ANNALES DE L'I. H. P., SECTION A

G. ROYER

Study of Dobrushin's critical coupling in rotator models

Annales de l'I. H. P., section A, tome 44, nº 1 (1986), p. 29-38
<http://www.numdam.org/item?id=AIHPA_1986__44_1_29_0>

Ann. Inst. Henri Poincaré,

Vol. 44, n° 1, 1986, p. 29-38. *Physique théorique*

Study of Dobrushin's critical coupling in rotator models

by

G. ROYER

Département de mathématiques appliquées,
Université de Clermont II, B.P. 45, 63170 Aubière, France

ABSTRACT. — We show that Dobrushin's uniqueness theorem yields an $(N(N-3))^{1/2}$ bound for critical interaction in general N components rotator models; related calculations show that at least an $(N(N-1))^{1/2}$ bound may be attainable in a similar way.

RÉSUMÉ. — Nous montrons que le théorème d'unicité de Dobrushin conduit à une majoration par $(N(N-3))^{1/2}$ de l'interaction critique dans les modèles généraux de rotateurs à N composantes ; des calculs voisins indiquent que la borne $(N(N-1))^{1/2}$ devrait pouvoir être atteinte.

INTRODUCTION

In the theory of N components rotator models, the basic estimate of the critical interaction strength J_c is: $J_c \geq N$ (Simon [*16*]. See also [*17*]). Although this result has been improved in particular models by means of hard analysis ([*4*]), it would be interesting to prove it via Dobrushin's uniqueness theorem, since this theorem holds in a more general setting and owns interesting corollaries ([*8*] [*9*]); the present article is an attempt in this direction: first, we obtain $J_c \geq (N(N-3))^{1/2}$ for $N \geq 3$; using a slightly different method, we also obtain

$$J_c \geq (N/2)^{1/2}\left(\sum_{k \geqslant N-1} k^{-2}\right)^{-1/2} \qquad \text{for} \qquad N \geq 3\,;$$

this last estimate is better only for $N = 3, 4$ and bad for high dimension. Nevertheless, we think its proof has a value to suggest how to obtain $J_c \geq (N(N-1))^{1/2}$ and perhaps a little more. These estimates improve on that given by Levin [*10*], namely $N/\sqrt{5}$, for $N \geq 3$.

NOTATION: in this article it is more convenient to use $n = N - 1$ as the dimension parameter.

§ 1. Let us recall Dobrushin's method. We denote by I a countable set (of sites) and by (S, r) a separable metric space; for each finite subset L of I and each configuration $y \in S^{I-L}$, a probability measure $\mu_L(.\mid y)$ on S^L is given and the associated Gibbs measures are those probability measures on S^I which admit, for each L, the kernel μ_L as conditional distribution of the restricted configuration x_L, under the condition $x_{I-L} = y$.

Dobrushin's condition uses the natural distance between probability measures induced by the distance r (Fortet, Kantorovitch, Vasershtein, see [*6*]); strictly speaking, R is defined only on the convex set of probability measures ν on S such that, 0 denoting some fixed point in S, $\int r(0, y)\nu(dy) < \infty$ (a condition which is irrelevant if S is compact) by:

$$R(\nu_1, \nu_2) = \sup\left\{ \int f d\nu_1 - \int f d\nu_2 \, ; \, \mathscr{L}(f) \leqslant 1 \right\}$$

where $\mathscr{L}(f)$ is the Lipschitz norm of the function f:

$$\mathscr{L}(f) = \sup_{\tilde{z} \neq z} (f(\tilde{z}) - f(z))/r(\tilde{z}, z).$$

The dependence coefficient of site i under site j, $\rho(i, j)$, is defined as the supremum of $R(\mu_{\{i\}}(.\mid y), \mu_{\{i\}}(.\mid \tilde{y}))/r(y_j, \tilde{y}_j)$ when the configurations $y, \tilde{y}$ coincide off j; if we put $\rho = \sup_{i \in I} \sum_j \rho(i, j)$, Dobrushin's theorem formulates as follows:

THEOREM 1.1. — *If $\rho < 1$, there exists at most one Gibbs measure γ such that* $\sup_{i \in I} \int r(0, x_i)\gamma(dx) < \infty$.

Here we aim to apply this result to rotator models: S is the unit sphere S^n with its natural (geodesic) distance r and μ_L takes on the following form:

$$\mu_L(dy \mid z) = Z^{-1} \exp(-\mathscr{H}_L(y, z)) \prod_{i \in L} \sigma_n(dy_i),$$

(σ_n denotes the uniform measure on S^n) normalized by Z into a probability measure and

$$\mathscr{H}_L(y, z) = (1/2) \sum_{i,j \in L} J(i, j) y_i . y_j + \sum_{\substack{i \in L \\ k \notin L}} J(i, k) y_i . z_k ,$$

where J is a symmetric function on I × I which vanishes on the diagonal $\left(\text{we may also add an « external field »} \sum_{i \in L} H(i) . y_i \text{ without altering our}\right.$ results). We put $J = \sup_i \sum_j | J(i, j) |$ and define $J_c(n)$ as the supremum of numbers M such that unicity holds in every n-dimensional rotator models such that $J < M$.

For each vector $h \in \mathbb{R}^{n+1}$ let $\nu(h)$ be the probability measure on S^n, the density of which is proportional to $x \to \exp(x . h)$; let us denote by $\operatorname{var}(f, \nu)$ the variance of a function f with respect to a bounded measure ν, i. e., with respect to the proportional probability mesure; our starting point is the following:

LEMMA 1.2. — *Let* $C(n, h)$ *be the best constant in the inequality*

$$\operatorname{var}(f, \nu(h)) \leqslant C(n, h)(\mathscr{L}(f))^2 , \qquad (1)$$

where f *is an arbitrary Lipschitz function on* S^n, *and* $C(n) = \sup_h C(n, h)$. *Then* $J_c \geqslant ((n + 1)/C(n))^{1/2}$.

As indicated in [*10*], we shall use a basic result of Dyson, Lieb and Simon [*15*]:

THEOREM 1.3. — *For any vectors h and z, the function z on* S^n, $y \to z . y$ *satisfies*

$$\operatorname{var}(z ., \nu(h)) \leqslant | z |^2/(n + 1) . \qquad (2)$$

Proof of 1.2. — Let K, z_1, z_2, ζ be respectively a real number, unit vectors and a vector. Let us prove the following inequality:

$$R(\nu(Kz_1 + \zeta), \nu(Kz_2 + \zeta)) \leqslant | K | (C(n)/(n+1))^{1/2} r(z_1, z_2) . \qquad (3)$$

We consider a minimal geodesic $z(t)$ from z_1 to z_2, parametrized by arc-length; a straightforward calculation gives for an arbitrary Lipschitz function f

$$\left| \frac{d}{dt} \int f(y) \nu(Kz(t) + \zeta, dy) \right| = | K \operatorname{cov}(f(y), y . z'(t)) | \qquad (4)$$

(this covariance being calculated with respect to $\nu(Kz(t) + \zeta)$); from the Schwarz inequality, (1), (2) and $| z'(t) | = 1$, we may majorize by $(C(n)/(n + 1))^{1/2} \mathscr{L}(f)$ and integration in t gives (3).

To finish the proof it suffices to remark that the « *a priori* spin distribution » $\mu_{\{i\}}(.\,|\,y)$ is, as a function of y_j, of the form $\nu(\mathrm{K}y_j + \zeta)$ where $\mathrm{K} = -\mathrm{J}(i,j)$ and $\zeta = -\sum_{k \neq i,j} \mathrm{J}(i,k)y_k$; therefore (3) yields

$$\rho(i,j) \leqslant (\mathrm{C}(n)/(n+1))^{1/2}\,|\,\mathrm{J}(i,j)\,| \qquad \text{so} \qquad \rho \leqslant (\mathrm{C}(n)/(n+1))^{1/2}\mathrm{J}$$

and from theorem 1.1 follows uniqueness when $\mathrm{J} < (n+1)^{1/2}(\mathrm{C}(n))^{-1/2}$. ■

§ 2. USE OF INEQUALITIES OF POINCARE'S TYPE

Let us consider the best constant $\chi(n,h)$ in the following inequality:

$$\operatorname{var}(f, \nu(h)) \leqslant \chi(n,h) \int_{\mathrm{S}^n} |\operatorname{grad}(f)|^2 d\nu(h),$$

say for $f \in \mathscr{C}^1(\mathrm{S}^n)$; it is well-known that this constant is the inverse of the least non-zero eigenvalue of the self-adjoint operator of $\mathrm{L}^2(\nu(h))$, $f \to -\Delta f - \operatorname{grad}(\mathrm{U}).\operatorname{grad}(f)$ with $\mathrm{U}(y) = h.y$; in particular, equality in (4) is obtained with at least one non-zero $\mathscr{C}^\infty$ function. For $h = 0$ the operator reduces to Laplacian and $\chi(n,0) = 1/n$. Theorem 3.1 below suggests that $\chi(n,h) \leqslant \chi(n,0)$, an inequality which would yield $\mathrm{J}_c \geqslant (n(n+1))^{1/2}$; an easy way to approach this conjecture is to use the following estimate of Brascamp-Lieb ([*3*]):

THEOREM 2.1. — *If* W *is a strictly convex* $\mathscr{C}^2$ *function on* $]a,b[$ *such that* $\exp(-\mathrm{W}(t))$ *vanishes at a and b, then*

$$\operatorname{var}(f, \exp(-\mathrm{W}(t)dt) \leqslant \mathrm{Z}^{-1} \int_a^b f'^2(t)(\mathrm{W}''(t))^{-1} \exp(-\mathrm{W}(t))dt$$

where
$$\mathrm{Z} = \int_a^b \exp(-\mathrm{W}(t))dt\,.$$

Let us sketch a pedagogical slight variation of the original proof: we may always suppose $\mathrm{Z} = 1$. We have:

$$\operatorname{var}(f) = \int_{a \leqslant x \leqslant y \leqslant b} (f(y) - f(x))^2 \exp -(\mathrm{W}(y) + \mathrm{W}(x))dxdy\,; \qquad (5)$$

Schwarz's inequality gives us the following inequality

$$(f(y) - f(x))^2 \leqslant (\mathrm{W}'(y) - \mathrm{W}'(x)) \int_x^y f'^2(t)/\mathrm{W}''(t)dt\,,$$

which we insert in (5); using Fubini's theorem, it suffices to show:

$$\int_{\substack{y \leqslant t \leqslant b \\ a \leqslant x \leqslant t}} (W'(y) - W'(x)) \exp - (W(y) + W(x)) dx dy = \exp(- W(t))$$

and this is immediate from integration by parts. ■

For $a \in \mathbb{R}$, we denote by $\rho_{a,n}$ or simply ρ_a the probability on $(-\pi/2, \pi/2)$ the density of which is proportional to $\cos^{n-1}(\theta) \exp(a \sin \theta)$.

LEMMA 2.2. — *For $n > 2$, and non constant g,*

$$\operatorname{var}(g, \rho_a) < (n - 2)^{-1} \int g'^2(\theta) \rho_a(d\theta).$$

Proof. — By the change $t = \sin(\theta)$ we have to estimate the variance of $t \to g(\arcsin(t))$ with respect to the measure

$$(1 - t^2)^{n/2 - 1} \exp(at) dt \quad \text{on} \quad]-1, +1[;$$

elementary calculations show that 2.1 applies and yields:

$$\operatorname{var}(g, \rho_a) \leqslant (Z(n - 2))^{-1} \int_{-1}^{+1} g'^2(\arcsin(t)) \frac{(1 - t^2)^{n/2}}{1 + t^2} \exp(at) dt =$$

$$= (n-2)^{-1} \int g'^2(\theta) \cos^2(\theta)(1 + \sin^2(\theta))^{-1} \rho_a(d\theta) < (n-2)^{-1} \int g'^2(\theta) \rho_a(d\theta). \quad ■ \tag{6}$$

THEOREM 2.3. — *For any vector h, if $n > 2$, $\chi(n, h) < (n - 2)^{-1}$.*

Proof. — Let us consider the axis defined by h as a south-north diameter; we denote by θ and φ the corresponding latitude and longitude; $\theta \in [-\pi/2, \pi/2]$; $\varphi \in S^{n-1}$ is defined as follows: we fix a bijection from the parallel S_θ at latitude θ onto $S_0 = S^{n-1}$ by shift along meridians; the distribution law of θ with respect to $\nu(h)$ is ρ_a with $a = |h|$ and the conditional law of φ when θ is fixed is the uniform law σ_{n-1} on S^{n-1}; hence, like in [3], we have:

$$\operatorname{var}(f, \nu(h)) = \operatorname{var}(g, \rho_a) + \int \operatorname{var}(f(\theta, .), \sigma_{n-1}) \rho_a(d\theta) \tag{7}$$

where

$$g(\theta) = \int f(\theta, \varphi) \sigma_{n-1}(d\varphi).$$

We choose for f a non-constant function which realizes equality in (4).

From lemma 2.2 and Poincaré's inequality on S^{n-1}, we get

$$\operatorname{var}(f, \nu_h) \leqslant (n - 2)^{-1} \int g'^2 d\rho_a + (n - 1)^{-1} \int \left| \frac{\partial f}{\partial \varphi} \right|^2 \sigma_{n-1}(d\varphi) \rho_a(d\theta);$$

we use Jensen's inequality in first integral and get

$$< (n-2)^{-1}\int\left(\left(\frac{\partial f}{\partial\theta}\right)^2 + \left|\frac{\partial f}{\partial\varphi}\right|^2\right)dv_h \leqslant (n-2)^{-1}\int |\operatorname{grad}(f)|^2 dv_h$$
$$= (\chi(n,h))^{-1}(n-2)^{-1}\operatorname{var}(f, v_h);$$

we have used strict inequality in 2.2 (and in $(n-2) < (n-1)$) and

$$|\operatorname{grad}(f)|^2 = \left(\frac{\partial f}{\partial\theta}\right)^2 + \cos^{-2}(\theta)\left|\frac{\partial f}{\partial\varphi}\right|^2 \qquad \blacksquare$$

COROLLARY 2.3. — *The critical interaction* J_c *is no less than* $((n+1)(n-2))^{1/2}$.

Proof. — The space of $\mathscr{C}^1$-Lipschitz functions is dense in the space of Lipschitz functions; so $C(n, h) \leqslant \chi(n, h) < (n-2)^{-1}$; thus $C(n) \leqslant (n-2)^{-1}$ and we return to lemma 1.2. $\blacksquare$

Remarks. — 1) $\chi(n, h)$ is a continuous function of h which tends likely to zero when $|h| \to \infty$; the same proof should give strict inequality.

2) Roughly speaking, log-concavity estimates are accurate in high dimensions: this is not surprizing since the measure σ_n behaves then like a Gaussian and the situation becomes similar to that of [*14*].

§ 3. USE OF LIPSCHITZ NORMS

In what follows we shall work with inequalities of the same type as (1). Let us shorten the notation var (Identity, v) into var (v).

THEOREM 3.1. — *The function* $a \to \operatorname{var}(\rho_{a,n})$ *is even and decreasing for* $a \geqslant 0$, *for any real number* $n \geqslant 1$.

The parity statement is obvious; to prove the other let us recall a classical probabilistic inequality (Chebychev, generalized by Preston [*13*] for F. K. G. inequalities).

LEMMA 3.2. — *Let* ρ_1 *and* ρ_2 *be two probability measures on an interval* I *with positive densities* g_1, g_2; *in order that* ρ_2 *may be stochastically greater than* ρ_1 $\left(\textit{i. e.} \int f d\rho_1 < \int f d\rho_2 \textit{ for any increasing function } f\right)$ *it is sufficient that*:

$$g_1(t)/g_1(s) < g_2(t)/g_2(s) \qquad \textit{for} \qquad \textit{any} \qquad s < t.$$

Proof of theorem 3.1. — We have

$$\operatorname{var}(\rho_a)=1/2\int(x-y)^2\rho_a(dx)\rho_a(dy)$$
$$=(2Z_a)^{-2}\int(x-y)^2\cos^{n-1}(x)\cos^{n-1}(y)\exp(a(\sin(x)+\sin y))dxdy.$$

Put $(y-x)/2=u(y+x)/2=v$; then

$$\operatorname{var}(\rho_a)=4Z_a^{-2}\int_{-\pi/2}^{\pi/2}du\int_{|u|-\pi/2}^{\pi/2-|u|}u^2(\cos(2u)+\cos(2v))^{n-1}\exp(2a\cos(u)\sin(v))dv.$$

By using symmetries, if $t=(\pi/2)-u$ for $u>0$, we get

$$\operatorname{var}(\nu_a)=16\,Z_a^{-2}\int_0^{\pi/2}(t-\pi/2)^2 g_a(t)dt \quad \text{where:}$$
$$g_a(t)=\int_0^t(\cos(2v)-\cos(2t))^{n-1}\cosh(2a\sin(t)\sin(v))dv.$$

By lemma 3.2, since $(t-\pi/2)^2$ decreases we only need to show that for $0\leqslant a,\ s\leqslant t,\ a\to g_a(t)/g_a(s)$ is increasing; it is sufficient to prove

$$\mathrm{I}:=g_a'(t)g_a(s)-g_a'(s)g_a(t)>0,\qquad \text{for}\qquad a>0.$$

Let us write I as a double integral:

$$\mathrm{I}=2\int_{\substack{0\leqslant v\leqslant t\\ 0\leqslant w\leqslant s}}\varphi(v,w)\Psi(v,w)dvdw$$

where

$$\varphi(v,w)=((\cos(2v)-\cos(2t))(\cos(2w)-\cos(2s)))^{n-1},$$
$$\Psi(v,w)=\sin(v)\sin(t)\sinh(2a\sin(t)\sin(v))\cosh(2a\sin(s)\sin(w))$$
$$-\sin(w)\sin(s)\sinh(2a\sin(s)\sin(w))\cosh(2a\sin(t)\sin(v)).$$

We remark that $\Psi(v,w)>0$ if $w<v$, since tanh is increasing; hence the part of I coming from the domain $s\leqslant v\leqslant t, 0\leqslant w\leqslant s$ is positive; the other part may be written:

$$\tilde{\mathrm{I}}=2\int_{0\leqslant w\leqslant v\leqslant s}(\varphi(v,w)\Psi(v,w)+\varphi(w,v)\Psi(w,v))dv.$$

In the new domain of integration, $\Psi(v, w) \geqslant 0$; on the other hand $\varphi(v, w) \geqslant \varphi(w, v)$; indeed a simple derivative calculation implies:

$$(\cos(2v) - \cos(2t))/(\cos(2w) - \cos(2t)) \geqslant$$
$$\geqslant (\cos(2w) - \cos(2t))/(\cos(2w) - \cos(2s))$$

for $0 \leqslant w \leqslant v \leqslant s$, $0 \leqslant s \leqslant t \leqslant \pi/2$ and we recall that $n - 1 \geqslant 0$.

Therefore to finish the proof it is sufficient to show $\Psi(v, w) + \Psi(w, v) \geqslant 0$; denoting $2\sin(s)$, $2\sin(t)$, $\sin(w)$, $\sin(v)$ respectively by x, y, X, Y, we have $0 \leqslant x \leqslant y$, $0 \leqslant \mathrm{X} \leqslant \mathrm{Y}$ and we want:

$$y\mathrm{Y}\sinh(ay\mathrm{Y})\cosh(ax\mathrm{X}) - x\mathrm{X}\sinh(ax\mathrm{X})\cosh(ay\mathrm{Y}) +$$
$$+ y\mathrm{X}\sinh(ay\mathrm{X})\cosh(ax\mathrm{Y}) - x\mathrm{Y}\sinh(ax\mathrm{Y})\cosh(ay\mathrm{X}) \geqslant 0.$$

We expand in powers of a; the coefficient of a^m cancels for m even and, for m odd, we get up to a positive factor, owing to the formula

$$2\sinh(u)\cosh(v) = \sinh(u + v) + \sinh(u - v):$$
$$(x\mathrm{X} + y\mathrm{Y})^m(y\mathrm{Y} - x\mathrm{X}) + (y\mathrm{Y} - x\mathrm{X})^m(y\mathrm{Y} + x\mathrm{X})$$
$$+ (y\mathrm{X} + x\mathrm{Y})^m(y\mathrm{X} - x\mathrm{Y}) + (y\mathrm{X} - x\mathrm{Y})^m(y\mathrm{X} + x\mathrm{Y}).$$

This coefficient is non-negative: if we set $\mathrm{A} = \mathrm{Y}/\mathrm{X}$, $\alpha = y/x$, we get up to a positive factor:

$$(\mathrm{A}\alpha - 1)(\mathrm{A}\alpha + 1)[(\mathrm{A}\alpha - 1)^{m-1} + (\mathrm{A}\alpha + 1)^{m-1}]$$
$$- (\mathrm{A} - \alpha)(\mathrm{A} + \alpha)\ [(\mathrm{A} - \alpha)^{m-1} + (\mathrm{A} + \alpha)^{m-1}];$$

if $\mathrm{A} \leqslant \alpha$ all terms are non-negative; if not, we note that $\mathrm{A}\alpha + 1 \geqslant \mathrm{A} + \alpha$ and $\mathrm{A}\alpha - 1 \geqslant \mathrm{A} - \alpha$ from $\mathrm{A} \geqslant 1$, $\alpha \geqslant 1$. ■

Remark. — Theorem 3.1 is also valid for $0 < n \leqslant 1$; similar calculations have to be done with the new integration variable $z = v/t$.

So the maximum of var $(\rho_{a,n})$ is var $(\rho_{0,n})$ and we call it v_n.

COROLLARY 3.3. — *For any Lipschitz function g on $(-\pi/2, \pi/2)$* var $(g, \rho_{a,n}) \leqslant v_n(\mathscr{L}(g))^2$ *and v_n is the best constant in this inequality.*

Proof. — We write one more time var (g, ρ_a) as a double integral to get

$$\operatorname{var}(g, \rho_a) \leqslant \operatorname{var}(\mathrm{Id}, \rho_a)(\mathscr{L}(g))^2 \leqslant v_n(\mathscr{L}(g))^2. \quad ■$$

Now we return to the sphere; let us recall the definition of, say, « the re-arrangement $\tilde{f}$ of a function f, increasing with respect to latitude »: once a north pole on S^n has been chosen, $\tilde{f}$ is characterized by two properties (except sometimes on negligible sets):

1) $\tilde{f}$ depends only on latitude θ and is non-decreasing
2) $\tilde{f}$ has the same distribution function as f: $\sigma(\tilde{f} > t) = \sigma(f > t)$.

THEOREM 3.4. — *If f is Lipschitzian so is $\tilde{f}$ and $\mathscr{L}(\tilde{f}) \leqslant \mathscr{L}(f)$; similarly $f \to \tilde{f}$ is a contraction for Dirichlet's integral.*

The first statement is a Corollary to lemma 1 in [*2*]; to get the second we may use theorem 2 in [*2*] as in [*12*]; see also [*7*] and its bibliography.

COROLLARY 3.5. — *We have* $C(n, 0) = v_n$ *and* $v_n \leqslant 1/n$.

Proof. — By re-arrangement, var (f, σ) is not modified and the Lipschitz norm decreases; we may consider $\tilde{f}$ as a function on $(-\pi/2, \pi/2)$ and Corollary 3.3 gives $C(n, 0) = v_n$. Poincaré's inequality on S^n yields $v_n \leqslant 1/n$. ■

Actually v_n can be calculated explicitly:

$$v_n = \left(\int_{-\pi/2}^{\pi/2} \theta^2 \cos^{n-1}(\theta) d\theta\right)\left(\int_{=\pi/2}^{\pi/2} \cos^{n-1}(\theta) d\theta\right)^{-1}$$

hence:

$$v_n = v_{n-2} - 2(n-1)^2,$$

$$v_{2p+1} = \pi^2/12 - 2\sum_{n=1}^{p}(2n)^{-2}$$

$$v_{2p} = \pi^2/4 - 2\sum_{n=1}^{p}(2n-1)^{-2};$$

finally, we find

$$v_n + v_{n-1} = 2\sum_{k \geqslant n} k^{-2}.$$

THEOREM 3.6. — *For* $n > 1$, $C(n) < v_n + v_{n-1}$ *and therefore*

$$J_c > (n+1)^{1/2}\left(2\sum_{k \geqslant n} k^{-2}\right)^{-1/2}$$

Proof. — From compactness with respect to point-wise convergence of $\{ f ; \mathscr{L}(f) \leqslant 1 \text{ and } f(0) = 1 \}$, follows the existence of a function f which realizes equality in relation (1) of lemma 1.2 and such that $\mathscr{L}(f) = 1$. Again we use relation (7) above; obviously we also have $\mathscr{L}(g) \leqslant 1$; on the other hand, the Lipschitz norm of $\varphi \to f(\theta, \varphi)$ is not greater than $\cos(\theta)$. So from (7), 3.3, 3.5 we get

$$\operatorname{var}(f, \nu(X)) \leqslant v_n + v_{n-1} \int_{-\pi/2}^{\pi/2} \cos^2(\theta) \rho_a(d\theta).$$

An application of lemma 3.2 shows that the last integral is maximum for $a = 0$, in which case its value is

$$\Gamma^2((n+1)/2)\Gamma^{-1}(n/2)\Gamma^{-1}(n/2+1)$$

which is less than 1 since Γ is log-convex. ■

Finally, by analogy with 3.1, one might think that $C(n, h) \leqslant C(n, 0)$; this conjecture would give $J_c \geqslant (n+1)^{1/2} v_n^{-1/2}$ and should be true if equality in (1) is realizable with a function depending only on latitude (defined with respect to h).

NUMERICAL VALUES $\quad A = (n+1)^{1/2}(v_n + v_{n-1})^{-1/2}$, $\quad B = (n+1)^{1/2}(n-2)^{1/2}$

$n = N - 1$	2	3	4	5	10	$\to \infty$
A	1.52	2.25	2.96	3.68	7.23	$\sim n/\sqrt{2}$
B	0	2	3.16	4.24	9.38	$\sim n$
v_n^{-1}	1.21	2.13	4.07	5.06	10.03	$\sim n$

ACKNOWLEDGMENTS

I should like to thank R. Berthuet and F. Dunlop for helpful conversations.

REFERENCES

[1] M. B. AIZENMAN, B. SIMON, *Commun. Math. Phys.*, t. **77**, 1980, p. 137-143.
[2] A. BAERNSTEIN, B. A. TAYLOR, *Duke Math. Journal*, t. **43**², 1976, p. 245-269.
[3] H. BRASCAMP, E. H. LIEB, *J. Funct. Anal.*, t. **22**, 1976, p. 366-389.
[4] O. BRYDGES, J. FRÖHLICH, T. SPENCER, *Commun. Math. Phys.*, t. **83**, 1982, p. 123-150.
[5] R. L. DOBRUSHIN, *Theor. Prob. Appl.*, t. **15**³, p. 1970, p. 458.
[6] R. M. DUDLEY, Probabilites and metrics, *Lecture Notes*, Series 45, Aarhus Universität.
[7] A. EHRHARD, *Ann. Scient. Ec. Norm. Sup.*, t. **17**, 1984, p. 317-322.
[8] H. FÖLLMER, *J. Funct. Anal.*, t. **46**, 1982, p. 387-395.
[9] H. KÜNSCH, *Commun. Math. Phys.*, t. **84**, 1982, p. 207-222.
[10] S. L. LEVIN, *Commun. Math. Phys.*, t. **781**, 1980, p. 65-74.
[11] E. H. LIEB, *Commun. Math. Phys.*, t. **77**, 1980, p. 127-135.
[12] C. MUELLER, E. WEISSLER, *J. Funct. Anal.*, t. **48**, 1982, p. 252-283.
[13] R. HOLLEY, *Commun. Math. Phys.*, t. **38**, 1974, p. 227-231.
[14] G. ROYER, *J. Math. Pures et Appl.*, t. **58**, 1977, p. 455-478.
[15] F. J. DYSON, E. H. LIEB, B. SIMON, *J. Stat. Phys.*, t. **18**, 1978, p. 335-383.
[16] B. SIMON, *J. Stat. Phys.*, t. **22**⁴, 1980, p. 491-492.
[17] W. DRIESSLER, L. LANDAU, J. FERNANDO-PEREZ, *J. Stat. Phys.*, t. **20**, 1979, p. 123.

(Manuscrit reçu le 6 juin 1985)