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Global existence theorems for hyperbolic harmonic maps

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INTRODUCTION

Harmonic maps between properly riemannian manifolds have a long mathematical history, with global results of great interest in geometry and physics (cf. [*1*] [*2*] [*16*]). The harmonic maps with source a riemannian manifold of hyperbolic (lorentzian) signature have been studied more recently, however such maps appear in numerous problems of physics, from the harmonic gauge of General Relativity to the non-linear σ models and many others, as pointed out in [*15*] [*17*] and [*14*].

The natural problem for such « hyperbolic harmonic maps » is the Cauchy problem, that is the determination of the map from its value, and the value of its first derivative, on a space like submanifold of the source. A local existence theorem of a solution of the problem for harmonic maps from M^{n+1}, Minkowski space time of arbitrary dimension, into various compact riemannian manifold has been proved by Ginibre and Velo [*10*], using I. Segal's theory of non linear semi-group [*12*]. They also prove for such problems a global existence theorem when $n = 1$, by using energy estimates. A global existence from M^2 into a complete riemannian has been proved, by a different method, for smooth data by Gu Chao Hao [*11*].

In this article we prove a local existence theorem for harmonic maps from a globally hyperbolic manifold (M, g) into a riemannian manifold (N, h) both arbitrary except for some regularity conditions. The proof uses the embedding of (N, h) in an euclidean space ($\mathbb{R}^q$, e), like Ginibre and Velo [*10*]. Another proof, which used only the standard local existence and uniqueness results for hyperbolic equations had been indicated in [*3*], which treated harmonic gauges in General Relativity.

We prove in § 5 a global existence theorem in the case $n = 1$, using the second order equation satisfied by the differential of an harmonic mapping and, like Ginibre and Velo, the resulting *a priori* estimates.

In § 6 we prove a global existence theorem for $(M, g) = M^{n+1}$, with n odd, arbitrary if the Cauchy data are sufficiently near from those of a constant map. We use the method of conformal transformation as in [*6*] [*7*] and [*8*]. The theorem is valid for $n = 1$, because of conformal invariance, and for $n = 3$ because the operator satisfies an analogue of the condition indicated as sufficient by Christodoulou in [*8*] (which treats scalar systems).

It results from counter examples constructed by Shatah [*9*] that this last theorem cannot be true for arbitrary, large, data.

1. DÉFINITIONS

Let (M, g) and (N, h) be two smooth riemannian manifolds of arbitrary signature and dimensions. Let

$$f : M \to N$$

be a smooth map. The *differential* of f at $x \in M$ is a linear map

$$\nabla f(x) : T_x M \to T_{f(x)} N$$

it is therefore an element of $T_x^* M \otimes T_{f(x)} N$. The differential itself, ∇f is a mapping $x \to \nabla f(x)$, that is a section of the vector bundle with base M and fiber at x the vector space $T_x^* M \otimes T_{f(x)} N$. This vector bundle—one forms on M with values at x in $T_{f(x)} N$—is denoted $T^*M \otimes f^{-1}N$. The vector bundle with base M and fiber $T_{f(x)}N$ at x is denoted $f^{-1}TN$. If (x^α) and (y^a) are respectively local coordinates in M and N, and f is represented in these coordinates by

$$y^a = f^a(x^\alpha)$$

the derivative f is represented by

$$(x^\alpha) \to \left(\frac{\partial f^a}{\partial x^\beta}(x^\alpha)\right).$$

The metrics g on M and h on N endow the fiber at x of the vector bundle $E = T^*M \otimes f^{-1}TN$ with a scalar product $G(x) = g^\#(x) \otimes h(f(x))$, where $g^\#$ is the contravariant tensor canonically associated with g. In coordinates, if u and v are two sections of E :

$$G(x)(u, v) = g^{\alpha\beta}(x^\lambda)h_{ab}(f^c(x^\lambda))u^a_\alpha(x^\lambda)v^b_\beta(x^\lambda)\,. \tag{1}$$

The vector bundle $E \equiv T^*M \otimes f^{-1}TN$ is endowed with a linear connection ∇, mapping sections of E into sections of $T^*M \otimes E$, by the usual rules: if s is a section of $f^{-1}TN$ and t a section of T^*M we have:

$$\nabla_v(t \otimes s) = {}^g\nabla_v t \otimes s + t \otimes f^{*h}\nabla_v s$$

with ${}^g\nabla$ and ${}^h\nabla$ the riemannian covariant derivatives in the metrics g and h respectively. In local coordinates if $(x^\alpha) \mapsto (u^a(x^\alpha))$ is a section of E, we have:

$$\nabla_\alpha u^a_\beta(x^\lambda) = \partial_\alpha u^a_\beta(x^\lambda) + \frac{\partial f^b}{\partial x^\alpha}\Gamma_b{}^a{}_c(f^d(x^\lambda))u^c_\beta(x^\lambda) - \Gamma_\alpha{}^\mu{}_\beta(x^\lambda)u^a_\mu(x^\lambda) \tag{2}$$

where $\Gamma_b{}^a{}_c$ and $\Gamma_\alpha{}^\lambda{}_\beta$ denote respectively the riemannian connexions of g and h. The mapping f is called *harmonic* if

$$\mathrm{tr}_g \nabla^2 f = 0 \tag{3}$$

that is, in local coordinates

$$g^{\alpha\beta}\nabla_\alpha\partial_\beta f^a \equiv g^{\alpha\beta}(\partial^2_{\alpha\beta}f^a - \Gamma_\alpha{}^\lambda{}_\beta\partial_\lambda f^a + \Gamma_b{}^a{}_c\partial_\alpha f^b\partial_\beta f^c) = 0\,.$$

If f satisfies (3) it is a critical point of the functional

$$f \mapsto \mathrm{E}(f) = \int_{\mathrm{M}} \mathrm{G}(\nabla f, \nabla f)d\mu(g) = \int_{\mathrm{M}} g^{\alpha\beta}(x^\lambda)h_{ab}(f^c(x^\lambda))\partial_\alpha f^a\partial_\beta f^b d\mu(g).$$

2. HYPERBOLIC HARMONIC MAPS. ENERGY INTEGRAL

When the metrics g and h are properly riemannian the integral (4) is called the energy of the mapping f. When g is of hyperbolic signature we will define another integral as the energy of f, like for usual wave equations.

We define the stress energy tensor of the map f as the covariant 2-tensor on M given by

$$\mathrm{T} = (h \circ f)(\nabla f, \nabla f) - \frac{1}{2}g(g^\# \otimes h \circ f)(\nabla f \otimes \nabla f)$$

that is

$$\mathrm{T}_{\alpha\beta} = (h_{ab} \circ f)\partial_\alpha f^a\partial_\beta f^b - \frac{1}{2}g_{\alpha\beta}g^{\lambda\mu}(h_{ab} \circ f)\partial_\lambda f^a\partial_\lambda f^b \tag{2.1}$$

We have on M

$$\nabla_\alpha \mathrm{T}^\alpha{}_\beta \equiv (h_{ab} \circ f)\partial_\beta f^a\nabla^\alpha\partial_\alpha f^b \tag{2.2}$$

that is

$$\nabla.\mathrm{T} = (h \circ f)(\nabla f, \mathrm{tr}_g\nabla^2 f)\,.$$

Thus $\nabla.\mathrm{T} = 0$ if f is a harmonic map.

We suppose that (M, g) is a hyperbolic manifold with $\mathrm{M} = \mathrm{S} \times \mathbb{R}$, $\mathrm{S}_t \equiv \mathrm{S} \times \{t\}$ space-like, we denote by n their unit time like normal. Let X be a time like vector field. We define the energy density of f relative to S_t and X by:

$$e(f) = \mathrm{T}(\mathrm{X}, n) = \mathrm{X}^\alpha n^\beta \mathrm{T}_{\alpha\beta}$$

we have

$$e(f) = \frac{1}{2}\gamma^{\alpha\beta}(h_{ab} \circ f)\partial_\alpha f^a\partial_\beta f^b = \frac{1}{2}\gamma^\# \otimes (h \circ f)(\nabla f, \nabla f) \tag{2.3}$$

where $\gamma^{\#}$ is the quadratic form

$$\gamma^{\alpha\beta} = n^{\alpha}X^{\beta} + n^{\beta}X^{\alpha} - g^{\alpha\beta}X^{\lambda}n_{\lambda}.$$

It is well known that this form is positive definite if g of hyperbolic signature $(+ - - \ldots)$ with X and n time like.

We deduce from

$$\nabla_{\alpha}T^{\alpha\beta} = 0$$

when f is a harmonic map that

$$\nabla_{\alpha}(X_{\beta}T^{\alpha\beta}) = \frac{1}{2}T^{\alpha\beta}(\nabla_{\alpha}X_{\beta} + \nabla_{\beta}X_{\alpha}). \tag{2.4}$$

By integration of 2-4 on $S \times [0, t]$ we obtain the following:

PROPOSITION. — Let f be a smooth map such that $\nabla f\,|_{S_t}$ has a compact support for $0 \leqq \tau \leqq t$ then, if f is harmonic it satisfies the identity:

$$\int_{S_t} Ne(f)d\mu_t \equiv \int_{S_0} Ne(f)d\mu_0 + \frac{1}{2}\int_0^t\int_{S_\tau} N(T.LX)d\mu_\tau \tag{2.5}$$

$d\mu_t$ denotes the volume element of the metric $\bar{g}_t$ induced on S_t by g, N is the lapse function, $N = g(X, n)$, that is $N = (g^{00})^{-1/2}$ if X is the tangent vector to the curves $\{x\} \times \mathbb{R}$ and the coordinates are adapted to the product $S \times \mathbb{R}$: the volume element of (M, g) is

$$d\mu(g) = N d\mu_{x^0} dx^0.$$

DEFINITION 1. — The manifold (M, g) is said *regularly hyperbolic* if

1) M and g are smooth and $M = S \times \mathbb{R}$, the metrics $\bar{g}_t$ induced on $S_t = S \times \{t\}$ are (properly) riemannian ([1]), and uniformly equivalent to the metric $\bar{g}_0$ which is complete.

2) The tangent vector X to the lines $\{x\} \times \mathbb{R}$ is time like, and there exists strictly positive numbers a and b such that

$$\inf_M g(X, X) \geqq a \geqq 0 \qquad \text{and} \qquad \operatorname{Sup}_M N \leqq b$$

we then have also, since $N = g(X, n) \geqq (g(X, X))^{1/2}$

$$\inf_M N \geqq a^{1/2} \qquad \text{and} \qquad \operatorname{Sup}_M g(X, X) \leqq b^2.$$

If (M, g) is regularly hyperbolic the metric γ on $M = S \times \mathbb{R}$ defined by 2-3 is uniformly equivalent to the metric $\Gamma = (dx^0)^2 - \bar{g}_0$.

We shall then add to the definition of regular hyperbolicity the following.

3) The riemann curvature of g, together with as many of its covariant derivatives as is relevant, is bounded in Γ-norm.

([1]) These metrics are negative definite: $-\bar{g}_t$ is positive definite.

DEFINITION 2. — A (properly) riemannian manifold is said to be regular if it is smooth, has a non zero injectivity radius (thus is complete), and has a bounded riemannian curvature, as well as its covariant derivatives up to the relevant order.

Tensor products of the metrics Γ and h give scalar products and norms in the fiber at $x \in \mathrm{M}$ of vector bundles $(\otimes \mathrm{T}_*\mathrm{M})^p (\otimes f^{-1}\mathrm{TN})^q$ or their duals. We denote this norm by $|\ \ |$. We have, for instance,

$$|\nabla f|^2 = \Gamma^{\alpha\beta}(h_{ab} \circ f)\partial_\alpha f^a \partial_\beta f^b .$$

If s and u are two sections of such vector bundles we have, at a point $x \in \mathrm{M}$

$$|s \otimes u| = |s|\,|u|, \qquad |s.u| \leqq |s|\,|u|$$

if $s.u$ is some contracted tensor product

Therefore, in particular

$$|\mathrm{LX}.\mathrm{T}| \leqq |\mathrm{LX}|\,|\mathrm{T}| .$$

It results from the definition that $|g|$ and $|g^{\#}|$ are uniformly bounded if (M, g) is regularly hyperbolic (cf. [5]). Thus, due to the expression of T, there exists a constant C such that

$$|\mathrm{T}| \leqq \mathrm{C}e(f) \tag{2.6}$$

and also, if $|\mathrm{X}|$ is uniformly bounded on M, a constant still denoted C such that

$$|\mathrm{LX}.\mathrm{T}| \leqq \mathrm{C}e(f) .$$

From the equality (2.5) results then the inequality (C_0 and C positive constants)

$$y(t) \leqq \mathrm{C}_0 y(0) + \mathrm{C}\int_0^t y(\tau)d\tau \tag{2.7}$$

with

$$y(t) = \int_{\mathrm{S}_t} |\nabla f|^2 d\mu_0 .$$

If y is a continuous function of t we deduce from (2.7), by the Gromwall lemma

$$y(t) \leqq \mathrm{K}(t)y(0) \tag{2.8}$$

with K(t) the continuous function of t

$$\mathrm{K}(t) = \mathrm{C}_0 e^{\mathrm{C}t} .$$

3. SECOND ORDER EQUATION FOR f

PROPOSITION. — Every smooth harmonic map f: (M, g) $\rightarrow$ (N, h) satisfies the equation

$$(\nabla.\nabla)(\nabla f) - \mathrm{Ricc}(g)\nabla f + \mathrm{tr}_g(f^* \mathrm{Riem}(h).\nabla f) = 0$$

that is, in local coordinates

$$(3.1)\qquad \nabla^{\lambda}\nabla_{\lambda}\partial_{\alpha}f^{a} - R^{\beta}_{\alpha}\partial_{\beta}f^{a} + R_{cd}{}^{a}{}_{b}\partial_{\alpha}f^{c}\partial_{\beta}f^{d}\partial^{\beta}f^{b} = 0 .$$

The proof, independant of signature, is straightforward and given in [1]. We now consider the case g hyperbolic and h properly riemannian. We set (h_{ab} stands always for $h_{ab} \circ f$)

$$(3.2)\qquad T^{(1)}_{\alpha\beta} = e^{\lambda\mu}h_{ab}\left\{ \nabla_{\alpha}\partial_{\lambda}f^{a}\nabla_{\beta}\partial_{\mu}f^{b} - \frac{1}{2}g_{\alpha\beta}g^{\rho\sigma}\nabla_{\rho}\partial_{\lambda}f^{a}\nabla_{\sigma}\partial_{\mu}f^{b} \right\}$$

We have identically, after use of the Ricci identity

$$(3.3)\qquad \begin{aligned} \nabla_{\alpha}T^{\alpha\beta}_{(1)} \equiv\ & e^{\lambda\mu}h_{ab}\nabla^{\alpha}\nabla_{\alpha}\partial_{\lambda}f^{a}\nabla_{\beta}\partial_{\mu}f^{b} \\ & + e^{\lambda\mu}h_{ab}\nabla^{\alpha}\partial_{\lambda}f^{a}(-R^{\beta\rho}_{\alpha}{}_{\mu}\partial_{\rho}f^{b} + \partial_{\alpha}f^{c}\partial^{\beta}f^{d}\partial_{\mu}f^{e}R_{cd}{}^{b}{}_{e}) \\ & + h_{ab}(\nabla^{\alpha}e^{\lambda\mu})\left\{ \nabla_{\alpha}\partial_{\lambda}f^{a}\nabla^{\beta}\partial_{\mu}f^{b} - \frac{1}{2}g_{\alpha}{}^{\beta}g^{\rho\sigma}\nabla_{\rho}\partial_{\lambda}f^{a}\nabla_{\sigma}\partial_{\mu}f^{b} \right\}. \end{aligned}$$

Using (3.1) we see that, for a harmonic map, $\nabla_{\alpha}T^{\alpha\beta}_{(1)}$ is a polynomial $Q(f, \nabla f, \nabla^{2} f)$ of degree 2 in $\nabla^{2} f$, with coefficients of degree 1 or 3 [respectively 0] in ∇f for the terms of degree 1 [respectively 2] in $\nabla^{2} f$. The mapping f itself appears through $h \circ f$ and Riem $(h) \circ f$.

On the other hand we have:

$$e_{1,f} \equiv T^{(1)}_{\alpha\beta}X^{\alpha}n^{\beta} = \frac{1}{2}\gamma^{\alpha\beta}\gamma^{\lambda\mu}h_{ab}\nabla_{\alpha}\partial_{\lambda}f^{a}\nabla_{\beta}\partial_{\mu}f^{b} \geqq 0 .$$

Integrating the identity:

$$\nabla_{\alpha}(X_{\beta}T^{\alpha\beta}_{(1)}) \equiv X_{\beta}\nabla_{\alpha}T^{\alpha\beta}_{(1)} + \frac{1}{2}T^{\alpha\beta}_{(1)}(\nabla_{\alpha}X_{\beta} + \nabla_{\beta}X_{\alpha})$$

with $T^{(1)}_{\alpha\beta}$ and $\nabla_{\alpha}T^{\alpha\beta}_{(1)}$ given by (3.2) and (3.3) gives for a harmonic map, with compact support in space, using (3.1), an equality:

$$(3.4)\qquad \int_{S_t} Ne_{1,f}d\mu(\bar{g}_t) = \int_{S_0} Ne_{1,f}d\mu(\bar{g}_0) + \int_0^t \int_{S_\tau} NQ_1(f, \nabla f, \nabla^2 f)d\mu(\bar{g}_\tau)d\tau$$

with Q_1 of the type

$$Q_1(f, \nabla f, \nabla^2 f) \equiv \Sigma k\nabla^2 f.\{ \text{Riem}(g).\nabla f + \text{Riem}(h) \circ f.(\otimes \nabla f)^3 + \nabla^2 f \}$$

with k polynomial in $g^{\#}$, h, X and ∇X.

If (M, g) is regularly hyperbolic, and (N, h) regularly riemannian we deduce from (3.4) an inequality, as in § 2; with C_0, C_1, C_2, C_3 positive constants:

$$(3.5)\qquad y_1(t) \leqq C_0 y_1(0) + C_1 \int_0^t y_1(\tau)d\tau + \int_0^t \int_{S_\tau} (C_2 | \nabla^2 f | \, | \nabla f | + C_3 | \nabla^2 f | \, | \nabla f |^3)d\mu_0 d\tau$$

where, by definition

$$y_1(t) = \int_{S_t} |\nabla^2 f|^2 d\mu_0$$

and

$$|\nabla^2 f|^2 = e^{\lambda\mu} e^{\alpha\beta} (h_{ab} \circ f) \nabla_\alpha \partial_\lambda f^a \nabla_\beta \partial_\mu f^b .$$

4. LOCAL EXISTENCE

Let N be a submanifold of the riemannian manifold (Q, q) and h be the metric induced on N by q. We shall suppose that N is defined by p equations

$$\mathrm{N} : \Phi^{\mathrm{I}}(z) = 0, \qquad z \in \mathrm{Q}, \qquad \mathrm{I} = 1, \ldots, p$$

where $\phi = (\Phi^{\mathrm{I}}) : \mathrm{Q} \to \mathbb{R}^p$ is a smooth map of rank p at each point of N. The matrix $m = (m^{\mathrm{IJ}})$ given by

$$m = q(\nabla\phi, \nabla\phi), \qquad \text{i. e.} \quad m^{\mathrm{IJ}} = (q^{\mathrm{AB}} \partial_{\mathrm{A}} \phi^{\mathrm{I}} \partial_{\mathrm{B}} \phi^{\mathrm{J}}) \circ f$$

(x^{A} coordinates in Q) is then positive definite on M when f takes its values in N. We denote by $m^{-1} = m_{\mathrm{IJ}}$ the inverse matrix.

LEMMA 1. — A necessary and sufficient condition for the mapping f: $\mathrm{M} \to \mathrm{N} \subset \mathrm{Q}$ to be a harmonic map from (M, g) into (N, h) is that, as a mapping $\mathrm{M} \to \mathrm{Q}$ it satisfies the equations which read in local coordinates x^α in M and x^{A} in Q:

$$\hat{\nabla}^\alpha \nabla_\alpha f^{\mathrm{A}} + \lambda_{\mathrm{I}} (q^{\mathrm{AB}} \partial_{\mathrm{B}} \Phi^{\mathrm{I}}) \circ f = 0 \tag{4.1}$$

where $(\hat{\nabla}^\alpha \nabla_\alpha f^{\mathrm{A}})$ is the tension field of the map $f : (\mathrm{M}, g) \to (\mathrm{Q}, q)$ and

$$\lambda_{\mathrm{I}} = m_{\mathrm{IJ}} g^{\alpha\beta} \partial_\alpha f^{\mathrm{A}} \partial_\beta f^{\mathrm{B}} (\hat{\nabla}_{\mathrm{B}} \partial_{\mathrm{A}} \phi^{\mathrm{J}}) \circ f \tag{4.2}$$

together with the conditions

$$\phi \circ f = 0 .$$

Proof (cf. a particular case in Ginibre and Velo [*10*]).

A mapping $f : \mathrm{M} \to \mathrm{N}$ defines a mapping $\mathrm{F} : \mathrm{M} \to \mathrm{Q}$ by

$$\mathrm{F} = i \circ f \tag{4.3}$$

where i denotes the embedding (identity map) $\mathrm{N} \to \mathrm{Q}$.

The integral constructed with $\mathrm{F} : \mathrm{M} \to \mathrm{Q}$

$$\mathrm{E(F)} = \int_{\mathrm{M}} (g^{\#} \otimes q)(\nabla\mathrm{F}, \nabla\mathrm{F}) d\mu(g)$$

is equal to the integral (1.4) constructed with f since

$$\nabla\mathrm{F} = \nabla i . \nabla f, \qquad \text{i. e.} \quad \partial_\alpha \mathrm{F}^{\mathrm{A}} = \partial_a i^{\mathrm{A}} \partial_\alpha f^a$$

and h is the metric induced by i on N; that is $h_{ab} = q_{AB}\partial_a i^A \partial_b i^B$, and

$$\text{(4.4)} \qquad E(f) = E(F).$$

Thus a critical point of $E(f)$ is a critical point of E(F) with the constraint $\phi(F) = 0$, that is a solution of equations of the form:

$$\text{(4.5)} \qquad \hat{\nabla}^\alpha \nabla_\alpha F^A + \lambda_I q^{AB} \partial_B \Phi^I = 0$$

where the λ_I (Lagrange multipliers) are determined by derivating twice the conditions

$$\Phi^I \circ F = 0$$

with $\Phi^I \circ F$ considered as a mapping $M \to Q \to \mathbb{R}^p$, and contracting with g:

$$\text{(4.6)} \qquad \nabla^\alpha \nabla_\alpha (\Phi^I \circ f) \equiv \partial_A \Phi^I \hat{\nabla}^\alpha \partial_\alpha F^A + g^{\alpha\beta} \partial_\alpha F^A \partial_\beta F^B \nabla_B \partial_A \Phi^I = 0$$

comparing (4.5) and (4.6) gives

$$\text{(4.7)} \qquad \lambda_I q^{AB} \partial_B \Phi^I = g^{\alpha\beta} \partial_\alpha F^A \partial_\beta F^B \nabla_B \partial_A \Phi^I$$

(4.7) is equivalent to (4.2) ($F = i \circ f = f$, since i is the identity mapping $N \to N \subset Q$.

LEMMA 2. — The equations (4.1) satisfied by a mapping $M \to Q$, with λ_I given by (4.2), imply that the mapping $\phi \circ f : M \to \mathbb{R}^p$ satisfies the homogeneous wave equation on M

$$\text{(4.8)} \qquad \nabla^\alpha \nabla_\alpha (\phi \circ f) = 0 .$$

Proof. — (4.6) implied by (4.1) and (4.2).

DEFINITION 3. — A submanifold N of $\mathbb{R}^q$ given by $N = \{ y \in \mathbb{R}^q, \phi(y) = 0 \}$ with ϕ a smooth map $\mathbb{R}^q \to \mathbb{R}^p$ is said to be regularly defined by ϕ if there exists $a > 0$ and $\varepsilon > 0$ such that

$$\inf_{y \in N_\varepsilon} | \det m^{IJ}(y) | \geqq a > 0, \qquad N_\varepsilon = \{ y \in \mathbb{R}^q, d(y, N) < \varepsilon \}$$

d denotes the euclidean distance. The definition means that ϕ is uniformly of rank p in some uniform neighbourhood of N.

DEFINITION 4. — We denote by $H_s(S)$ the Sobolev space of $\mathbb{R}^q$-valued functions on the regularly riemannian manifold $(S, \bar{g}_0)$, closure of C^∞, $\mathbb{R}^q$ valued functions with compact support on S in the norm:

$$\| \varphi \|^2_{H_s} = \int_S \sum_{k=0}^{s} | D^k \varphi |^2 d\mu_0$$

where D is the covariant derivative in the metric $\bar{g}_0$ for each scalar valued map $\varphi^A : S \to \mathbb{R}$, and $|\ |_e$ is the $\bar{g}_0$ and e norm of the set $D^k\varphi = (D^k\varphi^A)$, for instance

$$|D^2\varphi|_e^2 = e_{AB}\bar{g}_0^{ij}\bar{g}_0^{hk}D_{ih}^2\varphi^A D_{jk}^2\varphi^B, \qquad e_{AB} = \delta_{AB}.$$

THEOREM (local existence). — Let (M, g), $M = S \times \mathbb{R}$, be a regularly hyperbolic manifold of dimension $n + 1$ (definition 1).

Let (N, h) be a regular riemannian manifold, regularly defined by a mapping $\phi : \mathbb{R}^q \to \mathbb{R}^p$. Let $\varphi, \dot{\varphi}$ be mappings $S \to \mathbb{R}^q$, $\varphi \in H_s(S)$, $\dot{\varphi} \in H_{s-1}(S)$, $s > \frac{n}{2} + 1$ such that $\Phi \circ \varphi = 0$, $(\nabla\phi \circ \varphi).\dot{\varphi} = 0$.

Then there exists $l > 0$ and on $S \times (-l, l)$ a harmonic map f: $(M, g) \to (N, h)$, with h the metric induced on N by the euclidean metric e of $\mathbb{R}^q$, such that

$$f|_{S_0} = \varphi, \qquad \partial_0 f|_{S_0} = \dot{\varphi}. \tag{4.9}$$

Proof. — We apply lemma 1 with $(Q, q) = (\mathbb{R}^q, e)$.
Equations (4.1) reads then:

$$\nabla^\alpha\nabla_\alpha f^A + (\nabla\phi^T\nabla\phi)_{IJ}^{-1}(f)g^{\alpha\beta}\partial_\alpha f^A\partial_\beta f^B(\partial_{AB}^2\Phi^J)(f) = 0; \tag{4.10}$$

they are a system of q, numerical, quasi-linear, quasi-diagonal second order hyperbolic equations on M, with smooth coefficients if $d(f, N) < \varepsilon$. The local existence theorem, on $S \times (-l, l)$ is a standard result, since $d(\varphi, N = 0$. The solution $f : S \times (-l, l) \to \mathbb{R}^q$ satisfies $\phi \circ f = 0$ because $\phi \circ f$ satisfies the homogeneous wave equation (4.8) with zero Cauchy data:

$$\phi \circ f|_S = \phi \circ \varphi|_{=0}, \qquad \partial_0(\phi \circ f)|_S = (\nabla\phi \circ f).\partial_0 f|_S = (\nabla\phi \circ \varphi).\dot{\varphi} = 0.$$

REMARK 1. — The local existence theorem for a numerical hyperbolic system gives that the interval of existence depends continuously on the $H_{s_0} \times H_{s_0-1}$ norm, s_0 smallest integer such that $s_0 > \frac{n}{2} + 1$, of the Cauchy data, and tends to infinity when these norms tend to zero. If N is a submanifold of $\mathbb{R}^q$, we can always, by translation, take the origin of $\mathbb{R}^q$ at some arbitrary given point y_0 of N. The $H_s(S_0)$ norm of a map $\varphi : S_0 \to \mathbb{R}^q$ is by definition

$$\|\varphi\|_s = \left\{\sum_{k=0}^{s}\int_S |\nabla^k\varphi|^2 d\mu_0\right\}^{1/2}$$

with

$$|\varphi|^2 = \sum_{A=1}^{q} |\varphi^A|^2 .$$

Small H_s norm for φ means then nearness of φ from the constant map $M \to y_0$.

REMARK 2. — Every riemannian manifold (N, h) can be isometrically embedded in a space $(\mathbb{R}^q, e)$-we have supposed moreover that N is given by equations $\Phi^I = 0$. We could have proceeded without this hypothesis, either inspired by techniques used by Eells and Sampson in the elliptic case, either by using atlases on M and N, together with local existence and uniqueness theorems (cf. an indication of such a proof in [*3*]).

5. GLOBAL EXISTENCE WHEN $n = 1$

The global existence of a solution of the Cauchy problem for harmonic maps from 2-dimensional Minkowski space time M^2 into a complete riemannian manifold (N, h) has been proved by Gu Chao Hao [*11*] for smooth initial data. It has been proved by Ginibre and Velo [*10*] from the two dimensional Minkowski space into various compact riemannian manifolds for $H_2 \times H_1$ Cauchy data. This result can be generalized:

THEOREM. — Let (M, g), $M = S \times \mathbb{R}$, be a regularly hyperbolic manifold of dimension 2, and (N, h) be a regular riemannian manifold, regularly defined as a submanifold of $(\mathbb{R}^q, e)$ by mapping $\phi : \mathbb{R}^q \to \mathbb{R}^p$, $N = \{ y \in \mathbb{R}^q, \phi(y) = 0. \}$

Let

$$\varphi \in H_s(S), \qquad \dot{\varphi} \in H_{s-1}(S), \qquad s \geqq 2$$

be given maps $\varphi : S \to \mathbb{R}^q$, $\dot{\varphi} : S \to \mathbb{R}^q$, such that $\phi \circ \varphi = 0$, $(\nabla\phi \circ \varphi).\dot{\varphi} = 0$. Then there exists on M a harmonic map $f : (M, g) \to (N, h)$ taking on $S_0 = S \times \{0\}$ these Cauchy data.

Proof. — In the case $n = 1$ the local existence theorem is valid with $s \geqq 2$. The solution $f : S \times (-l, l) \to N \subset \mathbb{R}^q$ admits a restriction on each $S_t = S \times \{t\}$, $|t| < l$ which is a mapping $f_t : S_t \to N \subset \mathbb{R}^q$ which belongs to $H_2(S)$ (definition 4). The derivative $\partial_0 f$ admits a restriction $(\partial_0 f)_t$: $S_t \to f^{-1}TN$ by $x \to T_{f_t(x)}N \subset \mathbb{R}^q$, $(\partial_0 f)_t \in H_1(S)$. The energy inequality (2.9) implies, when $f = i \circ f$ is considered as a mapping into N :

$$y(t) \leqq K(t)y(0) \tag{5.1}$$

with

$$y(t) = \int_S |\nabla f|_h^2 d\mu_0$$

with, due to the definition of Γ:

$$(5.2) \qquad |\nabla f|_h^2 = (h_{ab} \circ f)(\partial_0 f^a \partial_0 f^b - \bar{g}_0^{ij} \partial_i f^a \partial_j f^b)$$

but we have, since $\underline{f} = i \circ f$ and $h = i^* e$

$$(5.3) \qquad |\nabla f|_h^2 = |\nabla \underline{f}|_e^2 = e_{AB}(\partial_0 \underline{f}^A \partial_0 \underline{f}^B - \bar{g}_0^{ij} \partial_i \underline{f}^A \partial_j \underline{f}^B).$$

The inequality (2.9) implies therefore the non-blow up of the $L^2(S, \bar{g}_0)$ norm of $D\underline{f}_t$ and of $(\partial_0 \underline{f})_t$, as mappings $S \to \mathbb{R}^q$, and thus also of $\underline{f}_t$: the norms $\| \underline{f} \|_{H_1(S)}$ (and thus $\| \underline{f}_t \|_{C_b^0(S)}$) are bounded by continuous functions of t which extend to $t = +\infty$. To prove the non blow up of the second derivatives we look at the identity (3.4). Due to the regularity hypothesis we have, with C a constant

$$(5.4) \qquad \int_{S_t} Q_1 d\mu_t \leqq C \int_{S_t} |\nabla^2 f|_h (|\nabla f|_h + |\nabla f|_h^3 + |\nabla^2 f|_h) d\mu_0$$

using the fact that $\underline{f} = i \circ f$ we find inequalities of the form (C_1 and C_2 positive constants)

$$(5.5) \qquad \begin{aligned} |\nabla^2 f|_h^2 &\leqq |\hat{\nabla}^2 \underline{f}|_e^2 + C_1 |\nabla f|^4 \\ |\hat{\nabla}^2 \underline{f}|_e^2 &\leqq |\nabla^2 f|_h^2 + C_2 |\nabla f|^4 \end{aligned}$$

(recall that $\hat{\nabla}$ denotes the covariant derivative of $\underline{f}$ as a mapping $(M, g) \to (\mathbb{R}^q, e)$ and ∇ as mapping $M \to N$).

We deduce from (5.4), by the Cauchy-Schwartz inequality

$$(5.6) \qquad \int_{S_t} Q_1 d\mu_t \leqq C \int_{S_t} |\nabla^2 f|_h^2 d\mu_0 + \left\{ \int_{S_t} |\nabla^2 f|_h^2 d\mu_0 \right\}^{1/2} \left\{ \left(\int_{S_t} |\nabla f|^2 d\mu_0 \right)^{1/2} + \left(\int_{S_t} |\nabla f|^6 d\mu_0 \right)^{1/2} \right\}.$$

Considering $\underline{f}$ as mapping $M \to \mathbb{R}^q$, and setting

$$z(t) = \int_{S_t} |\hat{\nabla}^2 \underline{f}|_e^2 d\mu_0$$

we obtain, using (3.5) (5.5) and (5.6), with C some constant

$$z(t) \leqq C_0 y_1(0) + C \int_0^t \left\{ z(\tau) + \int_{S_\tau} (|\nabla f|^2 + |\nabla f|^4 + |\nabla f|^6) d\mu_0 \right\} d\tau.$$

By the inequality (2.9) we know that

$$\int_{S_\tau} |\nabla f|^2 d\mu_0 = y(t) \leqq K(\tau) y(0).$$

We bound the integrals $\int_{S_\tau} |\nabla f|^4 d\mu_0$ and $\int_{S_\tau} |\nabla f|^6 d\mu_0$ by using the following Sobolev inequality, valid in any dimension if $a = \frac{n(p-1)}{p}$ and S admits a uniformly locally finite atlas:

$$\| u \|_{L^p(S)} \leqq C \| Du \|^{2}_{L^1(S)} \| u \|^{1-a}_{L^1(S)}$$

by taking

$$u = |\nabla f|^2_t.$$

If $n = 1$, $a = \frac{p-1}{p}$, $p = 1$, 2 or 3 we have therefore:

$$\int_{S_t} |\nabla f|^{2p} d\mu_0 = (\| u \|_{L^p})^p \leqq C \| Du \|^{(p-1)/p}_{L^1(S)} \| u \|^{1/p}_{L^1(S)}$$

we have

$$\| u \|_{L^1(S)} = y(t)$$

and

$$\| Du \|^2_{L^1(S)} \leqq C y(t) z(t).$$

We then obtain an integral inequality for $z(t)$, with coefficients continuous functions of t extendable for all t, and at most of degree one in $z(t)$. The non-blow up of $z(t)$ follows.

6. GLOBAL EXISTENCE WHEN $(M, g) = M^{n+1}$, SMALL DATA

Met $M^{n+1} = (\mathbb{R}^{n+1}, \eta)$ be $n + 1$ dimensional Minkowski space time. Let $\Sigma^{n+1} = (S^n \times \mathbb{R}, g)$ be the Einstein cylinder with its canonical metric. M^{n+1} is known to be conformal to a subset V of Σ^{n+1}, that is:

(6.1) $$g = \Omega^2 \eta \qquad \text{on} \qquad V = \mathbb{R}^{n+1}$$

the identification of $V \subset \Sigma^{n+1}$ with $\mathbb{R}^{n+1}$ being given in canonical polar coordinates respectively $(t, r, \ldots)$ on $\mathbb{R}^{n+1}$ and $(T, \alpha, \ldots)$ on Σ^{n+1} by (cf. [7])

(6.2)
$$T = \operatorname{Arctg}(t + r) + \operatorname{Arctg}(t - r)$$
$$\alpha = \operatorname{Arctg}(t + r) - \operatorname{Arctg}(t - r)$$
$$V : \alpha - \Pi < T < \Pi - \alpha$$

We have

(6.3) $g = dT^2 - d\alpha^2 - \sin^2 \alpha d\sigma^2 = \Omega^2\eta = \Omega^2(dt^2 - dr^2 - r^2 g_{S^{n-1}})$
$$\Omega = \cos T + \cos$$

$g_{S^{n-1}}$ metric of the sphere S^{n-1}.

We remark that Ω extends to an analytic function on Σ^{n+1}, which vanishes on ∂V; the submanifold $t = 0$ (i. e. $\mathbb{R}^n \times \{0\}$) is mapped diffeomorphically onto $T = 0$ (i. e. $S^n \times \{0\}$), minus its north pole $\alpha = \Pi$. On $S^n \times \{0\}$ we have ($\alpha = \Pi$ is $r = +\infty$)

$$\Omega|_{T=0} = 1 + \cos\alpha = 2(1 + r^2)^{-1}.$$

To a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ corresponds, by the previous diffeomorphism a mapping, still denoted φ, defined almost everywhere on S^n.

THEOREM. — Let (N, h) be a riemannian submanifold of ($\mathbb{R}^q$, e) regularly defined by a smooth map $\phi : \mathbb{R}^q \to \mathbb{R}^p$

$$N = \{ y \in \mathbb{R}^q, \phi(y) = 0 \}.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ and $\dot\varphi : \mathbb{R}^n \to \mathbb{R}^q$ be given mappings such that

(6.4) $$\phi \circ \varphi = 0, \qquad (\nabla\phi \circ \varphi).\dot\varphi = 0$$

with

$$(1+\cos\alpha)\varphi \in H_s(S^n), \qquad (1+\cos\alpha)\dot\varphi \in H_{s-1}(S^n), \qquad s > \frac{n}{2} + 1$$

then there exists a harmonic map $f : M^{n+1} \to (N, h)$ such that

$$f|_{\mathbb{R}^n} = \varphi, \qquad \partial_0 f|_{\mathbb{R}^n} = \dot\varphi$$

if the mappings φ and $\dot\varphi$ are sufficiently near in the relevant norms respectively from a constant map and zero.

Proof. — It is inspired from the proofs of [*7*] and [*8*].

We do not restrict the generality by supposing that N passes through the origin of $\mathbb{R}^q$, that is $\phi(0) = 0$.

To a mapping $f : \mathbb{R}^{n+1} \to \mathbb{R}^q$ we associate $\tilde f : \mathbb{R}^{n+1} \to \mathbb{R}^q$ defined on V by:

(6.5) $$\tilde f = \Omega^{(1-n)/2} f.$$

We deduce then from (6.1), since the scalar curvature of Σ^{n+1} is $n(n-1)$ that

(6.6) $$\Box_g \tilde f - \frac{(n-1)^2}{4} \tilde f = \Omega^{-(3+n/2)} \Box_\eta f$$

where $\square_g$ and $\square_\eta$ are the wave operators in the metrics g and η respectively. We see that f satisfies (4.1) (4.2), with $(Q, q) = (\mathbb{R}^q, e)$ that is

$$(6.6\,a) \qquad \square_\eta f^A + \lambda_I[(\partial_A \phi^I) \circ f] = 0\,,$$

$$(6.7\,b) \qquad \phi(f) = 0$$

with

$$\lambda_I = m_{IJ}(f)\eta^{\alpha\beta}\partial_\alpha f^A \partial_\beta f^B[(\partial^2_{AB}\phi^I) \circ f)], \qquad m^{IJ} = \sum_{A=1}^{q} (\partial_A \phi^I \partial_B \phi^J) \circ f$$

if and only if

$$(6.8) \qquad \square_g \tilde{f} - \frac{(n-1)^2}{4} \tilde{f} + \Omega^{-(3+n/2)}\lambda_I[(\partial_A \phi^I) \circ \Omega^{(n-1/2}\tilde{f}] = 0$$

with

$$\lambda_I = m_{IJ}\Omega^2 g^{\alpha\beta}\partial_\alpha(\Omega^{(n-1)/2}\tilde{f}^A)\partial_\beta(\Omega^{(n-1)/2}\tilde{f}^B)[(\partial^2_{AB}\phi^I) \circ (\Omega^{(n-1)/2}\tilde{f})]$$

and

$$\phi(\Omega^{(n-1)/2}\tilde{f}) = 0\,.$$

We have

$$\partial_\alpha(\Omega^{(n-1)/2}\tilde{f}^A) = \Omega^{(n-1)/2}\partial_\alpha \tilde{f}^A + (n-1)/2\Omega^{(n-3)/2}\partial_\alpha\Omega \tilde{f}^A$$

$\nabla\Omega$ extends to a bounded function on Σ^{n+1}, and so does $\Omega^{-1}g(\nabla\Omega, \nabla\Omega)$ (cf. [*8*]) since

$$g(\nabla\Omega, \nabla\Omega) = g^{\alpha\beta}\partial_\alpha\Omega\partial_\beta\Omega = \cos^2\alpha - \cos^2 T = \Omega(\cos\alpha - \cos T)\,.$$

Therefore the equation (6.8) extends to a semi-linear, semi-diagonal second order system with smooth coefficients for a mapping $\tilde{f}$ from an open set U of Σ^{n+1} into $\mathbb{R}^p$ if on the one hand n is odd and if, on the other hand, $\tilde{f}$ is such that

$$d((\Omega^{(n-1)/2}\tilde{f})(X), N) < \eta \qquad \forall X \in U$$

this last property will be *a fortiori* satisfied since $0 \in N$ if

$$\sup_{X\in U} |\,\Omega^{(n-1)/2}(X)\tilde{f}(X)\,|_e < \eta$$

thus if

$$\sup_{X\in U} |\,\tilde{f}(X)\,|_e < \eta\,.$$

The existence of an open set $U = S^n \times (-l, l)$ where the equation (6.8) has a solution $\tilde{f}$ taking the Cauchy data:

$$\tilde{f}\,|_{S^n\times\{0\}} = (1 + \cos\alpha)^{(1-n)/2}\varphi$$

$$\partial_0 \tilde{f}\,|_{S^n\times\{0\}} = (1 + \cos\alpha)^{-(1+n)/2}\dot{\varphi}$$

is then a consequence of the local existence theorem, and Sobolev inequalities, if $s > \frac{n}{2} + 1$. The length depends continuously on the norms of the Cauchy data, and we have $l > \Pi$ if these norms are small enough.

The mapping $f = \Omega^{(n-1)/2} f$ is defined on M^{n+1}, satisfies (4.1), and also (4.2) (lemma 2).

REMARK. — The hypothesis (6.4) on φ implies that φ tends to the constant map $\mathbb{R}^n \to 0 \in N$, when r tends to infinity.

From the theorem follow decay estimates for f on M^{n+1} (i. e. rate of approximating the constant map $M^{n+1} \to 0 \in N$) when t or r tend to infinity.

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