \tag{4.2}$$

Let $v(r) > 0$ be defined according to (2.5). Note that $v_a, v \to 0$ for $r \to \infty$. That $v_a$ and $v$ with the above properties exist can be seen from Lemma 3.1. Application of the Comparison theorem 3.1 to $v_a$ and $v$ leads to

$$v_a(r) \geq k(a)\frac{v_a(\overline{\mathrm{R}})}{v(\overline{\mathrm{R}})} v(r) \qquad \text{for} \qquad r \geq \overline{\mathrm{R}} \geq \mathrm{R}, \tag{4.3}$$

with some $k(a) > 0$. Therefore it suffices to prove that for some $a > 0$

$$\psi_{av} \geq c(a)v_a \qquad \text{for} \qquad r \geq \overline{\mathrm{R}}, \tag{4.4}$$

with some $c(a) > 0$. This is the aim of the following considerations:

We define

$$u_a(r\omega) = \psi(r\omega) . v_a(r)^{-1} \tag{4.5}$$

and suppose indirectly that for all $a > 0$

$$\varliminf_{r \to \infty} \int_{\mathrm{S}^{n-1}} u_a^2(r\omega)d\omega = 0. \tag{4.6}$$

This implies that for all $a > 0$ a monotonously increasing sequence $\{ \mathrm{R}_m^{(a)} \}$ exists with $\mathrm{R}_m^{(a)} \to \infty$ for $m \to \infty$ such that

$$\left( \int_{\mathrm{S}^{n-1}} u_a^2(\mathrm{R}_m^{(a)}\omega)d\omega \right)^{1/2} \leq \frac{1}{m} \qquad \text{for} \qquad m \geq m_0(a) \tag{4.7}$$

with $m_0$ sufficiently large. For simplicity we shall frequently suppress the $a$-dependence of $\mathrm{R}_m$ and shall make it explicit just were it is necessary.

Il we choose $\overline{R} = R_m$ in (4.3) and in the upper bound to $\psi_{av}$ in (3.38) and combine these inequalities with (4.7), then

$$\left(\int_{S^{n-1}} u_a^2(r\omega)d\omega\right)^{1/2} \leq \frac{c_+}{k(a)}\left(\int_{S^{n-1}} u_a^2(R_m\omega)d\omega\right)^{1/2} \leq \frac{c_+}{k(a)m} \qquad (4.8)$$

for $r \geq R_m$, $m \geq m_0$ follows.

Let $P_a$ denote the following formal differential operator

$$P_a = -\Delta - ar^{-1-\varepsilon} + V_2 + V_3 - \frac{2v_a'}{v_a}\frac{\partial}{\partial r}. \qquad (4.9)$$

It is easily verified that

$$P_a u_a = 0 \quad \text{on} \quad \Omega_R \quad \text{in the distributional sense}. \qquad (4.10)$$

Next we need the following identity:

LEMMA 4.1. — For all functions $\phi$ which can be written as

$$\phi = \chi\tilde{\phi} \quad \text{with} \quad \chi \in C_0^\infty(\Omega_R), \qquad \tilde{\phi} \in H_{loc}^{2,2}(\Omega_R)$$

$$\operatorname{Re}((A-1)r^{-M}\phi, r^{-M}P_a\phi) = (\phi', 2rG_a r^{-2M}\phi') + \left(\phi, \frac{G_a'}{2}r^{-2M}\phi\right)$$
$$+\left(\phi, \left(-V_2 - \frac{1}{2}x.\nabla V_2 + a\frac{1-\varepsilon}{2}r^{-1-\varepsilon}\right)r^{-2M}\phi\right) + \left(\phi' r - \frac{1}{2}\phi, V_3 r^{-2M}\phi\right) \qquad (4.11)$$

where $M = \dfrac{n-1}{2}$,

$$G_a = -\left(\frac{v_a'}{v_a} + \frac{n-1}{2r}\right), \qquad G_a' = G_a^2 - V_a - \frac{(n-1)(n-3)}{4r^2} + E. \qquad (4.12)$$

*Proof.* — The identity is easily verified by partial integration and by using the commutator identity $[-\Delta + V_2, A] = -2\Delta - x.\nabla V_2$. See also the proof of Lemma 3.1 in Froese *et al.* [*12*].

Now we choose $\phi = \chi u_a$ where $\chi \in C_0^\infty(\Omega_R)$ is radially symmetric and $\chi \geq 0$. Denoting

$$J(\chi u_a) \equiv ((A-1)r^{-M}\chi u_a, r^{-M}P_a\chi u_a) \qquad (4.13)$$

we have due to (4.10)

$$J(\chi u_a) = \left(r^{-2M}\left((\chi u_a)'r - \frac{1}{2}\chi u_a\right), -u_a\Delta\chi - 2\chi' u_a' + \left(2G_a + \frac{n-1}{r}\right)\chi' u_a\right) \qquad (4.14)$$

which leads to

$$J(\chi u_a) = -2(u_a', r^{-2M+1}\chi\chi' u_a') - (u_a, r^{-2M}(rG_a' + 2G_a)\chi\chi' u_a)$$
$$+\left(u_a, r^{-2M}\left\{\frac{r\chi\chi'''}{2} + \frac{3}{2}r\chi'\chi'' + \chi\chi''\left(-G_a r + \frac{1}{2}\right) + \chi'^2\left(rG_a + \frac{1}{2}\right)\right\}u_a\right). \qquad (4.15)$$

Next we choose a sequence $\{ \chi_m \}$ for $\chi$ given by:

DEFINITION 4.1. — Let $R_0 \geq R$ and $R_1 > R_0 + 1$. Let $\chi^{(1)}, \chi^{(2)} \in C_0^\infty(\Omega_{R_0})$, radially symmetric and monotonously nonincreasing, with

$$\chi^{(1)}(r) = \begin{matrix} 1 & \text{for} & r \leq R_1 \\ 0 & \text{for} & r \geq 2R_1 \end{matrix}, \qquad \chi^{(2)}(r) = \begin{matrix} 1 & \text{for} & r \leq R_0 \\ 0 & \text{for} & r \geq R_0 + 1 \end{matrix}.$$

Denote

$$\chi_m(r) = \chi^{(1)}\left(\frac{rR_1}{R_m}\right) - \chi^{(2)}(r) \qquad \text{for} \qquad m \geq m_0,$$

where $\{ R_m \}$ is given according to (4.7), with $m_0$ sufficiently large chosen such that $R_1 \leq R_{m_0}$. Let further $\tilde{R}, \hat{R} \in (R_0, R_0 + 1)$ and let $\chi_m'' \geq 0$ in $[R_0, \tilde{R}]$, $\chi_m'' \leq 0$ in $[\tilde{R}, R_0 + 1]$ and $\chi_m'' \geq 0$ in $[R_0, \hat{R}]$.

Note that $\chi_m$ depends on $a$, since $R_m$ depends on $a$, but that

$$\chi_m(r) = (\chi^{(1)} - \chi^{(2)})(r) \qquad \text{for} \qquad r \in [R_0, R_0 + 1] \qquad \text{for all } a. \quad (4.16)$$

Now we proceed by the following strategy: In step $(i)$ we estimate $J(\chi_m u_a)$ from below by a positive expression not depending on $m$ with the aid of Lemma 4.1 and by choosing $R_0$ and $a \geq a_0(R_0)$ large enough. In step $(ii)$ we derive an upper bound to $J(\chi_m u_a)$. Thereby we use the fact that

$$| d^k\chi_m/dr^k | \leq d_k(a)R_m^{-k} \quad (k = 1, 2, 3) \qquad \text{for} \qquad r \geq R_0 + 1$$

and further we apply (4.8) and (4.15). Comparison of the upper and the lower bound to $J(\chi_m u_a)$ for $a$ and $m$ sufficiently large will lead to the desired contradiction.

In order to estimate $J(\chi_m u_a)$ we need

LEMMA 4.2. — Let $V_a$ and $v_a$ be defined according to (4.1) and (4.2) and let $G_a$ be given as in (4.12). Then

$$(V_1 - E + (a - b_2)r^{-1-\varepsilon} - ab_3 r^{-2-\varepsilon})^{1/2} \leq G_a(r) \leq (b_1 + a\bar{R}^{-1-\varepsilon})^{1/2} \quad (4.17)$$

$$G_a'(r) \geq -(b_4 r^{-1-\varepsilon} + ab_3 r^{-2-\varepsilon}) \quad (4.18)$$

for $r \geq \bar{R}$, $\bar{R}$ sufficiently large and for all $a \geq a_0(\bar{R})$. Thereby $b_i$ $(1 \leq i \leq 4)$ are suitable positive constants depending on $\varepsilon$, $\bar{R}$ and $\inf_{r \geq \bar{R}} (V_1 - E)$, resp. $\sup_{r \geq \bar{R}} (V_1 - E)$.

*Proof.* — Let $\phi = r^{(n-1)/2} v_a$ and $W = V_a - E + \frac{(n-1)(n-3)}{4} r^{-2}$ then $\phi'/\phi = -G_a$ and (4.2) implies $-\phi'' + W\phi = 0$ for $r \geq R$. Since W is absolutely continuous, $W > 0$ for $r \geq \bar{R}$, $\bar{R}$ sufficiently large and $\phi \to 0$

for $r \to \infty$ Lemma 3.2 can be applied to obtain bounds on $\phi'/\phi$: To derive an upper bound to $G_a$ define $\bar{g}(r)=e^{-mr}$ where

$$m^2 \geq \sup_{r \geq \bar{R}} \left( V_1 - E + \frac{(n-1)(n-3)}{4r^2} \right) + a\bar{R}^{-1-\varepsilon}.$$

Then $-\bar{g}'' + W\bar{g} \leq 0$ for $r \geq \bar{R}$ and application of Lemma 3.2 implies $-\bar{g}'/\bar{g} = m \geq G_a$ verifying the upper bound in (4.17).

For the lower bound to $G_a$ define $\underline{g}(r) = \exp\left( - \int_{\bar{R}}^{r} \sqrt{F(x)}dx \right)$ with

$$F(x) \equiv V_1 - E + \left( a - \frac{c_1}{2\alpha} \right) x^{-1-\varepsilon} - a\frac{(1+\varepsilon)}{2\alpha} x^{-2-\varepsilon} + bx^{-2} \quad (4.19)$$

where $c_1$ is given in condition $(i)$ to $V_1$, $\alpha^2 \equiv \inf_{r \geq \bar{R}} (V_1 - E) > 0$ for $\bar{R}$ sufficiently large and $b \leq \frac{(n-1)(n-3)}{4}(1 + 1/\alpha\bar{R})^{-1}$.

Note that $F \geq \alpha^2$ for large $\bar{R}$ and $a > a_0(\bar{R})$ with $a_0 = \frac{c_1}{2\alpha}\left(1 - \frac{1+\varepsilon}{2\alpha\bar{R}}\right)^{-1}$

Then in the distribution sense $-\underline{g}'' + \left( F - \frac{1}{2\sqrt{F}} F' \right)\underline{g} = 0$ for $r \geq \bar{R}$. Obviously $\underline{g}$ and $F - \frac{1}{2\sqrt{F}} F'$ are absolutely continuous, $\underline{g} \to 0$ for $r \to \infty$ and it is easily verified that $F - \frac{1}{2\sqrt{F}} F' \leq W$ for $r \geq \bar{R}$ for $\bar{R}$ large enough. Hence applying Lemma 3.2 we obtain $\underline{g}'/\underline{g} \geq \phi'/\phi$ for $r \geq \bar{R}$ and therefore $\sqrt{F} \leq G_a$ for $r \geq \bar{R}$ verifying (4.17).

The lower bound to $G_a'$ follows directly from (4.12) and (4.17). □

STEP $(i)$. — We derive a lower bound to $J(\chi_m u_a)$ by using identity (4.11) with $\phi = \chi_m u_a$: Application of the geometric arithmetic inequality gives

$$|(\phi, r^{-2M+1}V_3\phi')| \leq \frac{1}{2}(\phi', r^{-2M+1-\varepsilon}\phi') + \frac{1}{2}(\phi, r^{-2M+1+\varepsilon}V_3^2\phi) \quad (4.20)$$

$$(\phi, |V_3| r^{-2M}\phi) \leq \frac{1}{2}(\phi, r^{-2M+1+\varepsilon}V_3^2\phi) + \frac{1}{2}(\phi, r^{-2M-1-\varepsilon}\phi). \quad (4.21)$$

It is easily verified that condition $(iii)$ on $V_3$ implies that

$$(\phi, r^{-2M+1+\varepsilon}V_3^2\phi) \leq k_1(\phi', r^{-2M+1-\varepsilon}\phi') + k_2(\varepsilon)(\phi, r^{-2M-1-\varepsilon}\phi) \quad (4.22)$$

with some $0 < k_1, k_2(\varepsilon) < \infty$. Further using condition $(ii)$ on $V_2$ we have

$$\left(\phi, r^{-2M}\left(V_2 + \frac{1}{2}x.\nabla V_2\right)\phi\right) \leq k_3(\phi, r^{-2M+1-\varepsilon}\phi) + k_4(\phi, r^{-2M-1-\varepsilon}\phi) \quad (4.23)$$

for some $0 < k_3, k_4 < \infty$.

Combining the foregoing inequalities (4.20)-(4.23) with density (4.11) yields

$$J(\phi)\geq 2(\phi', r^{-2M+1}G_a\phi')+\frac{1}{2}(\phi, r^{-2M}G_a'\phi)-\left(\frac{1}{2}+\frac{3k_1}{4}+k_3\right)(\phi', r^{-2M+1-\varepsilon}\phi')$$
$$+\left(\frac{1-\varepsilon}{2}a-k_4-\frac{1+3k_2(\varepsilon)}{4}\right)(\phi, r^{-2M-1-\varepsilon}\phi). \quad (4.24)$$

Because of (4.19) and (4.17) we have for $R_0$ sufficiently large and for $a > a_0(R_0)$

$$G_a(r) \geq \alpha > 0. \quad (4.25)$$

Inserting (4.25) and (4.18) into (4.24) leads to

$$J(\phi) \geq \left(2\alpha - \left(\frac{1}{2} + \frac{3k_1}{4} + k_3\right)R_0^{-\varepsilon}\right)(\phi', r^{-2M+1}\phi')$$
$$+ \left(\frac{1-\varepsilon}{2}a - k_4 - \frac{1+3k_2}{4} - \frac{b_4}{2} - ab_3R_0^{-1}\right)(\phi, r^{-2M-1-\varepsilon}\phi). \quad (4.26)$$

Hence for $R_0$ and $a > a_0(R_0)$ large enough we obtain the desired positive lower bound

$$J(\chi_m u_a) \geq ak_0(R_0)(\chi_m u_a r^{-2M-1-\varepsilon}\chi_m u_a) \quad (4.27)$$

with some $k_0(R_0) > 0$.

STEP (*ii*). — We derive an upper bound to $J(\chi_m u_a)$ with the aid of identity (4.15).

DEFINITION 4.2. — Let $f$, $g \in L^1_{loc}(\Omega_R)$ and let $R \leq r_1 \leq r_2$, then we denote

$$(f, g)_{(r_1, r_2)} \equiv \int_{r_1}^{r_2}\int_{S^{n-1}} f^* g d\omega dr. \quad (4.28)$$

Noting that the support of the derivatives of $\chi_m$ is contained in

$$[R_0, R_0 + 1] \cup [R_m, 2R_m],$$

identity (4.15) can be written as

$$J(\chi_m u_a) = B(R_0, R_0 + 1) + B(R_m, 2R_m) \quad (4.29)$$

where

$$B(r_1, r_2) \equiv -2(u_a', r\chi_m\chi_m' u_a')_{(r_1,r_2)} - (u_a, (rG_a' + 2G_a)\chi_m\chi_m' u_a)_{(r_1,r_2)}$$
$$+\left(u_a, \left\{\frac{1}{2}r\chi_m\chi_m''' + \frac{3}{2}r\chi_m'\chi_m'' + r\chi_m\chi_m''\left(-G_a + \frac{1}{2r}\right) + r\chi_m'^2\left(G_a + \frac{1}{2r}\right)\right\}u_a\right)_{(r_1,r_2)}. \quad (4.30)$$

Now let $\lambda > 0$ and identify $(r_1, r_2)$ with either $(R_0, R_0 + 1)$ or $(R_m, 2R_m)$.

Since due to (4.10) $(u_a, r\chi_m\chi'_m P_a u_a)_{(r_1,r_2)} = 0$ we obtain by partial integration

$$\begin{aligned} B(r_1, r_2) &= B(r_1, r_2) + \lambda(u_a, r\chi_m\chi'_m P_a u_a)_{(r_1,r_2)} \\ &= (\lambda - 2)(u'_a, r\chi_m\chi'_m u'_a)_{(r_1,r_2)} + \lambda(u_a, \chi_m\chi'_m r(V_2 + V_3)u_a)_{(r_1,r_2)} \\ &+ \lambda(u_a, r^{-1}\chi_m\chi'_m L^2 u_a)_{(r_1,r_2)} - a\lambda(u'_a\chi_m\chi'_m r^{-\varepsilon}u_a)_{(r_1,r_2)} \\ &- (u_a, \chi_m\chi'_m \{ rG'_a(1 + \lambda) + G_a(2 + \lambda) \} u_a)_{(r_1,r_2)} \\ &+ (u_a, rG_a \{ (1 - \lambda)\chi'^2_m - (1 + \lambda)\chi_m\chi''_m \} u_a)_{(r_1,r_2)} \\ &+ \left(u_a, \left\{ (\chi'^2_m + \chi_m\chi''_m)\left(\frac{1}{2} - \lambda\right) + \frac{3}{2}\chi'_m\chi''_m r(1-\lambda) + \frac{r}{2}\chi_m\chi'''_m(1 - \lambda) \right\} u_a\right)_{(r_1,r_2)} \end{aligned} \tag{4.31}$$

where $-L^2$ denotes the angular part of the $n$-dimensional Laplacian.

To estimate $B(R_0, R_0 + 1)$ from above we need

LEMMA 4.3. — For some $C(R_0)$ and for all $v > 0$ we have

$$\begin{aligned} \left(u_a, \chi_m\chi'_m r\left(V_2 + V_3 + \frac{L^2}{r^2}\right)u_a\right)_{(R_0,R_0+1)} &\le C(R_0) \{ (u_a, (\chi'^2_m + v\chi_m\chi'_m)u_a)_{(R_0,R_0+1)} \\ &+ \frac{1}{v}(u'_a, r\chi_m\chi'_m u'_a)_{(R_0,R_0+1)} + (u_a, \chi_m\chi''_m u_a)_{(R_0,\tilde{R}_0)} \Big\} \end{aligned} \tag{4.32}$$

where $\tilde{R}$ is defined in Def. 4.1.

*Proof.* — Noting that obviously for some $C(R_0) < \infty$

$$\left(\psi, \left(\int_{S^{n-1}} \psi^2 d\omega\right)^{-1} r\left(V_2 + V_3 + \frac{L^2}{r^2}\right)\psi\right)_{(R_0,R_0+1)} \le C(R_0)$$

we obtain

$$\left(u_a, \chi_m\chi'_m r\left(V_2 + V_3 + \frac{L^2}{r^2}\right)u_a\right)_{(R_0,R_0+1)} \le C(R_0) \sup_{R_0 \le r \le R_0+1} \left(\chi_m\chi'_m \int_{S^{n-1}} u_a^2 d\omega\right). \tag{4.33}$$

But clearly

$$\chi_m\chi'_m \int_{S^{n-1}} u_a^2 d\omega = (u_a, (\chi'^2_m + \chi_m\chi''_m)u_a)_{(R_0,r)} + 2(u_a, \chi_m\chi'_m u'_a)_{(R_0,r)}$$

and further for all $v > 0$

$$\begin{aligned} \sup_{R_0 \le r \le R_0+1} \left(\chi_m\chi'_m \int_{S^{n-1}} u_a^2 d\omega\right) \le \sup_{R_0 \le r \le R_0+1} &\{ (u_a, (\chi'^2_m + \chi_m\chi''_m + v\chi_m\chi'_m)u_a)_{(R_0,r)} \\ &+ \frac{1}{v}(u'_a, \chi_m\chi'_m u'_a)_{(R_0,r)} \Big\}. \end{aligned} \tag{4.34}$$

Having in mind Def. 4.1 and combining (4.33) with (4.34) inequality (4.32) follows immediately. □

Now we are ready to show

LEMMA 4.4.

$$B(R_0, R_0 + 1) \leq k_1(R_0)\sqrt{a}(u_a, u_a)_{(R'_0, R_0+1)} \tag{4.35}$$

for all $a \geq a_0(R_0)$, with some $R'_0 \in (R_0, R_0 + 1)$ and some $k_1(R_0) > 0$.

*Proof.* — We choose $\lambda \in (1, 2)$ in identity (4.31), apply Lemma 4.3 and use that $G_a \geq 0$ and $\chi_m \chi'_m \geq 0$ in $(R_0, R_0 + 1)$. This yields

$$\begin{aligned} B(R_0, R_0 + 1) \leq & \left(-2 + \lambda + \frac{C(R_0)}{v}\lambda\right)(u'_a, \chi_m \chi'_m u'_a)_{(R_0, R_0+1)} \\ & + (u_a, \chi_m \chi'_m \{ - a\lambda r^{-\varepsilon} + \lambda v C(R_0) - (1 + \lambda) r G'_a \} u_a)_{(R_0, R_0+1)} \\ & + \left(u_a, \chi'^{\,2}_m \left\{ (1 - \lambda) r G_a + \lambda C(R_0) + \frac{1}{2} - \lambda \right\} u_a \right)_{(R_0, R_0+1)} \\ & + \frac{3}{2}(1 - \lambda)(u_a, \chi'_m \chi''_m r u_a)_{(R_0, R_0+1)} + \frac{1 - \lambda}{2}(u_a, \chi_m \chi'''_m r u_a)_{(R_0, R_0+1)} \quad (4.36) \\ & + \left(u_a, \chi_m \chi''_m \left\{ - (1 + \lambda) r G_a + \lambda C(R_0) + \frac{1}{2} - \lambda \right\} u_a \right)_{(R_0, \tilde{R}_0)} \\ & + \left(u_a, \chi_m \chi''_m \left\{ - (1 + \lambda) r G_a + \frac{1}{2} - \lambda \right\} u_a \right)_{(\tilde{R}_0, R_0+1)} . \end{aligned} \tag{4.36}$$

Now we apply the lower bounds to $G_a$ resp. $G'_a$ given in Lemma 4.2, take into account the properties of $\chi''_m$ and $\chi'''_m$ according to Def. 4.1 and choose $v > \lambda C(R_0)/(2 - \lambda)$. Then it is straightforward to see that

$$\begin{aligned} B(R_0, R_0 + 1) \leq & \frac{3}{2}(1 - \lambda)(u_a, \chi'_m \chi''_m u_a)_{(\tilde{R}_0, R_0+1)} \\ & + \frac{1 - \lambda}{2}(u_a, \chi_m \chi'''_m r u_a)_{(\hat{R}, R_0+1)} + \left(u_a, \left\{ -(1+\lambda) G_a r + \frac{1}{2} - \lambda \right\} \chi_m \chi''_m u_a \right)_{(\tilde{R}_0, R_0+1)} \end{aligned} \tag{4.37}$$

provided $R_0$ is sufficiently large and $a \geq a_0(R_0)$.

Next we take into account (4.16), which implies that

$$|\chi'_m \chi''_m + \chi_m \chi''_m| \leq c_1(R_0) \quad \text{for} \quad r \in [\tilde{R}_0, R_0 + 1]$$

and $|\chi_m \chi'''_m| \leq c_2(R_0)$ for $r \in [\hat{R}_0, R_0 + 1]$ for all $m \geq m_0$ and for all $a \geq a_0$. Finally applying these estimates and the upper bound to $G_a$ (given in Lemma 4.2) to inequality (4.37) we obtain inequality (4.35). □

LEMMA 4.5.

$$B(R_m, 2R_m) \leq k_2(a, R_0) \sup_{R_m \leq r \leq 2R_m} \int_{S^{n-1}} u_a^2 d\omega \tag{4.38}$$

with some $0 < k_2(a, R_0) < \infty$ and $\forall m \geq m_0$ and $a \geq a_0(R_0)$.

*Proof.* — We consider identity (4.31) with $(r_1, r_2) = (R_m, 2R_m)$ and choose $\lambda \geq 2$. According to Def. 4.1 we have for $m \geq m_0$, $\chi_m\chi_m' \leq 0$ in $(R_m, 2R_m)$, $|\chi_m'| \leq d_1/R_m$, $|\chi_m''| \leq d_2/R_m^2$ and $\forall \chi_m''' | \leq d_3/R_m^3$ for $r \geq R_0 + 1$ with some $d_i(a) > 0$ $(i = 1, 2, 3)$. Taking this into account and using the upper bound to $G_a$ in (4.17) it follows easily from (4.31) that

$$B(R_m, 2R_m) \leq \lambda(u_a, \chi_m\chi_m' r(V_2 + V_3)u_a)_{(R_m,2R_m)} - (u_a, \chi_m\chi_m' r G_a' u_a)_{(R_m,2R_m)} + d(a, R_0) \sup_{R_m \leq r \leq 2R_m} \int_{S^{n-1}} u_a^2 d\omega \quad (4.39)$$

for some $d(a, R_0) > 0$. Thereby we used that $\int_{S^{n-1}} u_a L^2 u_a d\omega > 0$. Further we conclude by condition (*iv*) on $V_2 + V_3$ that for $m \geq m_0$

$$|(u_a, \chi_m\chi_m' r(V_2 + V_3)u_a)_{(R_m,2R_m)}| \leq d_1(a) \int_{R_0}^{\infty} \sup_{\omega \in S^{n-1}} (V_2 + V_3)_- dr \sup_{R_m \leq r \leq 2R_m} \int_{S^{n-1}} u_a^2 d\omega. \quad (4.40)$$

Since $G_a' = (G_a')_+ - (G_a')_-$ it follows from (4.17) and (4.18) that $\forall m \geq m_0$

$$\int_{R_m}^{2R_m} |G_a'| dr = \int_{R_m}^{2R_m} (G_a' + 2(G_a')_-)dr \leq G_a(2R_m) - G_a(R_m) + ak_3(R_0) \int_{R_m}^{2R_m} r^{-1-\varepsilon} dr \leq ak_4(R_0) < \infty \quad (4.41)$$

with some suitable constants $k_3$, $k_4$ not depending on $m$.

Therefrom

$$|(u_a, \chi_m\chi_m' r G_a' u_a)_{(R_m,2R_m)}| \leq k_5(a, R) \sup_{R_m \leq r \leq 2R_m} \int_{S^{n-1}} u_a^2 d\omega \quad (4.42)$$

results immediately. Inserting (4.40) and (4.42) into (4.39) implies Lemma 4.5. □

Now we combine Lemma 4.4 and 4.5 with identity (4.29) and arrive at

$$J(\chi_m u_a) \leq k_1(R_0)\sqrt{a}(u_a, u_a)_{(R_0', R_0+1)} + k_2(a, R_0) \sup_{R_m \leq r \leq 2R_m} \int_{S^{n-1}} u_a^2 d\omega \quad (4.43)$$

for some $R_0' \in (R_0, R_0 + 1)$, for $m \geq m_0$ and $a \geq a(R_0)$.

Finally we combine the upper bound (4.43) to $J(\chi_m u_a)$ with the lower bound (4.27) which yields

$$k_1(R_0)\sqrt{a}(u_a, u_a)_{(R_0', R_0+1)} + k_2(a, R_0) \sup_{R_m \leq r \leq 2R_m} \int_{S^{n-1}} u_a^2 d\omega \geq ak_0(R_0)(\chi_m u_a, r^{-2M-1-\varepsilon}\chi_m u_a).$$

But having in mind (4.16)

$$(\chi_m u_a, r^{-2M-1-\varepsilon}\chi_m u_a) \geq k_6(R_0, R_0')(u_a, u_a)_{(R_0', R_0+1)}$$

follows easily for some $k_6 > 0$ and for $m \geq m_0, a \geq a_0(R_0)$. Hence we obtain with some $d(a, R_0) > 0$

$$\sup_{R_m \leq r \leq 2R_m} \int_{S^{n-1}} u_a^2 d\omega \geq d(a, R_0)(u_a, u_a)_{(R_0', R_0+1)} \tag{4.44}$$

for $m \geq m_0$ and sufficiently large $a \geq a_0(R_0)$. Finally we use our indirect assumption by applying inequality (4.8) to (4.44) which yields

$$\frac{1}{m^2} \geq c(a, R_0)(u_a, u_a)_{(R_0', R_0+1)} > 0 \tag{4.45}$$

for $m \geq m_0$ and a sufficiently large, with some $c > 0$ not depending on $m$. Hence for $m$ sufficiently large we obtain a contradiction, completing the proof of the lower bound to $\psi_{av}$.

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*(Manuscrit reçu le 20 mai 1984)*

*(Version révisée reçue le 18 octobre 1984)*