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Boundedness of two- and three-body resonances

by

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ABSTRACT. — We prove that the set of dilation-analytic resonances is bounded for two-body Schrödinger operators with dilation-analytic multiplicative potentials having short-range boundary values.

For the three-body problem with dilation-analytic multiplicative short-range interactions we prove that resonances are bounded in any strip between cuts associated with consecutive two-body thresholds. For pair potentials going to zero faster than r^{-2} at ∞ an upper bound on the real part of resonance energies is obtained.

RÉSUMÉ. — On démontre que l'ensemble des résonances, définies par la méthode d'analyticité par rapport aux dilatations, est borné pour les opérateurs de Schrödinger à deux corps avec des potentiels multiplicatifs analytiques, à valeurs au bord de courte portée.

Pour le problème à trois corps avec des potentiels à deux corps multiplicatifs, analytiques, à courte portée, on démontre que l'ensemble des résonances est borné dans chaque bande entre deux coupures associées aux seuils, soit des valeurs propres, soit des résonances des problèmes à deux corps associés. Pour des potentiels décroissant plus vite que r^{-2} à l'infini les énergies de résonance sont bornées.

Resonances are defined in the dilation-analytic theory as discrete eigenvalues of a complex-dilated Hamiltonian [*6*], [*13*]. In the two-body case and in the many-body case above thresholds it is easy to show that the set of resonances is bounded in any angle smaller than the maximal sector defined by the potential. This leaves the question open, whether the resonances remain bounded up to the limiting half line $e^{-2ia}\mathbb{R}^+$ (S_a is the analyticity sector of the potential) and between cuts in the many-body case.

In the two-body case examples of the type $\gamma r^{\beta}e^{-r^{\alpha}}, \alpha > 0$ are given in [14], which show that the set of dilation-analytic resonances may be bounded ($-2 < \beta < -1$) or unbounded ($\beta = 1, \alpha = 2$). These examples suggest that if the boundary operator $V(ia)$ is defined by a short-range potential (here $e^{i\frac{\pi\beta}{2\alpha}}r^{\beta}e^{-ir^{\alpha}}$), then the set of dilation-analytic resonances is bounded. This is in fact proved in Theorem 2.1 for dilation-analytic, multiplicative potentials with Δ-form-compact, short range boundary values.

In the many-body case boundedness of resonances in strips parallel to and at a positive distance from cuts was proved in [3] for pair potentials going to 0 at ∞ faster than $r^{-\varepsilon}, \varepsilon > 0$. Here we restrict the discussion to the three-body problem, where resonances are identical with singular points of the analytically continued Faddeev matrix $A(z, \zeta)$ (z dilation parameter, ζ energy variable) [4]. It is further shown in [4] that for short-range potentials $A(z, \zeta)$ has continuous boundary values $A_{\pm}(z, \zeta)$—in a suitably weighted L^2-space—on the cuts associated with two-body thresholds. This together with the two-body result suggests that three-body resonances are bounded in any closed strip bounded by two consecutive cuts (except the one starting from 0) for dilation-analytic, multiplicative, Δ-form-compact, short range pair potentials. This is precisely our main result, formulated in Theorem 3.1. It is proved by showing that $\|A^2(z, \zeta)\| \to 0$ for $\zeta \to \infty$ within any such strip in a certain weighted L^2-space. This result could also be formulated so as to include the boundary values $A_{\pm}(z, \zeta)$, and hence Theorem 3.1 includes any resonances that might lie on the cuts bounding the given strip (Remark 4.7). This is useful for a possible extension of the local scattering theory of [4] to a global theory. We also note that, under a weak additional condition on the potentials, the estimate of $\| A^2(z, \zeta) \|$ is uniform in the dilation angle φ, away from 0 and $\frac{\pi}{2}$ (Theorems 4.2 and 4.4).

For pair potentials going to zero faster than $r^{-2-\varepsilon}$ at ∞ stronger results are obtained. The zero-channel is included (Theorem 5.1), and the uniform estimates hold for φ up to 0 (Theorem 5.6). This means that there are no three-body resonances above a certain energy. For this result we have to assume that the two-body systems have no zero-energy resonance or positive energy bound states. Related results for such potentials are given in [15].

We emphasize, that the results of Theorems 2.1 and 3.1 are concerned only with dilation-analytic resonances. The set of resonances, which may be studied by other methods of analytic continuation, may be unbounded. In the two-body case it is actually proved [14] for the potentials $\gamma r^{\beta} e^{-r^{\alpha}}$, $-2 < \beta < -1$, $\alpha > 1$, that the resonances converge to ∞ along $e^{-i\frac{\pi}{\alpha}}\mathbb{R}^+$ *outside* the dilation-analytic sector. In the three-body case with dilation-analytic potentials decaying faster than any exponential, the Faddeev matrix and the resolvent have analytic continuations to a much larger Riemann surface, extending to ∞ on all sheets, than the one defined by dilation-analyticity [5], leaving the possibility open of an unbounded set of resonances on this Riemann surface. It is tempting to conjecture that for a three-body problem with the above Gaussian type of pair potentials, resonances are unbounded and converge from the outside to the limiting dilation cuts.

1. DEFINITIONS AND NOTATIONS

For $t \in \mathbb{R}$ we introduce the following Hilbert spaces and Banach spaces of complex-valued functions on $\mathbb{R}^3$:

$$\mathscr{h}_t = \mathrm{L}^2_{\frac{t}{2}}(\mathbb{R}^3) = \left\{ g \mid \|g\|^2_{2,t} = \int_{\mathbb{R}^3} |g(x)|^2 (1+|x|)^t dx < \infty \right\};\ \mathscr{h} = \mathscr{h}_0$$

$$\mathrm{L}^1_t(\mathbb{R}^3) = \left\{ g \mid \|g\|^2_{1,t} = \int_{\mathbb{R}^3} |g(x)|(1+|x|)^t dx < \infty \right\};\ \mathrm{L}^1(\mathbb{R}^3) = \mathrm{L}^1_0(\mathbb{R}^3)$$

$$\mathrm{L}^\infty_t(\mathbb{R}^3) = \{ g \mid \|g\|_{\infty,t} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} |g(x)(1+|x|)^t| < \infty \};\ \mathrm{L}^\infty(\mathbb{R}^3) = \mathrm{L}^\infty_0(\mathbb{R}^3)$$

$$\mathrm{R} = \left\{ g \mid \|g\|^2_{\mathrm{R}} = \int_{\mathbb{R}^3\times\mathbb{R}^3} |g(x)|\,|g(y)|\,|x-y|^{-2} dx dy < \infty \right\}$$

$$\mathrm{H}^m(\mathbb{R}^3) = \left\{ g \mid \|g\|^2_{\mathrm{H}^m} = \sum_{|\alpha|\leq m} \|\mathrm{D}^\alpha g\|^2_2 < \infty \right\}, \qquad m = 1, 2.$$

The Sobolev space $\mathrm{H}^{-1}(\mathbb{R}^3)$ of order -1 is the dual of $\mathrm{H}^1(\mathbb{R}^3)$. The weight function f_s is defined for $s > 1$ by

$$f_s(x) = (1 + |x|)^{-s}, \qquad x \in \mathbb{R}^3.$$

For $s > 1$ and $p > \frac{3}{2}$ we consider the function spaces

$$\mathrm{M}_s = \mathrm{R} \cap \mathrm{L}^1_{1-s} + \mathrm{L}^\infty_s$$
$$\mathrm{M}^p_s = \mathrm{L}^p \cap \mathrm{L}^1_s + \mathrm{L}^\infty_s$$

We note that $\mathrm{R} \subset \mathrm{L}^1_{1-s}$ for $s \geq 3$ and $\mathrm{M}^p_s \subset \mathrm{M}_s$.

We shall make use of the following property of functions in M_s and M^p_s.

LEMMA 1.1. — Let $V \in M_s$. Then for every $\delta > 0$ there exist $V_1^\delta \in R \cap L^1_{1-s}$ and $V_2^\delta \in L_s^\infty$ such that

$$V = V_1^\delta + V_2^\delta, \qquad \| V_1^\delta \|_R < \delta, \qquad \| V_1^\delta \|_{L^1_{1-s}} < \delta .$$

Let $V \in M_s^p$. Then for every $\delta > 0$ there exist $V_1^\delta \in L^p \cap L_s^1$ and $V_2^\delta \in L_s^\infty$ such that

$$V = V_1^\delta + V_2^\delta, \qquad \| V_1^\delta \|_p < \delta, \qquad \| V_1^\delta \|_{1,s} < \delta .$$

Proof. — For a given decomposition $V = V_1 + V_2$ let

$$V_{1,n} = \chi_{\{x \mid |V(x)| \leq n\}} \cdot \chi_{\{|x| \leq n\}} V_1 .$$

If $V_1 \in R(L^1_{1-s}, L^p, L^1_s)$, by Lebesgue's dominated convergence theorem $V_{1,n} \to V_1$ in $R(L^1_{1-s}, L^p, L^1_s)$, and the Lemma follows.

By $\{ U(\rho) \mid \rho \in \mathbb{R}^+ \}$ we mean the dilation group on $\mathscr{h}$. We fix throughout this paper a, $0 < a \leq \frac{\pi}{2}$. It is emphasized that the restriction $a \leq \frac{\pi}{2}$ is only for convenience, the results can be extended to $a > \frac{\pi}{2}$. By S_a we mean $\{ z \neq 0 \mid |\operatorname{Arg} z| < a \}$, $\bar{S}_a = \{ z \neq 0 \mid |\operatorname{Arg} z| \leq a \}$.

In the two-body case the masses of the two particles (1 and 2) are denoted by m_1 and m_2 and their reduced mass by $\mu = (m_1^{-1} + m_2^{-1})^{-1}$. Then the free Hamiltonian in the center-of-mass system is $h_0 = -\frac{\Delta}{2\mu}$, and for $V \in R + L^\infty(\mathbb{R}^3)$ the total Hamiltonian h is constructed in a standard way [12] [13] using the closed quadratic form

$$\int_{\mathbb{R}^3} (2\mu)^{-1} |\nabla g|^2 + V |g|^2, \qquad g \in H^1(\mathbb{R}^3).$$

It is assumed V is $\bar{S}_a$-dilation-analytic [13], i. e. the $\mathscr{C}(H^1, H^{-1})$-valued function $V(\rho) = U(\rho) V U(\rho^{-1})$, $\rho \in \mathbb{R}^+$, has a continuous extension $V(z)$ to $\bar{S}_a$ such that $V(z)$ is analytic in S_a. Furthermore we assume that $V(e^{ia}) \in M_s$ for some $s > 1$. For the construction of the dilated Hamiltonians $h(z) = z^{-2} H_0 + V(z)$, $z \in \bar{S}_a$, we refer to [13]. The essential spectrum of $h(z)$ is $z^{-2}\overline{\mathbb{R}^+}$ and the non-real, discrete spectrum r_φ of $h(z)$ ($z = \rho e^{i\varphi}$) lies in the sector between $\mathbb{R}^+$ and $e^{-2i\varphi}\mathbb{R}^+$ and is otherwise z-independent [6] [13]. It is easy to see that $r_a = \bigcup_{0 < \varphi < a} r_\varphi$.

Since the result and all proofs in the two-body case are concerned with a fixed operator $h(e^{ia})$, we omit for simplicity of notation the variable e^{ia}, making the following change of notation.

$$-e^{-2ia}\frac{\Delta}{2\mu} \to h_0, \qquad V(e^{ia}) \to V, \qquad -e^{-2ia}\frac{\Delta}{2\mu} + V(e^{ia}) \to h,$$
$$\left(-e^{-2ia}\frac{\Delta}{2\mu} - \zeta\right)^{-1} \to r_0(\zeta) \quad \text{for} \quad \zeta \notin e^{-2ia}\overline{\mathbb{R}^+},$$
$$\left(-e^{-2ia}\frac{\Delta}{2\mu} + V(e^{ia}) - \zeta\right)^{-1} \to r(\zeta) \quad \text{for} \quad \zeta \notin e^{-2ia}\overline{\mathbb{R}^+} \cup r_a \cup \sigma_p(h(1)).$$

If we need to specify the variable φ, we write $V(\varphi)$ for $V(e^{i\varphi})$.
We factorize V as $V = AB$, where $A = |V|^{\frac{1}{2}} \operatorname{sign} V$, $B = |V|^{\frac{1}{2}}$.
The symmetrized resolvent equation is

$$r(\zeta) = r_0(\zeta) - r_0(\zeta)A(1 + Br_0(\zeta)A)^{-1}Br_0(\zeta) \tag{1.1}$$

valid for all ζ such that $(1 + Br_0(\zeta)A)^{-1}$ exists. It is easy to see that if $V \in M_s$, this is the case precisely for $\zeta \in \rho(h)$. In fact, the map $\phi \to B\phi$ is an isomorphism of $\mathcal{N}(h - \zeta)$ onto $\mathcal{N}(1 + Br_0(\zeta)A)$ $(Br_0(\zeta)A \in \mathcal{B}(\mathscr{h}))$. We shall return to the operator-function $Br_0(\zeta)A$ in Section 2, where it is proved that for $V \in M_s$ $\| Br_0(\zeta)A \|_{\mathcal{B}(\mathscr{h})} \to 0$ for $\zeta \to \infty$, $\zeta \notin e^{-2ia}\overline{\mathbb{R}^+}$, implying boundedness of r_a.

For the 3-body problem we use the following standard notations. Let particles 1, 2, 3 have masses m_1, m_2, m_3 and denote pairs (i, j) by α, β, etc. If $\alpha = (1, 2)$, for example, $m_\alpha^{-1} = m_1^{-1} + m_2^{-1}$, $n_\alpha^{-1} = (m_1 + m_2)^{-1} + m_3^{-1}$, $x_\alpha = x_2 - x_1$, $y_\alpha = x_3 - \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$.

The conjugate momenta are denoted by k_α, p_α.

Note that for $\alpha \neq \beta$, the change of variables is given by

$$\begin{pmatrix} x_\beta \\ y_\beta \end{pmatrix} = \begin{Bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{Bmatrix} \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} \tag{1.2}$$

where $t_2 = \pm 1$ and $t_i = t_i(\alpha, \beta) \neq 0$ for $i = 1, 3, 4$.

$\mathscr{H} = H^0(\mathbb{R}^6) = L^2(\mathbb{R}^6)$ and $H^2(\mathbb{R}^6)$ are the Sobolev-spaces of order 0 and 2 respectively.

The free Hamiltonian H_0 is given for each α by the operator $-\dfrac{\Delta_{x_\alpha}}{2m_\alpha} - \dfrac{\Delta_{y_\alpha}}{2n_\alpha}$ on the domain $H^2(\mathbb{R}^6)$.

The pair potentials $V_\alpha = V_\alpha(x_\alpha)$ are assumed to be real-valued functions in M_s^p for some $p > \dfrac{3}{2}$, $s > 1$ and S_a-dilation-analytic for some $a \leq \dfrac{\pi}{2}$. Moreover, we assume $V_\alpha(e^{i\varphi}) \in M_s^p$ for every $\varphi \in (-a, a)$.

The Hamiltonian H is defined (cf. [*13*]) through the quadratic form

$$\mathrm{H}[\mathrm{F}] = \int_{\mathbb{R}^6}\left(\frac{|\nabla x_\alpha \mathrm{F}|^2}{2m_\alpha} + \frac{|\nabla y_\alpha \mathrm{F}|^2}{2n_\alpha} + \sum_\alpha \mathrm{V}_\alpha(0)\,|\,\mathrm{F}\,|^2\right)dx_\alpha dy_\alpha$$

and similarly for the operators $\mathrm{H}_\alpha = \mathrm{H}_0 + \mathrm{V}_\alpha$.

$\{\,\mathrm{U}(\rho)\,\}_{\rho\in\mathbb{R}^+}$ now denotes the dilation group on $\mathscr{H}$ and the dilated operators $\mathrm{H}(\rho e^{i\varphi})$ and $\mathrm{H}_\alpha(\rho e^{i\varphi})$ are defined as in [*9*].

As in the 2-body case we then fix $z = e^{i\varphi}$, $0 < \varphi < a$ and omit the variable φ, using the short-hand notation

$$-e^{-2i\varphi}\left(\frac{\Delta x_\alpha}{2m_\alpha} + \frac{\Delta y_\alpha}{2n_\alpha}\right) \to \mathrm{H}_0, \quad \mathrm{V}_\alpha(e^{i\varphi}) \to \mathrm{V}_\alpha, \quad -e^{-2i\varphi}\left(\frac{\Delta x_\alpha}{2m_\alpha} + \frac{\Delta y_\alpha}{2n_\alpha}\right)$$
$$+ \sum_\alpha \mathrm{V}_\alpha(e^{i\varphi}) \to \mathrm{H}, \quad -e^{-2i\varphi}\left(\frac{\Delta x_\alpha}{2m_\alpha} + \frac{\Delta y_\alpha}{2n_\alpha}\right) + \mathrm{V}_\alpha(e^{i\varphi}) \to \mathrm{H}_\alpha,$$
$$-e^{-2i\varphi}\frac{\Delta x_\alpha}{2m_\alpha} + \mathrm{V}_\alpha(e^{i\varphi}) \to h_\alpha, \quad \mathrm{R}_0(\zeta) = (\mathrm{H}_0 - \zeta)^{-1},$$
$$\mathrm{R}(\zeta) = (\mathrm{H} - \zeta)^{-1}, \quad \mathrm{R}_\alpha(\zeta) = (\mathrm{H}_\alpha - \zeta)^{-1}, \quad r_\alpha(\zeta) = (h_\alpha - \zeta)^{-1}$$

for ζ in the resolvent sets of the various operators, indicated above.

To further simplify the presentation we assume that each two-body operator has exactly one 1-dimensional, negative eigenvalue λ_α. The extension to the general case is straightforward (cf. [*3*]), the basic estimates are the same. The eigenfunction of h_α corresponding to λ_α is taken to be $\phi_\alpha = \phi_\alpha(\varphi)$, where $\phi_\alpha(0)$ is a real normalized eigenfunction of $-\dfrac{\Delta x_\alpha}{2m_\alpha} + \mathrm{V}_\alpha(0)$ associated with the eigenvalue λ_α.

We let $\mathscr{h}_\alpha = \mathrm{L}^2(\mathbb{R}^3_{y_\alpha})$, $\mathscr{h}_{\alpha,-s} = \mathrm{L}^2_{-\frac{s}{2}}(\mathbb{R}^3_{y_\alpha})$.

The relative free Hamiltonian $\tilde{h}_{\alpha 0} = -e^{-2i\varphi}\dfrac{\Delta y_\alpha}{2n_\alpha} + \lambda_\alpha$ in $\mathscr{h}_\alpha$ has resolvent $\tilde{r}_\alpha(\zeta) = (\tilde{h}_{\alpha 0} - \zeta)^{-1} \in \mathscr{B}(\mathscr{h}_\alpha)$ for $\zeta \notin \lambda_\alpha + e^{-2i\varphi}\overline{\mathbb{R}^+}$.

The essential spectrum of H is $\bigcup_\lambda \{\,\lambda + e^{-2i\varphi}\overline{\mathbb{R}^+}\,\}$, where λ ranges over zero and all discrete eigenvalues and resonances of the two-body operators. The non-real discrete spectrum of H, denoted by $\mathscr{R}_\varphi$, is confined between the half-lines $\{\,\lambda_e + \mathbb{R}^+\,\}$ and $\{\,\lambda_e + e^{-2i\varphi}\mathbb{R}^+\,\}$, where λ_e is the smallest negative threshold.

To prove boundedness of $\mathscr{R}_\varphi$ along cuts associated with two-body resonances we have to make the restrictive assumption that these two-body resonances are simple poles of the corresponding resolvents. We shall call cuch poles *simple resonances.*

The set $\mathscr{R}_\varphi$ is identical with the set of singular points of the symmetrized

Faddeev matrix $A(\zeta)$, defined as follows (cf. [4]). Here we restrict the discussion to the negative eigenvalues λ_α. The case of thresholds defined by simple resonances is very similar.

We decompose $R_\alpha(\zeta)$ as

$$R_\alpha(\zeta) = R_\alpha^0(\zeta) + |\phi_\alpha > \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha|$$

where

$$(\langle \chi | g)(y_\alpha) = \int_{\mathbb{R}_x^3} \chi(x_\alpha) g(x_\alpha, y_\alpha) dx_\alpha \quad \text{for} \quad g \in \mathscr{H}, \quad \chi \in L^2(\mathbb{R}^3_{x_\alpha}).$$

$$(|\chi\rangle \sigma)(x_\alpha, y_\alpha) = \chi(x_\alpha)\sigma(y_\alpha) \quad \text{for} \quad \chi \in L^2(\mathbb{R}^3_{x_\alpha}), \quad \sigma \in \hbar_\alpha .$$

We factorize V_α as $V_\alpha = A_\alpha B_\alpha$, $A_\alpha = |V_\alpha|^{\frac{1}{2}} \operatorname{sign} V_\alpha$, $B_\alpha = |V_\alpha|^{\frac{1}{2}}$. Set for $\alpha \neq \beta$

$$Q_{\alpha\beta}(\zeta) = \begin{Bmatrix} B_\alpha R_\beta^0(\zeta) A_\beta & B_\alpha | \phi_\beta > \\ \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | V_\alpha R_\beta^0(\zeta) A_\beta & \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | V_\alpha | \phi_\beta > \end{Bmatrix} \tag{1.3}$$

$$A(\zeta) = \begin{Bmatrix} 0 & Q_{\alpha\beta}(\zeta) & Q_{\alpha\gamma}(\zeta) \\ Q_{\beta\alpha}(\alpha) & 0 & Q_{\alpha\gamma}(\zeta) \\ Q_{\gamma\alpha}(\zeta) & Q_{\gamma\beta}(\gamma) & 0 \end{Bmatrix} \tag{1.4}$$

We introduce the auxiliary spaces $\tilde{\mathscr{H}}$ and $\tilde{\mathscr{H}}_{-s}$ defined by

$$\tilde{\mathscr{H}} = \sum_{\alpha}{}_{\oplus} (\mathscr{H} \oplus \hbar_\alpha), \qquad \tilde{\mathscr{H}}_{-s} = \sum_{\alpha}{}_{\oplus} (\mathscr{H} \oplus \hbar_{\alpha,-s})$$

It will be proved in Lemmas 3.3-3.6 that if $V_\alpha \in M_s^p$ then $A(\zeta) \in \mathscr{B}(\tilde{\mathscr{H}})$ for $\zeta \in \mathbb{C} \backslash \sigma_e(H)$. Using this we shall now sketch a proof of the fact that $\mathscr{R}_\varphi$ is the set of singular points of $1 + A(\zeta)$.

LEMMA 1.2. — Assume that $V_\alpha \in M_s^p$ for some $p > \frac{3}{2}$, $s > 1$ and all α, and let $\zeta \in \mathbb{C} \backslash \sigma_e(H)$. Then $\mathscr{N}(H - \zeta)$ and $\mathscr{N}(1 + A(\zeta))$ ($A(\zeta) \in \mathscr{B}(\tilde{\mathscr{H}})$) are isomorphic via the maps

$$\mathscr{N}(H - \zeta) \ni \psi \to L\psi = \sum_{\alpha}{}_{\oplus} (u_\alpha \oplus \sigma_\alpha) \in \mathscr{N}(1 + A(\zeta)),$$

where

$$u_\alpha = B_\alpha(1 + R_0(\zeta)V_\alpha)\psi, \qquad \sigma_\alpha = \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | A_\alpha u_\alpha$$

and

$$\mathscr{N}(1 + A(\zeta))) \ni \Phi = \sum_{\alpha}{}_{\oplus} (u_\alpha \oplus \sigma_\alpha) \to \psi = K\Phi \in \mathscr{N}(H - \zeta)$$

where

$$K\Phi = -\sum_{\alpha} R_\alpha(\zeta) A_\alpha u_\alpha .$$

Proof. — It is easy to see that L maps H^1 into $\tilde{\mathscr{H}}$ and K maps $\tilde{\mathscr{H}}$ into H^1. The algebraic verification that $L\psi \in \mathcal{N}(1 + A(\zeta))$ and $K\Phi \in \mathcal{N}(H - \zeta)$ as well as $KL\psi = \psi$ and $LK\Phi = \Phi$ is carried out in [*4*].

For $V_\alpha \in M_s^p$ we denote by $V_{\alpha_1}^\delta$ and $V_{\alpha_2}^\delta$ functions chosen in accordance with Lemma 1.1, such that

$$V_\alpha = V_{\alpha_1}^\delta + V_{\alpha_2}^\delta, \quad \| V_{\alpha_1}^\delta \|_{L^p} < \delta, \quad \| V_{\alpha_1}^\delta \|_{L_s^1} < \delta, \quad V_{\alpha_2}^\delta \in L_s^\infty .$$

For $\varepsilon > 0$, $0 < \varphi < a$, we define the half-planes S_ε and $S_{-\varepsilon}$ by

$$S_\varepsilon = \{ \zeta = s + e^{-2i\varphi}t \mid s \geq \varepsilon,\ t \in \mathbb{R} \}$$
$$S_{-\varepsilon} = \{ \zeta = s + e^{-2i\varphi}t \mid s \leq -\varepsilon,\ t \in \mathbb{R} \}$$

We set $\mathscr{R}_{\varphi,-\varepsilon} = \mathscr{R}_\varphi \cap S_{-\varepsilon}$, $\mathscr{R}_{\varphi,\varepsilon} = \mathscr{R}_\varphi \cap S_\varepsilon$ for $\varepsilon > 0$, $0 < \varphi < a$.

In the 3-body case we need the following assumption on the 2-body eigenfunctions ϕ_α and resonance functions ψ_α. We formulate this as a condition on the two-body system with potential V for any eigenfunction ϕ associated with a discrete eigenvalue.

CONDITION A. — For some C, $k > 0$

$$\sup_{x \in \mathbb{R}^3} | e^{k|x|}\phi(x) | < C$$

If V is Δ-compact, condition A is always satisfied. In this case $\phi \in H^2(\mathbb{R}^3)$, and the standard boost-analytic argument yields $e^{k|x|}\phi \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ for k small, positive. Since $V \in M_s^p$ for some $p \geq 2$, $s > 1$ implies $V \in L^2(\mathbb{R}^3) + L_s^\infty(\mathbb{R}^3)$, we have

(*i*) If $V \in M_s^p$ for some $p \geq 2$, then A is satisfied for all discrete eigenfunctions and resonance functions.

In the case of negative eigenvalues, let I be a closed interval contained in $(-a, a)$; then clearly $k = k_I$ and $C = C_I$ can be chosen such that

$$\sup_{\varphi \in I} \sup_{x \in \mathbb{R}^3} | e^{k_I|x|}\phi(\varphi, x) | < C_I$$

where $\phi(\varphi, x)$ is the dilation-analytic extension of the eigenfunction ϕ. The same holds with suitably chosen I in the case of resonance thresholds.

If V is only known to be in M_s^p for some p, $\frac{3}{2} < p < 2$, $s > 1$, then V is Δ-form-compact, $\phi \in H^1(\mathbb{R}^3)$, and it is not known whether A holds in general. For radial potentials, however, one can prove the following result using ordinary differential equations techniques.

(*ii*) If V is radial and $V \in M_s^p$ for some p, $\frac{3}{2} < p < 2$, $s > 1$, then every discrete eigenfunction ϕ satisfies condition A. Moreover, a uniform estimate as above can be obtained for the dilation-analytic extension $\phi(\varphi)$ of ϕ. Similar results hold for resonance functions.

For non-radial potentials the following condition is given in [*11*], vol. IV, p. 200:

(*iii*) If $\hat{V} \in L^1 + L^q$ for some q, $2 < q < 3$, then A is satisfied for all discrete eigenfunctions. We note that the proof also works for resonance functions.

2. THE TWO-BODY CASE

The main result is the following

THEOREM 2.1. — Suppose $V = V(a) \in M_s$ for some $s > 1$. Then the set r_a is bounded.

We introduce four Lemmas and then prove Theorem 2.1.

LEMMA 2.2. — For every $s > 1$ there exists $K = K(s)$, such that

$$\| f_s^{\frac{1}{2}} r_0(\zeta) f_s^{\frac{1}{2}} \|_{\mathscr{B}(\mathscr{H})} < K \,|\, \zeta \,|^{-\frac{1}{2}} \quad \text{for} \quad |\, \zeta \,| > 1, \quad \zeta \notin e^{-2i\varphi}\overline{\mathbb{R}^+}\,.$$

Proof. — This follows from [*11*], III, p. 443, keeping track of the ζ-dependence of the various constants at all stages of the proof of this Lemma.

LEMMA 2.3. — For every $s > 1$ there exists $K_s < \infty$ such that

$$\sup_{y \in \mathbb{R}^3} \left\{ (1 + |\, y \,|)^{s-1} \int_{\mathbb{R}^3} |\, y - x \,|^{-2} f_s(x) dx \right\} = K_s^2$$

Proof. — A straightforward exercise using the decomposition

$$\int_{\mathbb{R}^3} = \int_{|x| < |y|/2} + \int_{|x| > |y|/2}$$

LEMMA 2.4. — Suppose $f, g \in R$. Then

$$\int_{\mathbb{R}^6} |\, f(x) g(y) \,|\, |\, x - y \,|^{-2} dx dy$$
$$\leq \left(\int |\, f(x) f(y) \,|\, |\, x - y \,|^{-2} dx dy \right)^{\frac{1}{2}} \left(\int |\, g(x) g(y) \,|\, |\, x - y \,|^{-2} dx dy \right)^{\frac{1}{2}}.$$

Proof. — See [*12*] (1.10, p. 14).

LEMMA 2.5. — Suppose $V \in M_s$, $s > 1$. Then

$$\| B r_0(\zeta) A \|_{\mathscr{B}(\mathscr{H})} \to 0 \quad \text{for} \quad \zeta \to \infty, \quad \zeta \notin e^{-2ia}\overline{\mathbb{R}^+}\,.$$

Proof. — Let $\varepsilon > 0$ be given. Since

$$|\, V \,|^{\frac{1}{2}} = |\, V_1^\delta + V_2^\delta \,|^{\frac{1}{2}} \leq |\, V_1^\delta \,|^{\frac{1}{2}} + |\, V_2^\delta \,|^{\frac{1}{2}} \quad \text{for every } \delta > 0\,,$$

we have for all $\delta_1, \delta_2 > 0$

$$
\begin{aligned}
\| \mathrm{B}r_0(\zeta)\mathrm{A} \| = \| \mathrm{B}r_0(\zeta)\mathrm{B} \| \\
&\leq \| \, | \mathrm{V}_1^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_1^1 |^{\frac{1}{2}} \| + \| \, | \mathrm{V}_1^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_2^1 |^{\frac{1}{2}} \| \\
&+ \| \, | \mathrm{V}_2^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_1^{\delta_2} |^{\frac{1}{2}} \| + \| \, | \mathrm{V}_2^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_2^{\delta_2} |^{\frac{1}{2}} \| \quad (2.1)
\end{aligned}
$$

We now estimate each of the four terms on the right hand side of (2.1), where δ_1 and δ_2 have to be chosen successively. First choose $\delta_1 > 0$ such that (K_s given as in Lemma 2.3)

$$\delta_1^{\frac{1}{2}} < \varepsilon\pi \min \{ \| \mathrm{V}_1^1 \|_{\mathrm{R}}^{-\frac{1}{2}}, \mathrm{K}_s^{-1} \| \mathrm{V}_2^1 \|_{\infty,s}^{-\frac{1}{2}} \} .$$

Then, because $\| \mathrm{V}_1^{\delta_1} \|_{\mathrm{R}}^{\frac{1}{2}} < \delta_1^{\frac{1}{2}}$, $\| \mathrm{V}_1^{\delta_1} \|_{\mathrm{L}_{1-s}^1}^{\frac{1}{2}} < \delta_1^{\frac{1}{2}}$, we get using at the last step Lemma 2.4

$$
\begin{aligned}
&\| \, | \mathrm{V}_1^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_1^1 |^{\frac{1}{2}} \| \leq \| \, | \mathrm{V}_1^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_1^1 |^{\frac{1}{2}} \|_{\mathrm{H.S.}} \leq \\
&\leq (4\pi)^{-1} \left\{ \int\!\!\int | \mathrm{V}_1^{\delta_1}(x) | | \mathrm{V}_1^1(y) | | x - y |^{-2} dx dy \right\}^{\frac{1}{2}} \leq (4\pi)^{-1} \| \mathrm{V}_1^{\delta_1} \|_{\mathrm{R}}^{\frac{1}{2}} \| \mathrm{V}_1^1 \|_{\mathrm{R}}^{\frac{1}{2}} < \frac{\varepsilon}{4}
\end{aligned}
$$

Also (2.2)

$$
\begin{aligned}
\| \, | \mathrm{V}_1^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_2^1 |^{\frac{1}{2}} \| &\leq \| \, | \mathrm{V}_1^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_2^1 |^{\frac{1}{2}} \|_{\mathrm{H.S.}} \leq \\
&\leq (4\pi)^{-1} \left\{ \int\!\!\int | \mathrm{V}_1^1(y)(1 + | y |)^{1-s} \right. \\
&\quad \left. \cdot \, | \mathrm{V}_2^1(x) f_s^{-1}(x) \, | \, (1 + | y |)^{s-1} f_s(x) \, | x - y |^{-2} dx dy \right\}^{\frac{1}{2}} \\
&\leq (4\pi)^{-1} \| \mathrm{V}_1^{\delta_1} \|_{\mathrm{L}_{1-s}^1}^{\frac{1}{2}} \| \mathrm{V}_2^1 \|_{\infty,s}^{\frac{1}{2}} \mathrm{K}_s \leq \frac{\varepsilon}{4} \qquad (2.3)
\end{aligned}
$$

Now choose $\delta_2 > 0$ such that

$$\delta_2^{\frac{1}{2}} < \varepsilon\pi \mathrm{K}_s^{-1} \| \mathrm{V}_2^{\delta_1} \|_{\infty,s}^{-\frac{1}{2}} .$$

Then

$$
\begin{aligned}
\| \, | \mathrm{V}_2^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_1^{\delta_2} |^{\frac{1}{2}} \| &\leq \| \, | \mathrm{V}_2^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | \mathrm{V}_1^{\delta_2} |^{\frac{1}{2}} \|_{\mathrm{H.S.}} \\
&\leq (4\pi)^{-1} \left\{ \int\!\!\int | \mathrm{V}_1^{\delta_2}(y)(1 + | y |)^{1-s} | \right. \\
&\quad \left. \cdot \, | \mathrm{V}_2^{\delta_1}(x) f_s^{-1}(x)(1 + | y |)^{s-1} f_s(x) \, | x - y |^{-2} dx dy \right\}^{\frac{1}{2}} \\
&\leq (4\pi)^{-1} \| \mathrm{V}_1^{\delta_2} \|_{\mathrm{L}_{1-s}^1}^{\frac{1}{2}} \| \mathrm{V}_2^{\delta_1} \|_{\infty,s}^{\frac{1}{2}} \mathrm{K}_s \leq \frac{\varepsilon}{4} \qquad (2.4)
\end{aligned}
$$

The estimates (2.2), (2.3) and (2.4) hold true for all ζ. Finally, by Lemma 2.2 there exists $\mathrm{R}_0 > 0$ such that

$$\| f_s^{\frac{1}{2}} r_0(\zeta) f_s^{\frac{1}{2}} \| \leq \frac{\varepsilon}{4} \| \mathrm{V}_2^{\delta_1} \|_{\infty,s}^{-\frac{1}{2}} \| \mathrm{V}_2^{\delta_2} \|_{\infty,s}^{-\frac{1}{2}} \quad \text{for} \quad | \zeta | > \mathrm{R}_0$$

and hence

$$\| \, | V_2^{\delta_1} |^{\frac{1}{2}} r_0(\zeta) \, | V_2^{\delta_2} |^{\frac{1}{2}} \| \leq \| \, | V_2^{\delta_1} | \, \|_{\infty,s}^{\frac{1}{2}} \, \| \, | V_2^{\delta_2} | \, \|_{\infty,s}^{\frac{1}{2}} \cdot \| f_s^{\frac{1}{2}} r_0(\zeta) f_s^{\frac{1}{2}} \|$$

$$\leq \frac{\varepsilon}{4} \quad \text{for} \quad | \zeta | > R_0 \qquad (2.5)$$

By (2.1)-(2.5)

$$\| B r_0(\zeta) A \| \leq \varepsilon \quad \text{for} \quad | \zeta | > R_0$$

and the Lemma is proved.

Proof of Theorem 2.1. — By Lemma 2.5 there exists $R_0 > 0$ such that

$$\| B r_0(\zeta) A \| < 1 \quad \text{for} \quad | \zeta | > R_0 .$$

Consequently (cf. (1.1)) the set of resonances is confined to $\{ \zeta \mid | \zeta | \leq R_0 \}$.

REMARK 2.6. — If $V(z)$ is known to have an analytic continuation from S_a to S_b for some $b > a$, then Theorem 2.1 is well known [*6*]

REMARK 2.7. — Suppose we drop the assumption that V is $\bar{S}_a$-dilation-analytic and only require that V be S_a-dilation-analytic. Then Theorem 2.1 is not true in the sense that $r = \bigcup_{0<\varphi<a} r_\varphi$ is bounded. For a counter-example we refer to [*14*].

Also an assumption like $V(ia) \in M_s$ seems necessary. For instance if it is only known that $V(ia) \in \mathscr{C}(H^1, H^{-1})$, then we do not expect Theorem 2.1 to hold true. (We believe that $V(r) = r^\beta e^{-r^\alpha}$, $-\frac{1}{2} < \beta < 0$, $\frac{2}{3}\beta + \frac{4}{3} > \alpha > 1$, represent counter-examples [*14*] (4.4)).

REMARK 2.8. — In the case $V \in R$, Lemma 2.5 is well-known (cf. [*10*], p. 274-276 and [*12*], Theorem I.23). The Lemma implies that the first Born approximation is good in the high-energy limit, that is $(1 + B r_0(\zeta) A)^{-1} \simeq 1 - B r_0(\zeta) A$.

3. THE THREE-BODY CASE

We formulate the main result:

THEOREM 3.1. — Assume that $V_\alpha = V_\alpha(\varphi) \in M_s^p$ for some $p > \frac{3}{2}$, $s > 1$, $| \varphi | < a$ and all α. Let $\varepsilon > 0$ be given and suppose that every two-body eigenfunction ϕ_α associated with an eigenvalue $\lambda_\alpha \leq -\varepsilon$ satisfies A. Then $\mathscr{R}_{\varphi,-\varepsilon}$ is bounded.

If all two-body resonances in S_ε are simple and every two-body eigenfunction ϕ_α associated with a positive eigenvalue $\lambda_\alpha \geq \varepsilon$ or a resonance $\lambda_\alpha \in S_\varepsilon$ satisfies A, then $\mathscr{R}_{\varphi,\varepsilon}$ is bounded.

The proof consists in showing that $\| A^2(\zeta) \| \to 0$ for $\zeta \to \infty$ in $S_{-\varepsilon}$ and S_ε respectively. We make the simplifying assumption of Section 1, that each two-body system has exactly one 1-dimensional eigenvalue in $S_{-\varepsilon}$. The following Lemmas (to be proved later) contain the basic estimates of the operators $Q_{\alpha\beta}(\zeta)$ $(\alpha \neq \beta)$ given by (1.3), constituting the Faddeev matrix (1.4). Using these Lemmas we shall prove Theorem 3.1, focusing on the case of $S_{-\varepsilon}$. The case of S_ε is then briefly discussed.

REMARK 3.2. — In the case of negative eigenvalues, the estimates of $\| A^2(\zeta) \|$ can be obtained uniformly for φ in a closed interval $I \subseteq (0, a)$, provided all $V_\alpha(\varphi)$ have decompositions $V_{\alpha_1}(\varphi) + V_{\alpha_2}(\varphi)$ with $V_{\alpha_1}(\varphi)$ a continuous L^p-valued (L^1_s-valued) and $V_{\alpha_2}(\varphi)$ a continuous L^∞_s-valued function on $(-a, a)$. We shall discuss this in Section 4. This extends in an obvious way to the case of simple resonance thresholds.

LEMMA 3.3. — For $\zeta \in S_{-\varepsilon}$

$$B_\alpha R^0_\beta(\zeta) A_\beta, \qquad B_\beta R^0_\beta(\zeta) A_\beta, \quad B_\alpha R_0(\zeta) A_\beta \in \mathscr{B}(\mathscr{H})$$

and

$$B_\alpha R^0_\beta(\zeta) A_\beta = B_\alpha R_0(\zeta)(1 - | \phi_\beta \rangle \langle \phi_\beta |) A_\beta - B_\alpha R_0(\zeta) A_\beta \,.\, B_\beta R^0_\beta(\zeta) A_\beta$$

Moreover

1) $\| B_\alpha R^0_\beta(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H})}, \quad \| B_\alpha R_0(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H})} \to 0$ for $\zeta \to \infty$, $\zeta \in S_{-\varepsilon}$

2) $\| B_\beta R^0_\beta(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H})} < C < \infty$ for $\zeta \in S_{-\varepsilon}$.

LEMMA 3.4. —

$$B_\alpha | \phi_\beta > \in \mathscr{B}(h_{\beta,-s}, \mathscr{H})$$

LEMMA 3.5. — For $\zeta \in S_{-\varepsilon}$, $\zeta \notin \lambda_\alpha + e^{-2i\varphi}\overline{\mathbb{R}^+}$

$$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | V_\alpha R^0_\beta(\zeta) A_\beta \in \mathscr{B}(\mathscr{H}, h_\alpha)\,.$$

Moreover

$$\| \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | V_\alpha R^0_\beta(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H}, h_{\alpha,-s})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}\,.$$

LEMMA 3.6. — For $\zeta \in S_{-\varepsilon}$, $\zeta \notin \lambda_\alpha + e^{-2i\varphi}\overline{\mathbb{R}^+}$

$$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | V_\alpha | \phi_\beta > \in \mathscr{B}(h_\beta, h_\alpha) \cap \mathscr{B}(h_{\beta,-s}, h_{\alpha,-s})\,.$$

Moreover

$$\| \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | V_\alpha | \phi_\beta > \|_{\mathscr{B}(h_{\beta,-s}, h_{\alpha,-s})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}\,.$$

Proof of Theorem 3.1. — By Lemma 1.2 for any resonance λ there exists $0 \neq \Phi \in N(1 + A(\lambda))\,(A(\lambda) \in \mathscr{B}(\tilde{\mathscr{H}}))$. Then $\Phi \in \tilde{\mathscr{H}}_{-s}$, and

$$\Phi \in N(1 - A^2(\lambda)) \quad (A^2(\lambda) \in \mathscr{B}(\tilde{\mathscr{H}}_{-s}))\,.$$

By Lemmas 3.3-3.6 there exists $R_0 > 0$ such that $\| A^2(\zeta) \|_{\mathscr{B}(\tilde{\mathscr{H}}_{-s})} < 1$ for $\zeta \in S_{-\varepsilon}, |\zeta| > R_0$. Hence $1 - A^2(\zeta)$ is invertible and there are no resonances for $\zeta \in S_{-\varepsilon}, |\zeta| > R_0$.

In the general case, where each h_α may have possibly infinitely many eigenvalues accumulating at 0, the proof is similar. For $\varepsilon > 0$ there is a finite number of finite-dimensional eigenvalues of the operators h_α below $-\varepsilon$. For each α we choose an orthonormal basis of the total eigenspace with energies below $-\varepsilon$ and obtain a Faddeev-matrix containing as elements a finite number of operators of the type treated in Lemmas 3.3-3.6 (cf. [*4*] [*5*]). Also the case of S_ε, where the cuts start from two-body resonances or positive eigenvalues, is treated in a very similar way. The assumption that all two-body resonances in S_ε are simple is required to prove the analogues of Lemmas 3.3 and 3.5. It is clear from the proofs of Lemmas 3.3 and 3.5 (to be given later) why this assumption is necessary.

REMARK 3.7. — It is not difficult to sharpen the results in Lemmas 3.3, 3.5, 3.6 as follows:

$B_\alpha R_\beta^0(\zeta) A_\beta \in \mathscr{C}(\mathscr{H})$	(Lemma 3.3)
$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha R_\beta^0(\zeta) A_\beta \in \mathscr{C}(\mathscr{H}, \hbar_\alpha)$	(Lemma 3.5)
$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha \mid \phi_\beta > \in \mathscr{C}(\hbar_\beta, \hbar_\alpha)$	(Lemma 3.6)

Using these facts we find that $A^2(\zeta) \in \mathscr{C}(\tilde{\mathscr{H}})$ and hence the singularities of $(1+A(\zeta))^{-1} = (1-A(\zeta))(1-A^2(\zeta))^{-1}$ are isolated poles in $\mathscr{C} \setminus \bigcup_\lambda \{ \lambda + e^{-2i\varphi}\overline{\mathbb{R}^+} \}$ where λ ranges over 0 and all discrete eigenvalues of two-body operators $h_\alpha(e^{i\varphi})$. This of course is well known [*13*].

For the purpose of proving Lemmas 3.3-3.6 we need the following Lemmas (3.8-3.16).

LEMMA 3.8. — For every $s > 1$ there exists $K = K(s)$ such that

$$\sup_{y \in \mathbb{R}^3} \| f_s^{\frac{1}{2}} r_0(\zeta) f_s^{\frac{1}{2}}(.+y) \|_{\mathscr{B}(\hbar)} < K |\zeta|^{-\frac{1}{2}} \quad \text{for} \quad |\zeta| > 1 .$$

Proof. — Compare with Lemma 2.2. We use the proofs of Lemmas 3, 4, 5 in [*11*], vol. III, p. 442.

LEMMA 3.9. — Let $s > 1$, $p > \dfrac{3}{2}$ and $V_\alpha \in M_s^p$ be given. Then

$$\sup_{y \in \mathbb{R}^3} \| f_s^{\frac{1}{2}} r_0(\zeta) A_\alpha(.+y) \|_{\mathscr{B}(\hbar)} \to 0 \quad \text{for} \quad \zeta \to \infty, \quad \zeta \notin e^{-2i\varphi}\overline{\mathbb{R}^+}$$

Proof. — Let $\varepsilon > 0$ be given. We choose $\delta > 0$ such that

$$\delta^{\frac{1}{2}} < 2\varepsilon\pi K_s^{-1} \quad (K_s \text{ defined in Lemma 2.3}).$$

Since

$$|A_\alpha| = |V_\alpha|^{\frac{1}{2}} = |V^\delta_{\alpha 1} + V^\delta_{\alpha 2}|^{\frac{1}{2}} \leq |V^\delta_{\alpha 1}|^{\frac{1}{2}} + |V^\delta_{\alpha 2}|^{\frac{1}{2}}$$

we have

$$\| f_s^{\frac{1}{2}} r_0(\zeta) A_\alpha(\,.\,+y) \| \leq \| f_s^{\frac{1}{2}} r_0(\zeta) \,|\, V^\delta_{\alpha 1}(\,.\,+y)\,|^{\frac{1}{2}} \| + \| f_s^{\frac{1}{2}} r_0(\zeta) \,|\, V^\delta_{\alpha 2}(\,.\,+y)\,|^{\frac{1}{2}} \| \tag{3.1}$$

Because $\| V^\delta_{\alpha 1}(\,.\,+y) \|_{L^1(\mathbb{R}^3)} < \delta$, we have for all $y \in \mathbb{R}^3$

$$\begin{aligned} \| f_s^{\frac{1}{2}} r_0(\zeta) \,|\, V^\delta_{\alpha 1}(\,.\,+y)\,|^{\frac{1}{2}} \| &\leq \| f_s^{\frac{1}{2}} r_0(\zeta) \,|\, V^\delta_{\alpha 1}(\,.\,+y)\,|^{\frac{1}{2}} \|_{\mathrm{H.S.}} \\ &\leq (4\pi)^{-1} \left\{ \iint f_s(x) \,|\, V^\delta_{\alpha 1}(z+y) \,|\, |x-z|^{-2} dx dz \right\}^{\frac{1}{2}} \\ &\leq (4\pi)^{-1} K_s \| V^\delta_{\alpha 1}(\,.\,+y) \|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)} < (4\pi)^{-1} K_s \delta^{\frac{1}{2}} < \frac{\varepsilon}{2} \end{aligned} \tag{3.2}$$

Furthermore, by Lemma 3.8 there exists $R_0 > 0$ such that

$$\sup_{y \in \mathbb{R}^3} \| f_s^{\frac{1}{2}} r_0(\zeta) f_s^{\frac{1}{2}}(\,.\,+y) \|_{\mathscr{B}(\mathscr{H})} < \frac{\varepsilon}{2} \| V^\delta_{\alpha 2} \|^{-\frac{1}{2}}_{\infty,s} \quad \text{for} \quad |\zeta| > R_0$$

Hence, if $|\zeta| > R_0$ we have for all $y \in \mathbb{R}^3$

$$\| f_s^{\frac{1}{2}} r_0(\zeta) \,|\, V^\delta_{\alpha 2}(\,.\,+y)\,|^{\frac{1}{2}} \| < \| f_s^{\frac{1}{2}} r_0(\zeta) f_s^{\frac{1}{2}}(\,.\,+y) \| \,.\, \| V^\delta_{\alpha 2} \|^{\frac{1}{2}}_{\infty,s} < \frac{\varepsilon}{2} \tag{3.3}$$

By (3.1)-(3.3)

$$\sup_{y \in \mathbb{R}^3} \| f_s^{\frac{1}{2}} r_0(\zeta) g(\,.\,+y) \| < \varepsilon \quad \text{for} \quad |\zeta| > R_0,$$

and the Lemma is proved.

LEMMA 3.10. — For $\zeta \notin \{ e^{-2i\varphi}\overline{\mathbb{R}^+} \} \cup \{ \lambda_\alpha + e^{-2i\varphi}\overline{\mathbb{R}^+} \}$

$$\tilde{r}_\alpha(\zeta) R_0(\zeta) = \left(-e^{-2i\varphi} \frac{\Delta_{x_\alpha}}{2m_\alpha} - \lambda_\alpha \right)^{-1} [\tilde{r}_\alpha(\zeta) - R_0(\zeta)].$$

Proof. — In momentum representation

$$\tilde{r}_\alpha(\zeta) R_0(\zeta) = \left(e^{-2i\varphi} \frac{p_\alpha^2}{2n_\alpha} + \lambda_\alpha - \zeta \right)^{-1} \left(e^{-2i\varphi} \frac{k_\alpha^2}{2m_\alpha} + e^{-2i\varphi} \frac{p_\alpha^2}{2n_\alpha} - \zeta \right)^{-1}$$

and

$$\begin{aligned} &\left(e^{-2i\varphi} \frac{k_\alpha^2}{2m_\alpha} - \lambda_\alpha \right)^{-1} [\tilde{r}_\alpha(\zeta) - R_0(\zeta)] \\ &\qquad = \left(e^{-2i\varphi} \frac{k_\alpha^2}{2m_\alpha} - \lambda_\alpha \right)^{-1} \left[\left(e^{-2i\varphi} \frac{p_\alpha^2}{2n_\alpha} + \lambda_\alpha - \zeta \right)^{-1} \right. \\ &\qquad\qquad \left. - \left(e^{-2i\varphi} \frac{k_\alpha^2}{2m_\alpha} + e^{-2i\varphi} \frac{p_\alpha^2}{2n_\alpha} - \zeta \right)^{-1} \right] \end{aligned}$$

LEMMA 3.11. — Let $g \in L^2(\mathbb{R}^3_{x_\alpha})$. Then

$$< g \mid \in \mathscr{B}(\mathscr{H}, \hbar_\alpha) \quad \text{and} \quad \| < g \mid \|_{\mathscr{B}(\mathscr{H}, \hbar_\alpha)} \leq \| g \|_{L^2(\mathbb{R}^3_{x_\alpha})}$$

Proof. — Fubini's theorem and Cauchy-Schwarz' inequality.

LEMMA 3.12 [8]. — Assume that $V_\alpha \in L^p(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for some $p > \frac{3}{2}$ and all α. Then $B_\alpha R_0(\zeta) A_\beta \in \mathscr{B}(\mathscr{H})$ and there exists $C_p < \infty$ (depending on p and m_i) such that

$$\| B_\alpha R_0(\zeta) A_\beta \| < C_p \max \{ \| V_\alpha \|_{L^p}^{\frac{1}{2}} . \| V_\beta \|_{L^p}^{\frac{1}{2}}, \| V_\alpha \|_{L^1}^{\frac{1}{2}} . \| V_\beta \|_{L^1}^{\frac{1}{2}} \}$$

for $\zeta \notin e^{-2i\varphi}\overline{\mathbb{R}^+}$ and all pairs α, β.

LEMMA 3.13. — Let V_α and C_p be as in Lemma 3.12 and assume

$$\varepsilon(\zeta) = | \operatorname{Im}(e^{2i\varphi}\zeta) | > 0 .$$

Then

$$\| B_\alpha R_0(\zeta) \| \leq \varepsilon^{-\frac{1}{2}}(\zeta) C_p^{\frac{1}{2}} \{ \max [\| V_\alpha \|_{L^p}, \| V_\alpha \|]_{L^1} \}^{\frac{1}{2}}$$

Proof. —

$$\begin{aligned}
\| B_\alpha R_0(\zeta) \| &= \| B_\alpha R_0(\zeta)(R_0(\zeta))^* B_\alpha^* \| \\
&= \left\| \frac{1}{2} \varepsilon^{-1}(\zeta) B_\alpha [R_0(\zeta) - (R_0(\zeta))^*] B_\alpha^* \right\|^{\frac{1}{2}} \leq \varepsilon^{-\frac{1}{2}}(\zeta) \| B_\alpha R_0(\zeta) B_\alpha \|^{\frac{1}{2}} \\
&\leq \varepsilon^{-\frac{1}{2}}(\zeta) C_p^{\frac{1}{2}} \{ \max [\| V_\alpha \|_{L^p}, \| V_\alpha \|_{L^1}] \}^{\frac{1}{2}}
\end{aligned}$$

where we have used Lemma 3.12.

LEMMA 3.14. — Let $\varepsilon > 0$, $s > 1$ and $\alpha \neq \beta$ be given. Then

$$\| (1 + | x_\alpha |)^{-s/2} R_0(\zeta)(1 + | x_\beta |)^{-s/2} \| \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon} .$$

Proof. — By a result of Agmon [1] we have for some b satisfying $\frac{1}{4} < b < \frac{1}{2}$ and a constant C for $| \zeta | > 1$, $\zeta \notin e^{-2i\varphi}\overline{\mathbb{R}^+}$,

$$\| (1 + | x_\alpha |)^{-s/2} R_0(\zeta)(1 + | x_\beta |)^{-s/2} \| < C \| R_0(\zeta)(1 + | x_\alpha |)^{-b}(1 + | x_\beta |)^{-s/2} \| . \tag{3.4}$$

Moreover, introducing $\varepsilon(\zeta) = | \operatorname{Im}(e^{2i\varphi}\zeta) |$,

$$\begin{aligned}
&\| R_0(\zeta)(1 + | x_\alpha |)^{-b}(1 + | x_\beta |)^{-s/2} \| \\
&\quad \leq \| R_0(\zeta)(1 + | x_\alpha |)^{-2b}(1 + | x_\beta |)^{-s} \|^{\frac{1}{2}} \| R_0(\zeta) \|^{\frac{1}{2}} \\
&\quad \leq \Big\| \frac{1}{2} \varepsilon^{-1}(\zeta)(1 + |x_\alpha|)^{-2b}(1 + |x_\beta|)^{-s} [(R_0(\zeta))^* - R_0(\zeta)] \\
&\qquad\qquad (1 + |x_\alpha|)^{-2b}(1 + |x_\beta|)^{-s} \Big\|^{1/4} . \| R_0(\zeta) \|^{\frac{1}{2}} \\
&\quad \leq \varepsilon^{-3/4}(\zeta) \| (1 + | x_\alpha |)^{-2b}(1 + | x_\beta |)^{-s} R_0(\zeta)(1 + | x_\alpha |^{-2b})(1 + | x_\beta |)^{-s} \|^{1/4}
\end{aligned} \tag{3.5}$$

Since $\inf_{\zeta \in S_{-\varepsilon}} \varepsilon(\zeta) = \varepsilon \sin 2\varphi > 0$ and $2b,\ s > \frac{1}{2}$, we conclude from (3.4) and (3.5), using Lemma 2.2 (with $\mathbb{R}^3$ replaced by $\mathbb{R}^6$) that the Lemma holds true.

LEMMA 3.15. — Let $\varepsilon > 0$, $p > \frac{3}{2}$, $s > 1$ and $\alpha \neq \beta$ be given. Suppose $V_\alpha, V_\beta \in L^p(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) + L_s^\infty(\mathbb{R}^3)$.

Then $B_\alpha R_0(\zeta) A_\beta \in \mathscr{B}(\mathscr{H})$, and

$$\| B_\alpha R_0(\zeta) A_\beta \| \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}.$$

Proof. — Since

$$B_\gamma = | V_\gamma |^{\frac{1}{2}} = | V_{\gamma 1}^\delta + V_{\gamma 2}^\delta |^{\frac{1}{2}} \leq | V_{\gamma 1}^\delta |^{\frac{1}{2}} + | V_{\gamma 2}^\delta |^{\frac{1}{2}}$$

for all pairs γ and every $\delta > 0$, we have for all $\delta_1, \delta_2 > 0$,

$$\begin{aligned} \| B_\alpha R_0(\zeta) A_\beta \| \leq \| \, | V_{\alpha 1}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \, | V_{\beta 1}^1 |^{\frac{1}{2}} \| + \| \, | V_{\alpha 1}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \, | V_{\beta 2}^1 |^{\frac{1}{2}} \| + \\ + \| \, | V_{\alpha 2}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \, | V_{\beta 1}^{\delta_2} |^{\frac{1}{2}} \| + \| \, | V_{\alpha 2}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \, | V_{\beta 2}^{\delta_2} |^{\frac{1}{2}} \| \end{aligned} \tag{3.6}$$

Let $\varepsilon_0 > 0$ be given. First choose $\delta_1 > 0$ such that

$$\delta_1^{\frac{1}{2}} < \frac{\varepsilon_0}{4} \min \{ C_p^{-1}, \varepsilon_1^{\frac{1}{2}} C_p^{-\frac{1}{2}} \| V_{\beta 2}^1 \|_{\infty,s}^{-\frac{1}{2}} \}$$

where $\varepsilon_1 = \varepsilon \sin 2\varphi = \inf_{\zeta \in S_{-\varepsilon}} \varepsilon(\zeta)$ and C_p is given in Lemma 3.12.

Then by Lemma 3.12

$$\begin{aligned} \| \, | V_{\alpha 1}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \, | V_{\beta 1}^1 |^{\frac{1}{2}} \| < C_p \max \{ \| V_{\alpha 1}^{\delta_1} \|_{L^p}^{\frac{1}{2}} \cdot \| V_{\beta 1}^1 \|_{L^p}^{\frac{1}{2}}, \| V_{\alpha 1}^{\delta_1} \|_{L^1}^{\frac{1}{2}} \cdot \| V_{\beta 1}^1 \|_{L^1}^{\frac{1}{2}} \} \\ < C_p \delta_1^{\frac{1}{2}} < \frac{\varepsilon_0}{4} \end{aligned} \tag{3.7}$$

Also, by Lemma 3.13

$$\begin{aligned} \| \, | V_{\alpha 1}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \, | V_{\beta 2}^1 |^{\frac{1}{2}} \| \leq \| \, | V_{\alpha 1}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \| \; \| V_{\beta 2}^1 \|_{\infty,s}^{\frac{1}{2}} \\ < (\| V_{\beta 2}^1 \|_{\infty,s})^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}(\zeta) C_p^{\frac{1}{2}} \{ \max [\| V_{\alpha 1}^{\delta_1} \|_{L^p}, \| V_{\alpha 1}^{\delta_1} \|_{L^1}] \}^{\frac{1}{2}} \\ \leq (\| V_{\beta 2}^1 \|_{\infty,s})^{\frac{1}{2}} \varepsilon_1^{-\frac{1}{2}} C_p^{\frac{1}{2}} \delta_1^{\frac{1}{2}} \leq \frac{\varepsilon_0}{4} \end{aligned} \tag{3.8}$$

Now choose $\delta_2 > 0$ such that

$$\delta_2^{\frac{1}{2}} < \frac{\varepsilon_0}{4} \varepsilon_1^{\frac{1}{2}} C_p^{-\frac{1}{2}} \| V_{\alpha 2}^{\delta_1} \|_{\infty,s}^{-\frac{1}{2}}$$

Then by Lemma 3.13

$$\begin{aligned} \| \, | V_{\alpha 2}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) \, | V_{\beta 1}^{\delta_2} |^{\frac{1}{2}} \| \leq \| V_{\alpha 2}^{\delta_1} \|_{\infty,s}^{\frac{1}{2}} \| R_0(\zeta) \, | V_{\beta 1}^{\delta_2} |^{\frac{1}{2}} \| \\ \leq (\| V_{\alpha 2}^{\delta_1} \|_{\infty,s})^{\frac{1}{2}} \varepsilon_1^{-\frac{1}{2}} C_p^{\frac{1}{2}} \delta_2^{\frac{1}{2}} < \frac{\varepsilon_0}{4} \end{aligned} \tag{3.9}$$

Finally, by Lemma 3.14 there exists $R_0 > 0$ such that

$$\| (1+|x_\alpha|)^{-s/2} R_0(\zeta)(1+|x_\beta|)^{-s/2} \| < \frac{\varepsilon_0}{4} \| V_{\alpha 2}^{\delta_1} \|_{\infty,s}^{-\frac{1}{2}} \| V_{\beta 2}^{\delta_2} \|_{\infty,s}^{-\frac{1}{2}}$$

for $\zeta \in S_{-\varepsilon}$ and $|\zeta| > R_0$.

Hence

$$\begin{aligned} \| \, | V_{\alpha 2}^{\delta_1} |^{\frac{1}{2}} R_0(\zeta) | V_{\beta 2}^{\delta_2} |^{\frac{1}{2}} \| & \\ \leq \| V_{\alpha 2}^{\delta_1} \|_{\infty,s}^{\frac{1}{2}} \| V_{\beta 2}^{\delta_2} \|_{\infty,s}^{\frac{1}{2}} & \| (1+|x_\alpha|)^{-s/2} R_0(\zeta)(1+|x_\beta|)^{-s/2} \| \\ & < \frac{\varepsilon_0}{4} \quad \text{for} \quad \zeta \in S_{-\varepsilon} \quad \text{and} \quad |\zeta| > R_0 \end{aligned} \tag{3.10}$$

By (3.6)-(3.10)

$$\| B_\alpha R_0(\zeta) A_\beta \| < \varepsilon_0 \quad \text{for} \quad \zeta \in S_{-\varepsilon}, \quad |\zeta| > R_0$$

and the Lemma is proved.

LEMMA 3.16. — For ε, ζ and $R_\alpha^0(\zeta)$ as in Lemma 3.3, $B_\alpha R_\alpha^0(\zeta) A_\alpha \in \mathscr{B}(\mathscr{H})$ and there exists K such that

$$\| B_\alpha R_\alpha^0(\zeta) A_\alpha \| < K \quad \text{for all} \quad \zeta \in S_{-\varepsilon}$$

Proof. — By (1.1)

$$B_\alpha r_\alpha(\zeta) A_\alpha = q(\zeta) - q(\zeta)(1+q)(\zeta))^{-1} q(\zeta) \tag{3.11}$$

where $q(\zeta) = B_\alpha r_0(\zeta) A_\alpha$, valid for all ζ such that $(1+q(\zeta))^{-1}$ exists.

By Lemma 2.5,

$$\| q(\zeta) \| \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon} \tag{3.12}$$

Together with the assumption that h_α has exactly one 1-dimensional eigenvalue $\lambda_\alpha < 0$, (3.11) and (3.12) imply that

$$\| B_\alpha r_\alpha(\zeta) A_\alpha \| \leq C_1 < \infty \quad \text{for} \quad \zeta \in S_{-\varepsilon} \backslash D \tag{3.13}$$

where

$$D = \left\{ \xi \mid |\xi - \lambda_\alpha| \leq \delta = \frac{\varepsilon}{2} \sin 2\varphi \right\}.$$

We write $r_\alpha(\zeta)$ as

$$r_\alpha(\zeta) = | \phi_\alpha > (\lambda_\alpha - \zeta)^{-1} < \overline{\phi_\alpha} | + r_\alpha^0(\zeta).$$

It is easy to see that the pole term of $r_\alpha(\zeta)$ satisfies

$$\| B_\alpha | \phi_\alpha > (\lambda_\alpha - \zeta)^{-1} < \overline{\phi_\alpha} | A_\alpha \| \leq C_2 < \infty \quad \text{for} \quad \zeta \in S_{-\varepsilon} \backslash D \tag{3.14}$$

For $|\zeta - \lambda_\alpha| < \frac{3\delta}{2}$, $B_\alpha r_\alpha^0(\zeta) A_\alpha$ is given by the norm-convergent integral

$$B_\alpha r_\alpha^0(\zeta) A_\alpha = -\frac{1}{2\pi i} \int_{|\zeta' - \lambda_\alpha| = \frac{3\delta}{2}} (\zeta - \zeta')^{-1} B_\alpha r_\alpha(\zeta') A_\alpha d\zeta' \tag{3.15}$$

Hence $B_\alpha r_\alpha^0(\zeta)A_\alpha$ is analytic for $|\zeta - \lambda_\alpha| < \frac{3\delta}{2}$. This implies that

$$\| B_\alpha r_\alpha^0(\zeta)A_\alpha \| \leq C_3 < \infty \quad \text{for} \quad \zeta \in D \tag{3.16}$$

By (3.13), (3.14) and (3.16)

$$\| B_\alpha r_\alpha^0(\zeta)A_\alpha \| \leq \max(C_1 + C_2, C_3) \quad \text{for} \quad \zeta \in S_{-\varepsilon} \tag{3.17}$$

In momentum representation

$$R_\alpha^0(\zeta) = r_\alpha^0\left(\zeta - \frac{\overline{z}^2 p_\alpha^2}{2n_\alpha}\right)$$

and hence for all $\zeta \in S_{-\varepsilon}$

$$B_\alpha R_\alpha^0(\zeta)A_\alpha = B_\alpha r_\alpha^0\left(\zeta - \frac{\overline{z}^2 p_\alpha^2}{2n_\alpha}\right)A_\alpha . \tag{3.18}$$

The Lemma follows from (3.17) and (3.18).
We now proceed to the proof of Lemmas 3.3-3.6

Proof of Lemma 3.3. — By the 2nd resolvent equation

$$B_\alpha R_\beta^0(\zeta)A_\beta$$
$$= B_\alpha R_0(\zeta)(1 - |\phi_\beta\rangle\langle\phi_\beta|)A_\beta - B_\alpha R_0(\zeta)A_\beta \,.\, B_\beta R_\beta^0(\zeta)A_\beta \quad \text{for} \quad \zeta \in S_{-\varepsilon} \tag{3.19}$$

Because $B_\alpha R_0(\zeta)A_\beta$, $B_\beta R_\beta^0(\zeta)A_\beta \in \mathscr{B}(\mathscr{H})$ by Lemmas 3.15 and 3.16, it follows from (3.19) that $B_\alpha R_\beta^0(\zeta)A_\beta \in \mathscr{B}(\mathscr{H})$. Furthermore, by Lemma 3.15

$$\| B_\alpha R_0(\zeta)A_\beta \|_{\mathscr{B}(\mathscr{H})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}$$

and by Lemma 3.16

$$\| B_\beta R_\beta^0(\zeta)A_\beta \|_{\mathscr{B}(\mathscr{H})} < C \quad \text{for} \quad \zeta \in S_{-\varepsilon}$$

Hence by (3.19), since ϕ_β satisfies condition A,

$$\| B_\alpha R_\beta^0\zeta)A_\beta \|_{\mathscr{B}(\mathscr{H})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon},$$

and the Lemma is proved.

Proof of Lemma 3.4. — First we prove that $B_\alpha | \phi_\beta > \in \mathscr{B}(\hbar_\beta, \mathscr{H})$. Let $g \in \hbar_\beta$. Then by (1.2) (assuming for simplicity $t_2 = 1$; $t_2 = -1$ is similar)

$$\| B_\alpha | \phi_\beta > g \|_{\mathscr{H}}^2 = \int dx_\beta dy_\beta \, | B_\alpha(y_\beta + t_1 x_\beta)\phi_\beta(x_\beta)g(y_\beta) |^2 \leq$$
$$\leq \| g \|_{\hbar_\beta}^2 \sup_{y_\beta \in \mathbb{R}^3} \int dx_\beta \, | B_\alpha(y_\beta + t_1 x_\beta)\phi_\beta)\phi_\beta(x_\beta) |^2 = K^2 \| g \|_{\hbar_\beta}^2$$

where $K^2 < \infty$ by Condition A. Hence $B_\alpha | \phi_\beta > \in \mathscr{B}(h_\beta, \mathscr{H})$.

We shall now prove that $B_\alpha | \phi_\beta > \in \mathscr{B}(h_{\beta,-s}, \mathscr{H})$. Let $g \in h_{\beta,-s}$. Then

$$\begin{aligned}\| B_\alpha | \phi_\beta > g \|^2 &\leq \int dx_\beta dy_\beta \, | V^1_{\alpha 1}(y_\beta + t_1 x_\beta) | \, . \, | \phi_\beta(x_\beta) g(y_\beta) |^2 \\ &+ \int dx_\beta dy_\beta \, | V^1_{\alpha 2}(y_\beta + t_1 x_\beta) | \, . \, | \phi_\beta(x_\beta) g(y_\beta) |^2 \\ &\leq \| g \|^2_{h_{\beta,-s}} \left\{ \sup_{y_\beta} \int dx_\beta \, | V^1_{\alpha 1}(y_\beta + t_1 x_\beta) | \, . \, | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \right. \\ &\left. + \sup_{y_\beta} \int dx_\beta \, | V^1_{\alpha 2}(y_\beta + t_1 x_\beta) | \, . \, | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \right\} \end{aligned} \tag{3.20}$$

We set

$$C_\beta = \sup_{x_\beta, y_\beta} f_s(y_\beta + t_1 x_\beta) \, . \, f_s^{-1}(y_\beta) | \phi_\beta(x_\beta) |^2 .$$

By Condition A, $C_\beta < \infty$ and hence

$$\begin{aligned}\sup_{y_\beta} \int dx_\beta \, | V^1_{\alpha 1}(y_\beta + t_1 x_\beta) | | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) &\leq \\ \leq C_\beta \sup_{y_\beta} \int dx_\beta \, | V^1_{\alpha 1}(y_\beta + t_1 x_\beta) | \, f_s^{-1}(y_\beta + t_1 x_\beta) & \\ = C_\beta | t_1 |^{-3} \| V^1_{\alpha 1} \|_{L^1_s(\mathbb{R}^3)} < C_\beta | t_1 |^{-3} & \end{aligned} \tag{3.21}$$

Furthermore, we set

$$C = \sup_{x_\beta, y_\beta} f_s^{-1}(y_\beta) f_s(x_\beta) f_s(y_\beta + t_1 x_\beta) .$$

Then for all $x_\beta, y_\beta \in \mathbb{R}^3$

$$\begin{aligned}& | V^1_{\alpha 2}(y_\beta + t_1 x_\beta) | | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \\ &= | \phi_\beta(x_\beta) |^2 f_s^{-1}(x_\beta) \, . \, | V^1_{\alpha 2}(y_\beta + t_1 x_\beta) | f_s^{-1}(y_\beta + t_1 x_\beta) \\ &\quad . \, f_s^{-1}(y_\beta) f_s(x_\beta) f_s(y_\beta + t_1 x_\beta) \leq | \phi_\beta(x_\beta) |^2 f_s^{-1}(x_\beta) C \| V^1_{\alpha 2} \|^\infty_s .\end{aligned}$$

Hence

$$\sup_{y_\beta} \int dx_\beta \, | V^1_{\alpha 2}(y_\beta + t_1 x_\beta) | | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \leq \| \phi_\beta \|^2_{2,-s} C \| V^1_{\alpha 2} \|^\infty_s < \infty , \tag{3.22}$$

where we have used Condition A.

From (3.20)-(3.22) we obtain

$$B_\alpha | \phi_\beta > \in \mathscr{B}(h_{\beta,-s}, \mathscr{H})$$

and the Lemma is proved.

Proof of Lemma 3.5. — Using Lemma 3.3 it suffices to prove

$$\tilde{r}_\alpha(\zeta) < \bar{\phi}_\alpha | V_\alpha R_0(\zeta) A_\beta \in \mathscr{B}(\mathscr{H}, h_\alpha) \qquad \text{for} \qquad \zeta \in S_{-\varepsilon} \backslash \{ \lambda_\alpha + e^{-2i\varphi} \overline{\mathbb{R}^+} \}$$

and

$\| \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha R_0(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H}, \mathscr{h}_{\alpha, -s})} \to 0$ for $\zeta \to \infty$ in $S_{-\varepsilon} \setminus \{ \lambda_\alpha + e^{-2i\varphi}\overline{\mathbb{R}^+} \}$

Because

$$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha R_0(\zeta) A_\beta = \tilde{r}_\alpha(\zeta) < g_\alpha \mid B_\alpha R_0(\zeta) A_\beta,$$

where $g_\alpha := A_\alpha(\overline{z})\phi_\alpha(\overline{z}) \in L^2(R^3_{x_\alpha})$ by Condition A, we obtain from Lemmas 3.3 and 3.11

$$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha R_0(\zeta) A_\beta \in \mathscr{B}(\mathscr{H}, \mathscr{h}_\alpha)$$

We now prove the last part of the Lemma.

From Lemma 3.10 we obtain

$$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha R_0(\zeta) A_\beta = \langle \psi_\alpha \mid \tilde{r}_\alpha(\zeta) A_\beta - < \psi_\alpha \mid R_0(\zeta) A_\beta, \quad (3.23)$$

where

$$\psi_\alpha = \left(- e^{2i\varphi} \frac{\Delta x_\alpha}{2m_\alpha} - \overline{\lambda}_\alpha \right)^{-1} V_\alpha(\overline{z})\phi_\alpha(\overline{z}) \in L^2(\mathbb{R}^3_{x_\alpha}), \qquad z = e^{i\varphi}$$

(note that $\psi_\alpha \in L^2(\mathbb{R}^3_{x_\alpha})$ even though $V_\alpha \phi_\alpha$ may not lie in $L^2(\mathbb{R}^3_{x_\alpha})$; this follows from $V_\alpha \phi_\alpha \in L^1 + L^2$, since the operator is convolution by a function in $L^1 \cap L^2$).

By $f_{s\alpha} : \mathbb{R}^3_{x_\alpha} \oplus \mathbb{R}^3_{y_\alpha} \to \mathbb{C}$ we shall understand the function given by

$$f_{s\alpha}(x_\alpha, y_\alpha) = f_s(y_\alpha) \qquad \text{for} \qquad (x_\alpha, y_\alpha) \in \mathbb{R}^3_{x_\alpha} \oplus \mathbb{R}^3_{y_\alpha}$$

First we prove that

$$f_{s\alpha}^{\frac{1}{2}} \tilde{r}_\alpha(\zeta) A_\beta, \qquad f_{s\alpha}^{\frac{1}{2}} R_0(\zeta) A_\beta \in \mathscr{B}(\mathscr{H})$$

and

$$\| f_{s\alpha}^{\frac{1}{2}} \tilde{r}_\alpha(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H})}, \quad \| f_{s\alpha}^{\frac{1}{2}} R_0(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}.$$

Clearly $f_{s\alpha}^{\frac{1}{2}} R_0(\zeta) A_\beta \in \mathscr{B}(\mathscr{H})$, and from the proofs of Lemmas 3.12-3.15 we find that

$$\| f_{s\alpha}^{\frac{1}{2}} R_0(\zeta) A_\beta \|_{\mathscr{B}(\mathscr{H})} \to 0 \qquad \text{for} \qquad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}. \qquad (3.24)$$

Let $F \in L^2(\mathbb{R}^3_{x_\alpha} \oplus \mathbb{R}^3_{y_\alpha})$. Then for all $\zeta \in S_{-\varepsilon}$, $\zeta \notin \lambda_\alpha + e^{-2i\varphi}\overline{\mathbb{R}^+}$ (we assume $t_2 = 1$)

$$\begin{aligned} &\| f_{s\alpha}^{\frac{1}{2}} \tilde{r}_\alpha(\zeta) A_\beta F \|^2 \\ &= \int dx_\alpha dy_\alpha \left| f_s^{\frac{1}{2}}(y_\alpha) \left(- e^{-2i\varphi} \frac{\Delta y_\alpha}{2n_\alpha} + \lambda_\alpha - \zeta \right)^{-1} A_\beta(. + t_1 x_\alpha) F(x_\alpha, .) \right|^2 \\ &\leq K^2(\zeta) \int dx_\alpha \int dy_\alpha \mid F(x_\alpha, y_\alpha) \mid^2 = K^2(\zeta) \| F \|^2_{\mathscr{H}} \end{aligned} \qquad (3.25)$$

where (setting $x = t_1 x_\alpha$)

$$K(\zeta) = \sup_{x \in R^3} \| f_s^{\frac{1}{2}} \tilde{r}_\alpha(\zeta) A_\beta(. + x) \|_{\mathscr{B}(\mathscr{h}_\alpha)}$$

By Lemma 3.9 $K(\zeta) \to 0$ for $\zeta \to \infty$ in $S_{-\varepsilon}\backslash\{\lambda_\alpha + e^{-2i\varphi}\overline{\mathbb{R}^+}\}$.

From (3.23)-(3.25) and Lemma 3.11 we find that

$$\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid VR_0(\zeta)A_\beta \in \mathscr{B}(\mathscr{H}, \hbar_{\alpha,-s}) \qquad \text{for} \qquad \zeta \in S_{-\varepsilon}$$

and

$$\| \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha R_0(\zeta)A_\beta \|_{\mathscr{B}(\mathscr{H},\hbar_{\alpha,-s})} \to 0 \qquad \text{for} \qquad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}.$$

This concludes the proof of the Lemma.

Proof of Lemma 3.6. — To prove that $\tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha \mid \phi_\beta \rangle \in \mathscr{B}(\hbar_\beta, \hbar_\alpha)$ we remark that by Condition A $g_\alpha := V_\alpha(\overline{z})\phi_\alpha(\overline{z}) \in L^1(\mathbb{R}^3_{x_\alpha})$ and hence it suffices to prove that

$$\langle g_\alpha \mid \phi_\beta \rangle \in \mathscr{B}(\hbar_\beta, \hbar_\alpha). \tag{3.26}$$

Let $f \in \hbar_\beta$ be given. Then (assuming $t_2 = 1$)

$$\| \langle g_\alpha \mid \phi_\beta \rangle f \|^2_{\hbar_\beta} = \int dy_\alpha \int g_\alpha(x')\overline{\phi}_\beta(y_\alpha + t_1 x')\overline{f}(t_4 y_\alpha + t_3 x')dx' \cdot \int \overline{g}_\alpha(x'')\phi_\beta(y_\alpha + t_1 x'')f(t_4 y_\alpha + t_3 x'')dx''.$$

We use Fubini's theorem and Cauchy-Schwarz' inequality and find that

$$\| \langle g_\alpha \mid \phi_\beta \rangle f \|^2_{\hbar_\beta} \le | t_4 |^{-3} \| f \|^2_{\hbar_\beta} \{ \sup_{x \in \mathbb{R}^3} | \phi_\beta(x) | \}^2 \cdot \| g_\alpha \|^2_{L^1(\mathbb{R}^3)}.$$

Hence we have proved (3.26).

To prove that $\tilde{r}_\alpha(\zeta) \langle \overline{\phi}_\alpha \mid V_\alpha \mid \phi_\beta \rangle \in B(\hbar_{\beta,-s}, \hbar_{\alpha,-s})$ we remark that by Condition A

$$V_\alpha(\overline{z})\phi_\alpha(\overline{z}) = g_\alpha f_s^{\frac{1}{2}}, \qquad \text{where} \quad g_\alpha \in L^1(\mathbb{R}^3_{x_\alpha}). \tag{3.27}$$

Let $f \in \hbar_{\beta,-s}$ be given. Then

$$\| \tilde{r}_\alpha(\zeta) \langle \overline{\phi}_\alpha \mid V_\alpha \mid \phi_\beta \rangle f \|^2_{\hbar_{\alpha,-s}} = \| f_s^{\frac{1}{2}} \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha \mid V_\alpha \mid \phi_\beta \rangle f \|^2_{\hbar_\alpha} = \| \langle g_\alpha \mid F \|^2_{\hbar_\alpha},$$

where g_α is given by (3.27) and $F : \mathbb{R}^3_{x_\alpha} \oplus \mathbb{R}^3_{y_\alpha} \to \mathbb{C}$ is given by

$$F(x_\alpha, y_\alpha) = f_s^{\frac{1}{2}}(y_\alpha) f_s^{\frac{1}{2}}(x_\alpha) \tilde{r}_\alpha(\zeta)\phi_\beta(\cdot + t_1 x_\alpha) f(t_4 \cdot + t_3 x_\alpha)$$

We let $G : \mathbb{R}^3_{x_\alpha} \oplus \mathbb{R}^3_{y_\alpha} \to \mathbb{C}$ be given by

$$G(x_\alpha, y_\alpha) = f_s^{-\frac{1}{2}}(y_\alpha + t_1 x_\alpha)\phi_\beta(y_\alpha + t_1 x_\alpha) f_s^{\frac{1}{2}}(x_\alpha) f(t_4 y_\alpha + t_3 x_\alpha).$$

Clearly,

$$F(x_\alpha, y_\alpha) = f_s^{\frac{1}{2}}(y_\alpha)\tilde{r}_\alpha(\zeta) f_s^{\frac{1}{2}}(\cdot + t_1 x_\alpha) G(x_\alpha, \cdot)$$

and by Condition A there exists $C < \infty$ (depending on m_i and ϕ_β) such that

$$\sup_{x_\alpha \in \mathbb{R}^3} \left(\int dy_\alpha \mid G(x_\alpha, y_\alpha) \mid^2 \right)^{\frac{1}{2}} \le C \| f \|_{\hbar_{\beta,-s}}. \tag{3.28}$$

The right-hand side of

$$\| \langle g_\alpha | F \|_{\hbar_\alpha}^2 = \int dy_\alpha \int dx' g_\alpha(x') f_s^{\frac{1}{2}}(y_\alpha) \overline{\tilde{r}_\alpha(\zeta) f_s^{\frac{1}{2}}(\,.\,+t_1 x') G(x',.)}\,.$$

$$.\int dx'' \overline{g}_\alpha(x'') f_s^{\frac{1}{2}}(y_\alpha) \tilde{r}_\alpha(\zeta) f_s^{\frac{1}{2}}(\,.\,+t_1 x'') G(x'',.)$$

is now estimated using Fubini's theorem and Cauchy-Schwarz' inequality as follows:

$$\| \langle g_\alpha | F \|_{\hbar_\alpha}^2 \leq [K(\zeta)]^2 \int dx' \int dx'' \, | g_\alpha(x') g_\alpha(x'') |$$

$$\left(\int dy \, | G(x', y) |^2 \right)^{\frac{1}{2}} \left(\int dy \, | G(x'', y) |^2 \right)^{\frac{1}{2}},$$

where

$$K(\zeta) = \sup_{x \in \mathbb{R}^3} \| f_s^{\frac{1}{2}} \tilde{r}_\alpha(\zeta) f_s^{\frac{1}{2}}(\,.\,+x) \|_{\mathscr{B}(\hbar_\alpha)}\,.$$

We now use (3.28) and find the estimate

$$\| \tilde{r}_\alpha(\zeta) < \overline{\phi}_\alpha | V_\alpha | \phi_\beta > f \|_{\hbar_\alpha,-s}^2 \leq [K(\zeta)]^2 \| g_\alpha \|_{L^1(\mathbb{R}^3)}^2 C^2 \| f \|_{\hbar_\beta,-s}^2\,.$$

Hence $\tilde{r}_\alpha(\zeta) \langle \overline{\phi}_\alpha | V_\alpha | \phi_\beta \rangle \in \mathscr{B}(\hbar_{\beta,-s}, \hbar_{\alpha,-s})$ by Lemma 3.8. Moreover, since by the same Lemma $K(\zeta) \to 0$ for $\zeta \to \infty$ in $S_{-\varepsilon} \backslash \{ \lambda_\alpha + e^{-2i\varphi} \overline{\mathbb{R}^+} \}$, we obtain

$$\| \tilde{r}_\alpha(\zeta) \langle \overline{\phi}_\alpha | V_\alpha | \phi_\beta \rangle \|_{\mathscr{B}(\hbar_{\beta,-s}, \hbar_{\alpha,-s})} \to 0$$

for $\zeta \to \infty$ in $S_{-\varepsilon} \backslash \{ \lambda_\alpha + e^{-2i\varphi} \overline{\mathbb{R}^+} \}$.

The Lemma is proved.

4. UNIFORM ESTIMATES

LEMMA 4.1. — Let I be a closed interval contained in $[-a, a]$. Assume that for every $\varphi \in I$, $V(\varphi)$ has a decomposition

$$V(\varphi) = V_1(\varphi) + V_2(\varphi)$$

such that

$$V_1(\varphi) \in L^p \cap L_s^1, \qquad V_2(\varphi) \in L_s^\infty$$

and

$$V_1(\varphi) \text{ is a continuous map from I into } L^p \text{ and from I into } L_s^1$$

while

$$V_2(\varphi) \text{ is a continuous map from I into } L_s^\infty\,.$$

Assume moreover that for every $\varphi_0 \in I$ there exists $\delta(\varphi_0)$ and $F_{\varphi_0} \in L^p \cap L_s^1$ so that

$$| V_1(\varphi)(x) | \leq | F_{\varphi_0}(x) |, \quad | \varphi - \varphi_0 | \leq \delta(\varphi_0), \qquad x \in \mathbb{R}^3\,.$$

Then for every $\delta > 0$ there exists $C_I(\delta)$ such that for all $\varphi \in I$

$$V(\varphi) = V_1^\delta(\varphi) + V_2^\delta(\varphi)$$

where

$$V_1^\delta(\varphi) \in L^p \cap L_s^1, \qquad \| V_1^\delta(\varphi) \|_{L^p}, \| V_1^\delta(\varphi) \|_{L_s^1} < \delta$$

and

$$V_2^\delta(\varphi) \in L_s^\infty, \qquad \| V_2^\delta(\varphi) \|_{L_s^\infty} < C_I(\delta).$$

Proof. — Let $\varphi_0 \in I$. Then by the proof of Lemma 1.1 there exists N such that

$$\| (1 - \chi_N)V_1(\varphi_0) \|_{L^p} < \delta/2, \qquad \| (1 - \chi_N)V_1(\varphi_0) \|_{L_s^1} < \delta/2$$

where

$$\chi_N(x) = \chi_{\{x \mid |F_{\varphi_0}(x)| \leq N\}} \chi_{\{|x| \leq N\}}$$

By the continuity of $V_1(\varphi)$ and $V_2(\varphi)$ there exists $\eta > 0$ such that for

$$| \varphi - \varphi_0 | \leq \eta$$

$$\| V_1(\varphi) - V_1(\varphi_0) \|_{L^p} < \frac{\delta}{2}, \qquad \| V_1(\varphi) - V_1(\varphi_0) \|_{L_s^1} < \frac{\delta}{2}.$$

Then for $| \varphi - \varphi_0 | \leq \eta$

$$\begin{aligned} \| (1 - \chi_N)V_1(\varphi) \|_{L^p} &\leq \| (1 - \chi_N)(V_1(\varphi) - V_1(\varphi_0) \|_{L^p} + \| (1 - \chi_N)V_1(\varphi_0) \|_{L^p} \\ &\leq \| V_1(\varphi) - V_1(\varphi_0) \|_{L^p} + \| (1 - \chi_N)V_1(\varphi_0) \|_{L^p} < \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

and similarly with L^p replaced by L_s^1. Moreover, for $| \varphi - \varphi_0 | \leq \min \{ \eta, \delta(\varphi_0) \}$

$$\begin{aligned} \| \chi_N V_1(\varphi) + V_2(\varphi) \|_{L_s^\infty} &\leq \| \chi_N V_1(\varphi) \|_{L_s^\infty} + \| V_2(\varphi) \|_{L_s^\infty} \\ &\leq (1 + N)^s N + \max_{|\varphi - \varphi_0| \leq \eta} \| V_2(\varphi) \| \end{aligned}$$

A compactness argument concludes the proof.

THEOREM 4.2. — Let I be a closed interval contained in $(0, a)$ and assume that for some $p > \frac{3}{2}$, $s > 1$ all α and $\varphi \in I$ the functions $V_\alpha(\varphi)$ have decompositions as in Lemma 4.1. Assume moreover, that A is satisfied uniformly for $\varphi \in I$. Then for every $\varepsilon > 0$ the set of resonances $\bigcup_{\varphi \in I} \mathcal{R}_{\varphi, -\varepsilon}$ is bounded.

Proof. — The estimates of $\| A^2(\varphi, \zeta) \|$ given in Lemmas 3.3-3.6 can be obtained uniformly for $\varphi \in I$ by Lemma 4.1, the assumption that A holds uniformly and the fact that $\varepsilon_1 = \varepsilon \sin 2\varphi \geq D_I > 0$ for $\varphi \in I$. We need only replace K of Lemma 3.16 (used in the proof of Lemmas 3.3 and 3.5) by a constant K_I valid for all $\varphi \in I$, $\| V_{\alpha 2}^\delta \|_{\infty, s}$ by $C_I(\delta)$ (Lemma 4.1), k and C by k_I and C_I (Condition A(i)) and ε_1 by D_I in the various estimates. We illustrate this in the case of Lemma 3.16. We first of all note that (3.12) holds uniformly for $\varphi \in I$. This follows from a uniform version of Lemma 2.5, obtained by estimating the first term in (2.1) by $C\max\{\|V\|_{L^p}, \|V\|_{L^1_s}\}$ and by using in (2.3), (2.4) the trivial estimate $\|V_1^{\delta_1}\|_{L^1_{1-s}} \leq \|V_1^{\delta_1}\|_{L^1_s}$ together with the uniform estimates of Lemma 4.1. In (2.5) we use Lemma 4.1. It is easy to see that for every $R > 0$, $q(\varphi, \zeta)$ is jointly continuous in φ and ζ for $\varphi \in I$ and $\zeta \in S_{-\varepsilon}$, $|\zeta| \leq R$. It follows that $\|q(\varphi, \zeta)\| < C$ for $\varphi \in I$, $\zeta \in S_{-\varepsilon}$. Then (3.13) holds for $\varphi \in I$ with C_1 replaced by $C_{1,I}$. By continuity (3.14) holds for $\varphi \in I$ with C_2 replaced by $C_{2,I}$. Also the integrand in (3.15) is continuous in φ, ζ and ζ' for $\varphi \in I$, $\zeta \in D$ and $|\zeta' - \lambda_\alpha| = \frac{3\delta}{2}$. This implies (3.16) for $\varphi \in I$ with C_3 replaced $C_{3,I}$, and we obtain (3.17) uniformly for $\varphi \in I$, from which Lemma 3.16 follows with K replaced by K_I.

We finally discuss the case when the V_α are $\bar{S}_a$-dilation-analytic, $a < \frac{\pi}{2}$. We note the following simple fact.

REMARK 4.3. — Assume that V is $\bar{S}_a$-dilation-analytic and that $V(\varphi)$ has a decomposition as in Lemma 4.1 on $[-a, a]$. Then $V(\varphi)$ is continuous on $[-a, a]$ with values in $\mathscr{C}(H^1, H^{-1})$.

The estimates of $A^2(\zeta)$ are valid for the operator $H(e^{ia})$ as well as for $0 < \varphi < a$. The interval of Theorem 4.2 can then be chosen such that $I \subset (0, a]$. The constant $C_I(\delta)$ of Theorem 4.2 can be chosen accordingly, $\varepsilon \sin 2\varphi \geq C_I > 0$ for $\varphi \in I$, and we obtain the following result.

THEOREM 4.4. — Assume that for all α, V_α has a decomposition as in Lemma 4.1 on $[-a, a]$. Assume moreover that Condition A holds uniformly on $[-a, a]$. Then for every $\varepsilon > 0$ and every closed interval $I \subset (0, a]$ the set of resonances $\bigcup_{\varphi \in I} \mathscr{R}_{\varphi, -\varepsilon}$ is bounded.

REMARK 4.5. — The condition of Lemma 4.1 is satisfied, and V is $\overline{S_a}$-dilation-analytic, if V satisfies the following condition, expressed in polar coordinates.

There exists an $L^\infty(S^2)$-valued function $\tilde{V}(z)$, continuous on $\overline{S_a}$ and analytic in S_a, such that for some $p > \frac{3}{2}$

$$\int_0^1 \sup_{-a \leq \varphi \leq a} \{ \|\tilde{V}(re^{i\varphi})\|^p_{L^\infty(S^2)} \} r^2 dr < \infty ,$$

for some $t > 1$

$$\sup_{-a \leq \varphi \leq a} \ \sup_{1 < r < \infty} \{ \|V(re^{i\varphi})\|_{L^\infty(S^2)} r^t \} < \infty$$

and for $r \in \mathbb{R}^+$

$$V(r, .) = \tilde{V}(r) .$$

In this case $V(\varphi)$, $e^{i\varphi} \in \overline{S_a}$, is given by

$$V(\varphi)(r, .) = \tilde{V}(e^{i\varphi}r), \qquad r \in \mathbb{R}^+ .$$

Moreover

$$V_1(\varphi) \in L^p \cap L^1_s \qquad \text{and} \qquad V_2(\varphi) \in L^\infty_s,$$

where

$$V_1(\varphi)(r, .) = \tilde{V}(e^{i\varphi}r)\chi_{(0,1)}(r) \qquad r \in \mathbb{R}^+$$
$$V_2(\varphi)(r, .) = \tilde{V}(e^{i\varphi}r)\chi_{(1,\infty)}(r) \qquad r \in \mathbb{R}^+, \qquad \text{and} \qquad 1 < s < t .$$

REMARK 4.6. — The statement of Theorem 4.2 can be expressed as follows. For any two consecutive negative thresholds λ_1 and λ_2, consider the sheet F_{λ_1,λ_2} of the Riemann surface of dilation-analytic continuation attached to (λ_1, λ_2) and given by $\mathscr{F}_{\lambda_1,\lambda_2} = \bigcup_{\lambda_1 < \lambda < \lambda_2} \bigcup_{0 < \varphi < a} \{ \lambda + e^{-2i\varphi}\mathbb{R}^+ \}$. Then the set of resonances on $\mathscr{F}_{\lambda_1,\lambda_2}$ is bounded on the subset given by $0 < \delta \leq \varphi \leq a - \delta$ for any $\delta > 0$. Similarly for Theorem 4.4 for $0 < \delta \leq \varphi \leq a$ if $a < \frac{\pi}{2}$.

REMARK 4.7. — Using the well-known fact that $r_0(\zeta)$ has continuous boundary values on $\mathbb{R}^+$ in $(L^2_{\frac{s}{2}}, L^2_{-\frac{s}{2}})$, it is easy to show that $A(\zeta)$ has continuous boundary values $A_\pm(\zeta)$ in $\mathscr{B}(\tilde{\mathscr{H}}_{-s})$ on all non-zero cuts. The estimates of Lemmas 3.3-3.6 are then valid for $A^2(\zeta)$ including the boundary-values in each strip. Singular points of $A_\pm(\zeta)$ are identical with resonances on the respective sides of the cut [4]. Thus Theorem 3.1 extends to include resonances on the cut.

REMARK 4.8. — Replacing L^∞_s by L^∞_σ for some $\sigma > 0$, we can prove $\| A^2(\zeta) \|_{\mathscr{B}(\tilde{\mathscr{H}})} \to 0$ for $\zeta \to \infty$, keeping $\operatorname{dist}(\zeta, \sigma_e(H)) \geq c > 0$. This yields an improvement of the results of [3] in the three-body case allowing $r^{-2+\varepsilon}$-singularities instead of the $r^{-1+\varepsilon}$-singularities of [3].

5. BOUNDEDNESS OF RESONANCES ALONG THE ZERO-CUT

In this section we shall prove that if the pair potentials decrease roughly speaking faster than r^{-2} as $r \to \infty$, then the set of resonances is bounded also along $e^{-2i\varphi}\mathbb{R}^+$. Finally we establish the uniform estimates up to $\varphi = 0$.

DEFINITION. — We define $S_\pm$ and $\mathscr{R}_{\varphi\pm}$ by

$$S_{\underset{(-)}{+}} = \{ \zeta = s + e^{-2i\varphi}t \mid s \underset{(<)}{>} 0, t \in \mathbb{R} \}, \qquad \mathscr{R}_{\varphi\pm} = \mathscr{R}_\varphi \cap S_\pm .$$

THEOREM 5.1. — Assume that $V_\alpha = V_\alpha(\varphi) \in M_s^p$ for some $p > \frac{3}{2}$, $s \dot{>} 2$, $|\varphi| \leq a < \frac{\pi}{2}$ and all α. Suppose that every two-body eigenfunction ϕ_α associated with a negative eigenvalue satisfies A. Assume furthermore that for all α, $\mathcal{N}(1+B_\alpha r_0(0)A_\alpha) = \{0\}$. Then $\mathcal{R}_{\varphi-}$ is bounded.

Assume moreover that the number of two-body resonances in S_+ and the number of positive eigenvalues of h_α are finite for every α, that all two-body resonances in S_+ are simple and that every two-body eigenfunction associated with a positive eigenvalue satisfies A. Assume that there are no two-body resonances on $e^{-2i\varphi}\mathbb{R}^+$. Then $\mathcal{R}_{\varphi+}$ is bounded.

Proof. — This follows in the same way as Theorem 3.1 from the Lemmas 3.3-3.6 with $S_{\pm\varepsilon}$ replaced by $S_\pm$. Lemmas 3.3 and 3.5 are consequences of Lemmas 5.4 and 5.5, proved below.

REMARK 5.2. — $V_\alpha \in M_s^p$ (p, s as above) implies $V_\alpha \in L^p \cap L^q$ for $\frac{3}{s} < q < \frac{3}{2}$. It is well known, that under this assumption the number of negative eigenvalues of each h_α is finite (cf. [*12*], p. 86). Under the slightly stronger assumption that $(1 + |x|)^s V_\alpha \in L^p + L^\infty$ it is proved in [*7*], Prop. (3.5) that the number of positive eigenvalues is finite. Generically the number of resonances (and positive energy bound states) of each two-body problem is finite. This follows from the existence and compactness of $|V_\alpha|^{1/2} r_{\alpha 0}(0) V_\alpha^{1/2} = \lim_{\zeta \to 0} |\, |V_\alpha|^{1/2} r_{\alpha 0}(\zeta) V_\alpha^{1/2}$ in $(L^2(\mathbb{R}^3_{x_\alpha}))$, as noted by A. Jensen [*9*].

LEMMA 5.3. — (Iorio-O'Carroll [*8*], Ginibre-Moulin [*7*]).

(1) $B_\alpha R_0(\zeta) A_\beta$ is bounded and uniformly Hölder-continuous in $\mathcal{B}(\mathcal{H})$ for $\zeta \in \mathbb{C}$, including the boundary values on $e^{-2i\varphi}\overline{\mathbb{R}^+}$.

(2) $f_{s\alpha}^{1/2} R_0(\zeta) A_\beta$ is bounded and uniformly Hölder-continuous in $\mathcal{B}(\mathcal{H})$ for $\zeta \in \mathbb{C}$, where $f_{s\alpha}(x_\alpha, y_\alpha) = f_s(y_\alpha)$.

Proof. — The proof is given in [*7*] for the two-body problem; as indicated there (1) is proved in the same way, and this also holds for (2).

LEMMA 5.4.

$$(1) \qquad \| B_\alpha R_0(\zeta) A_\beta \|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{for} \quad |\zeta| \to \infty$$

$$(2) \qquad \| f_{s\alpha}^{1/2} R_0(\zeta) A_\beta \|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{for} \quad |\zeta| \to \infty$$

Proof. — Let $\varepsilon > 0$ be given. By Lemma 5.3 we can choose $\delta > 0$ such that for $0 \leq s \leq \delta$ and $t \in \mathbb{R}$

$$\| B_\alpha R_0((t - is)e^{-2i\varphi}) A_\beta - B_\alpha R_0((t - i\delta)e^{-2i\varphi}) A_\beta \| < \frac{\varepsilon}{2}$$

By Lemma 3.15 we can then choose $K = K(\delta) > 0$ so that

$$\| B_\alpha R_0((t - iu)e^{-2i\varphi})A_\beta \| < \frac{\varepsilon}{2} \quad \text{for} \quad u \geq \delta \quad \text{and} \quad |t| \geq K,$$

hence

$$\| B_\alpha R_0(\zeta)A_\beta \| < \varepsilon \quad \text{for} \quad \zeta \in S_-, \quad |\zeta| > K,$$

so

$$\| B_\alpha R_0(\zeta)A_\beta \| \to 0 \quad \text{for} \quad |\zeta| \to \infty \quad \text{in} \quad S_-.$$

Similarly we prove this in S_+, and (2) is proved in the same way.

LEMMA 5.5. —

$$\| B_\alpha R_\beta^0(\zeta)A_\beta \| \to 0 \quad \text{for} \quad |\zeta| \to \infty$$

and

$$B_\alpha R_\beta^0(\zeta)A_\beta = B_\alpha R_0(\zeta)(1 - |\phi_\beta\rangle\langle\phi_\beta|)A_\beta - B_\alpha R_0(\zeta)A_\beta \,.\, B_\beta R_\beta^0(\zeta)A_\beta$$

Proof. — This follows from Lemma 5.4 and Lemma 3.16, which can be proved also for $\varepsilon = 0$. Here we use the fact that $q(\zeta)$ has continuous boundary values in $\mathscr{B}(h)$ on $e^{-2i\varphi}\mathbb{R}^+$ and that these boundary values have no singular points on $e^{-2i\varphi}\mathbb{R}^+$. There are no zero-energy resonances by assumption. The fact that there are no singular points on $e^{-2i\varphi}\mathbb{R}^+$ is proved in [2] (note in the case of S_+ that by assumption $e^{-2i\varphi}\mathbb{R}^+$ does not contain two-body resonances).

We now proceed to discuss the extension of the uniform estimates of Section 4 to include the zero channel as well as φ near 0.

THEOREM 5.6. — Assume that $V_\alpha = V_\alpha(\varphi) \in M_s^p$ for some $p > \frac{3}{2}$, $s > 2$, $|\varphi| \leq a$ and all α, and that the conditions of Lemma 4.1 are satisfied for $\varphi \in I = [0, a]$. Suppose that every two-body eigenfunction ϕ_α associated with a negative eigenvalue satisfies A uniformly for $\varphi \in I$. Assume furthermore that for all α, h_α has no zero-energy resonance for any φ (or, equivalently, for one $\varphi \in I$) and no positive eigenvalues. Then $\bigcup_{\varphi \in I} \mathscr{R}_{\varphi-}$ is bounded.

Assume moreover that there exists $b \in (0, a]$ such that there are no two-body resonances for $\varphi \in [0, b]$ (This holds generically). Then $\bigcup_{\varphi \in [0,b]} \mathscr{R}_{\varphi+}$ is bounded.

Proof. — We sketch the proof in the case of $\mathscr{R}_{\varphi-}$, the proof for $\mathscr{R}_{\varphi+}$ is similar. The proof follows very closely that of Theorem 4.2, replacing $S_{-\varepsilon}$ by S_- as in Theorem 5.1. A main point to be elaborated further is the existence and joint continuity of $(1 + q(\varphi, \zeta))^{-1}$ in φ and ζ, uniformly for $\varphi \in I$ and $\zeta \in S_-$ $(= S_-(\varphi))$, where $q(\varphi, \zeta)$ is one of the operators $B_\alpha(\varphi)r_{\alpha 0}(\varphi, \zeta)A_\alpha(\varphi)$. As noted in the proof of Theorem 5.1 the continuous boundary values of $q(\varphi, \zeta)$ on $e^{-2i\varphi}\mathbb{R}^+$ have no singular points. By assumption, $1 + q(\varphi, 0)$ is non-singular. Finally the boundary values of $q(0, \zeta)$ on $\mathbb{R}^+$ have no singular points by the assumption that h_α has no positive eigenvalues. This implies that $(1 + q(\varphi, \zeta))^{-1}$ is continuous in φ and ζ for $\varphi \in \mathrm{I}$, $\zeta \in \mathrm{S}_-$. As in the proof of Theorem 4.2 this is used to show that $\mathrm{B}_\alpha(\varphi)\mathrm{R}_\alpha^0(\varphi, \zeta)\mathrm{A}_\alpha(\varphi)$ is bounded for $\varphi \in \mathrm{I}$, $\zeta \in \mathrm{S}_-$.

Another central points of the proof is to establish

$$\| \mathrm{B}_\alpha(\varphi)\mathrm{R}_0(\varphi, \zeta)\mathrm{A}_\beta(\varphi) \| \to 0 \quad \text{for} \quad | \zeta | \to \infty$$

uniformly for $\varphi \in \mathrm{I}$. This follows as in the proof of Theorem 5.1, using the estimates

$$\| \mathrm{V}_\alpha(\varphi) \|_{\mathrm{L}^p} < \mathrm{C}, \qquad \| \mathrm{V}_\alpha(\varphi) \|_{\mathrm{L}^q} < \mathrm{C} \qquad \text{for} \qquad \varphi \in \mathrm{I},$$

where $\dfrac{3}{s} < q < \dfrac{3}{2}$, leading to the uniform Hölder-continuity in ζ of $\mathrm{B}_\alpha(\varphi)\mathrm{R}_0(\varphi, \zeta)\mathrm{A}_\beta(\varphi)$ for $\varphi \in \mathrm{I}$ and $\zeta \in \mathrm{S}_-$.

A slight adaptation of the remaining part of the proof of Lemma 4.2 suffices to conclude the proof of the fact that $\| \mathrm{A}^2(\varphi, \zeta) \| \to 0$ for $| \zeta | \to \infty$, uniformly for $\varphi \in \mathrm{I}$, from which the Theorem follows.

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