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Partial *-algebras of closed operators and their commutants
I. General structure

by

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ABSTRACT. — Let $\mathscr{D}$ be a dense domain in a Hilbert space and $\mathfrak{M}$ a collection of closed operators defined on $\mathscr{D}$, together with their adjoints, and having $\mathscr{D}$ as a common core. We say that $\mathfrak{M}$ is a partial Op*-algebra on $\mathscr{D}$ if it is stable under suitable operations of involution, addition and (partially defined) multiplication. In this paper, the first of two, we introduce two classes of such objects, hereby generalizing previous results by W. Karwowski and one of us (JPA). We discuss their algebraic properties and their extensions by continuity to larger domains, and we describe various locally convex topologies that can be defined on them. The second paper will be devoted to commutants and bicommutants of partial Op*-algebras.

RÉSUMÉ. — Soient $\mathscr{D}$ un domaine dense dans un espace de Hilbert et $\mathfrak{M}$ une famille d'opérateurs fermés définis sur $\mathscr{D}$, en même temps que leurs adjoints, et ayant $\mathscr{D}$ pour cœur commun. La famille $\mathfrak{M}$ est appelée Op*-algèbre partielle sur $\mathscr{D}$ si elle est stable sous des opérations appropriées d'involution, d'addition et de multiplication (partiellement définie). Dans cet article (le premier de deux), on introduit deux classes de tels objets, généralisant des résultats antérieurs de W. Karwowski et l'un des auteurs (JPA). On discute leurs propriétés algébriques et leur extension par continuité à des domaines plus grands et on décrit différentes topologies localement convexes dont on peut les munir. La deuxième partie du travail sera consacrée aux commutants et bicommutants des Op*-algèbres partielles.

1. INTRODUCTION

In the algebraic approach to quantum theory, whose origins go back to Heisenberg's matrix mechanics, the basic object is the C*-algebra of observables, and symmetries are realized by automorphisms. Each state determines, through the familiar Gel'fand-Naimark-Segal construction, a representation of the algebra by *bounded* operators in a Hilbert space. A useful tool then is the von Neumann algebra generated by the representation. This language is by now standard e. g. in quantum statistical mechanics (see the monograph of Bratteli and Robinson [*1*] for a review).

Now many authors consider this framework as too narrow. On one hand, there are systems, typically spin systems with long range interaction, where the thermodynamical limit fails to exist in a C*-topology [*2*]. On the other hand, unbounded operators may be more natural: take, for instance, symmetry generators or boson field operators. Therefore structures more general than (normed) algebras of bounded operators have been proposed.

First appeared algebras of unbounded operators, originally introduced through the example of the field algebra [*3-5*], and subsequently developed into a full-fledged theory under the name of *Op*-algebras* [*6-8*]. Among these, a subclass that seems especially suited to the description of quantum observables « à la Dirac » is that of *V*-algebras* [*9*] [*10*]. In all cases the key point is that all the operators in the algebra are defined on the same dense domain and leave it invariant.

However, in several situations, obtaining a common invariant domain is difficult or unwieldy, sometimes impossible. Think, for instance, of a non-relativistic Hamiltonian $H = -\Delta + V(x)$, with V a non-smooth potential: Schwartz space $\mathcal{S}$ is in general contained in the domain of H, but it is not invariant under it; yet $\mathcal{S}$ is a very natural domain to use. Another example is a recent result of Horuzhy and Voronin [*12*] concerning a Wightman field theory: when the latter is formulated, as usual, in terms of the Op*-algebra of polynomials in the basic fields, on the Gårding domain $\mathcal{D}$, the von Neumann field algebras, both local and global, do not in general leave $\mathcal{D}$ invariant. Now, if we require only that our unbounded operators have a common dense domain, but do not necessarily leave it invariant, we cannot multiply them freely: the resulting structure is no longer an algebra, but only a *partial *-algebra*. This concept has been introduced and studied systematically by W. Karwowski and one of us (JPA) in a series of papers [*13-15*]. The aim of this work is to continue that analysis.

In the present paper, the first of two, we will study partial *-algebras of closed operators, both at the algebraic and topological level. In Part II, we will analyze systematically the various notions of commutants and bicommutants arising in that context, thus generalizing several results obtained in [*11*] and [*16*] for Op*-algebras. This may prove important e. g. for the representation theory of our objects (see [*15*] for a first discussion in that direction) and various aspects of the theory, familiar for W*-algebras [*1*]. We will follow mostly the terminology and notation of [*14*], except for some changes that have proven to be necessary; but the two papers are essentially self-contained.

Part I is organized as follows. In Section 2 we begin by a discussion of abstract partial *-algebras, with some emphasis on the problem of (non) associativity of a multiplication that is only partially defined. Section 3 introduces the main object of study, that we call *partial Op*-algebras*. To be more specific, let $\mathcal{H}$ be a Hilbert space, $\mathcal{D}$ a dense subspace of $\mathcal{H}$ and consider the following set of closed operators (A^* denotes the adjoint of A and D(A) its domain):

$$\mathfrak{C}(\mathcal{D}) = \{ A \text{ closed} \mid \mathcal{D} \subset D(A) \cap D(A^*), \mathcal{D} \text{ is core for } A \}. \qquad (1.1)$$

It turns out that *two* different structures of partial *-algebra may be defined on subsets of $\mathfrak{C}(\mathcal{D})$, leading respectively, to strong and weak partial Op*-algebras. We analyze their algebraic properties and their mutual relations. Section 4 is devoted to natural extensions of a partial Op*-algebra to larger domains. Finally, in Section 5, we describe various topologies that may be defined on partial *-algebras and will be needed in Part II. We conclude, in the Appendix, by describing a class of pathological examples, extending a construction due to Kürsten [*17*].

Throughout the realization of this work, we have benefited from discussions with W. Karwowski, J. Shabani, G. Epifanio and C. Trapani, and also from private communications from K-D. Kürsten and G. Lassner. It is a pleasure to thank them all.

2. ABSTRACT PARTIAL *-ALGEBRAS

2.A. General definitions.

The definition of an abstract partial *-algebra is due originally to Borchers [*18*]. We reproduce it for convenience.

DEFINITION 2.1. — A *partial *-algebra* is a (complex) vector space $\mathfrak{A}$, with an antilinear involution $x \mapsto x^+$ and a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ such that:

i) $(x, y) \in \Gamma$ iff $(y^+, x^+) \in \Gamma$;
ii) if $(x, y) \in \Gamma$ and $(x, z) \in \Gamma$, then $(x, \lambda y + \mu z) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$;

iii) whenever $(x, y) \in \Gamma$, there exists an element $x \circ y \in \mathfrak{A}$ with the usual properties of the product:

$$x \circ (y + \lambda z) = (x \circ y) + \lambda(x \circ z)$$
$$(x \circ y)^+ = y^+ \circ x^+ .$$

Notice that we do *not* assume the $\circ$ product to be associative (see § 2.B below).

Since not every product is defined in a partial *-algebra, it is natural to consider the set of elements that can multiply a given element, from the left or from the right. Similarly, for any subset $\mathfrak{N} \subset \mathfrak{A}$, we define the set of its *left*, resp. *right multipliers*:

$$\mathrm{L}\mathfrak{N} = \{ x \in \mathfrak{A} \mid (x, y) \in \Gamma \quad \text{for all} \quad y \in \mathfrak{N} \}$$
$$\mathrm{R}\mathfrak{N} = \{ x \in \mathfrak{A} \mid (y, x) \in \Gamma \quad \text{for all} \quad y \in \mathfrak{N} \} .$$

In particular, for single elements:

$$\mathrm{L}(z) \equiv \mathrm{L}\{z\}, \qquad \mathrm{R}(z) \equiv \mathrm{R}\{z\} .$$

This suggests to use a simpler notation:

$$(x, y) \in \Gamma \Leftrightarrow x \in \mathrm{L}(y) \Leftrightarrow y \in \mathrm{R}(x) .$$

The sets of multipliers $\mathrm{L}\mathfrak{N}$ and $\mathrm{R}\mathfrak{N}$ are vector subspaces of $\mathfrak{A}$. Also $x \in \mathrm{L}\mathfrak{N}$ iff $x^+ \in \mathrm{R}\mathfrak{N}^+$. Let now $\mathfrak{N}$ run over all subsets of $\mathfrak{A}$. Then the set of all spaces of multipliers exhibits a remarkable lattice structure [*13*] [*14*], due to the fact that the maps $\mathrm{L}: \mathfrak{N} \mapsto \mathrm{L}\mathfrak{N}$ and $\mathrm{R}: \mathfrak{N} \mapsto \mathrm{R}\mathfrak{N}$ form a Galois connection. The smallest of such spaces are $\mathrm{L}\mathfrak{A}$ and $\mathrm{R}\mathfrak{A}$, which are interchanged under the *involution*: $x \in \mathrm{L}\mathfrak{A}$ iff $x^+ \in \mathrm{R}\mathfrak{A}$. Their elements, the so-called *universal multipliers*, will play a crucial role in the sequel.

The element $e \in \mathfrak{A}$ is called a *unit* if $e^+ = e$, and for every $x \in \mathfrak{A}$ one has $e \in \mathrm{L}(x)$ and $e \circ x = x \circ e = x$. In this work we will consider only partial *-algebras with unit. This is in fact not a limitation, since every partial *-algebra $\mathfrak{A}$ without unit can be embedded in a larger one $\mathfrak{A}_e$ which has a unit, exactly as for *-algebras [*1*]. The extended partial *-algebra $\mathfrak{A}_e$ is defined as the set of pairs (x, α), $x \in \mathfrak{A}$, $\alpha \in \mathbb{C}$, with the following rules:

. vector space structure:

$$\lambda(x, \alpha) + \mu(y, \beta) = (\lambda x + \mu y, \lambda\alpha + \mu\beta)$$

. involution:

$$(x, \alpha)^+ = (x^+, \bar{\alpha})$$

. partial multiplication:

$$(x, \alpha) \in \mathrm{L}((y, \beta)) \text{ iff } x \in \mathrm{L}(y), \qquad \text{and} \qquad (x, \alpha) \circ (y, \beta) = (x \circ y + \alpha y + \beta x, \alpha\beta) .$$

It follows that $(0, 1)$ is a unit in $\mathfrak{A}_e$, and $\mathfrak{A}$ may be identified with the subset $\{ (x, 0), x \in \mathfrak{A} \}$ of $\mathfrak{A}_e$. As usual, we write $(0, 1) \equiv e$ and $(x, \alpha) \equiv x+\alpha e$, as suggested by the rules written above.

Given any mathematical structure, a substructure should be defined as a subset stable under all the operations at hand. Hence we put:

DEFINITION 2.2. — A **-subalgebra* of a partial *-algebra $\mathfrak{A}$ is a vector subspace $\mathfrak{M}$ of $\mathfrak{A}$ such that:

i) $e \in \mathfrak{M}$ (if any);

ii) $\mathfrak{M}^+ = \mathfrak{M}$;

iii) whenever $x, y \in \mathfrak{M}$ and $x \in \mathrm{L}(y)$, then $x \circ y \in \mathfrak{M}$.

It follows that the intersection of any family of *-subalgebras of $\mathfrak{A}$ is one again. Thus given any subset $\mathfrak{N} \subset \mathfrak{A}$, there exists a smallest *-subalgebra containing it, denoted $\mathfrak{M}[\mathfrak{N}]$, and called the *-subalgebra *generated by* $\mathfrak{N}$. For concrete examples, see Refs. [*13-15*].

Before proceeding, it is worth considering some examples of the abstract structure described so far.

Example 1. — The simplest class of partial *-algebras consists of those obtained from topological *-algebras by completion. Let $\mathfrak{A}_0[\tau]$ be a *non-complete* topological *-algebra. Then the multiplication $x \mapsto yx$, $x \mapsto xy$ is continuous for every $y \in \mathfrak{A}_0$, but not jointly continuous in general, i. e. $(x, y) \mapsto xy$ is in general not continuous from $\mathfrak{A}_0[\tau] \times \mathfrak{A}_0[\tau]$ into $\mathfrak{A}_0[\tau]$. In that case the multiplication cannot be continued to the whole completion $\mathfrak{A} \equiv \widetilde{\mathfrak{A}_0[\tau]}$: $\mathfrak{A}$ is only a partial *-algebra, since xy is defined only (by continuity) if one of the factors belongs to $\mathfrak{A}_0$. Thus $\mathrm{L}\mathfrak{A} = \mathrm{R}\mathfrak{A} = \mathfrak{A}_0$. Partial *-algebras of this type, introduced by Lassner [*19*], are called topological *quasi-*-algebras.* An example is the completion of a left Hilbert algebra [*20*, § 10.1]. Another one is the space $\mathfrak{A} \equiv \mathrm{L}^p[a, b]$ on a finite interval, obtained by completion in the L^p-norm of $\mathfrak{A}_0 \equiv \mathrm{C}[a, b]$, with pointwise multiplication [*19*].

Example 2. — *Operators on scales or lattices of Hilbert spaces.*

Let $(\mathscr{H}_n)_{n\in\mathbb{Z}}$ be a scale of Hilbert spaces (the argument is the same for a general lattice):

$$\mathscr{H}_\infty \equiv \bigcap_n \mathscr{H}_n \subset \ldots \subset \mathscr{H}_2 \subset \mathscr{H}_1 \subset \mathscr{H}_0 \subset \mathscr{H}_{\bar{1}} \subset \mathscr{H}_{\bar{2}} \subset \ldots \subset \mathscr{H}_{\bar{\infty}} \equiv \bigcup_n \mathscr{H}_n .$$

An operator on such a scale is defined by a unique *maximal* representative, that is, a bounded linear operator $\mathrm{A}: \mathscr{H}_p \to \mathscr{H}_q (p, q \in \mathbb{Z})$ with p minimal and q maximal, which in turn is extended by natural injections to a linear map $\mathrm{A}: \mathscr{H}_\infty \to \mathscr{H}_{\bar{\infty}}$. Clearly the product A.B of two such operators is well defined only if it can be factorized continuously through some $\mathscr{H}_s$:

$$\mathscr{H}_p \overset{\mathrm{B}}{\to} \mathscr{H}_s \overset{\mathrm{A}}{\to} \mathscr{H}_r .$$

Then the set of all operators on the scale is a partial *-algebra [*13*]. The same situation prevails for large classes of partial inner product spaces [*21*].

Example 3. — Closable operators on a fixed dense domain.
Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a fixed dense domain, and consider the following set of closable linear operators [*11*]:

$$\mathscr{C}(\mathcal{D}, \mathcal{H}) = \{ \text{A closable} \mid \mathcal{D} \subset \text{D(A)} \cap \text{D(A*)} \} \tag{2.1}$$

where A* is the adjoint of A and D(A) its domain. If we consider the subset of $\mathscr{C}(\mathcal{D}, \mathcal{H})$ consisting of those operators A such that $A\mathcal{D} \subset \mathcal{D}$, $A^*\mathcal{D} \subset \mathcal{D}$ and take their restrictions to $\mathcal{D}$, we get a *-algebra of (in general) unbounded operators, denoted $\mathscr{L}^+(\mathcal{D})$ by Lassner [*8*] and $\mathscr{C}_{\mathcal{D}}$ by Epifanio *et al.* [*6*] [*11*]. This is the arena for the theory of Op*-algebras. So $\mathscr{C}(\mathcal{D}, \mathcal{H})$ provides a natural generalization, and if turns out that several structures of partial *-algebras may be introduced on subsets of $\mathscr{C}(\mathcal{D}, \mathcal{H})$. This will occupy us for most of the sequel.

2.B. **Associativity.**

No requirement of associativity was made for the partial multiplication in Definition 2.1. The difficulty is that the usual rule does not make sense without qualifications. Relaxing it slightly, we encounter several possibilities. For instance, we may try the following:

DEFINITION 2.3. — The partial *-algebra $\mathfrak{A}$ is called *associative* if the following holds for any $x, y, z \in \mathfrak{A}$: whenever $x \in L(y)$, $y \in L(z)$ and $x \circ y \in L(z)$, then $y \circ z \in R(x)$ and one has:

$$(x \circ y) \circ z = x \circ (y \circ z) \tag{2.2}$$

Although it looks natural, this condition is too strong and rarely realized in practice, not even for quasi *-algebras (see below). However, for most purposes, a weaker notion is sufficient.

DEFINITION 2.4. — The partial *-algebra $\mathfrak{A}$ is called *semi-associative* if the conditions of Def. 2.3 are verified for all elements $x, y \in \mathfrak{A}$, $z \in R\mathfrak{A}$. In other words, if $y \in R(x)$ implies $y \circ z \in R(x)$ for every $z \in R\mathfrak{A}$ and Eq. (2.2) holds.

The last condition may be reformulated in any of the following equivalent forms:

i) $R\mathfrak{A}$ maps $R(y)$ into itself by right multiplication, for every $y \in \mathfrak{A}$.
ii) $y \in L(z)$ implies $x \circ y \in L(z)$ for every $x \in L\mathfrak{A}$.
iii) $L\mathfrak{A}$ maps $L(y)$ into itself by left multiplication, for every $y \in \mathfrak{A}$.

Thus, in a semi-associative partial *-algebra, the sets $L\mathfrak{A}$, $R\mathfrak{A}$ are in fact algebras. Notice also that, contrary to associativity, semi-associativity is not automatically inherited by a *-subalgebra $\mathfrak{M} \subset \mathfrak{A}$. since $\mathfrak{M}$ may have more universal right multipliers than $\mathfrak{A}$.

PROPOSITION 2.5. — Every quasi *-algebra is semi-associative.

Proof. — Let $\mathfrak{A}$ be a quasi-*-algebra, with distinguished algebra $\mathfrak{A}_0 = R\mathfrak{A} = L\mathfrak{A}$. For a given $x \in \mathfrak{A}$, we verify the statements of Def. 2.4.

i) If $x \in \mathfrak{A}_0$, $R(x) = \mathfrak{A}$ and there is no restriction on y or $y \circ z$.

ii) If $x \in \mathfrak{A} \backslash \mathfrak{A}_0$, $R(x) = \mathfrak{A}_0$, so that y, z and $y \circ z$ all belong to $\mathfrak{A}_0$.

In both cases at most one of the three elements x, y, z does not belong to $\mathfrak{A}_0$. Take, for instance, $x, z \in \mathfrak{A}_0$, $y \in \mathfrak{A} \backslash \mathfrak{A}_0$. Then $y = \lim_{\alpha \in I} y_\alpha$, $y_\alpha \in \mathfrak{A}_0$. For each $\alpha \in I$, the associativity relation holds in $\mathfrak{A}_0$, $x \circ (y_\alpha \circ z) = (x \circ y_\alpha) \circ z$, and remains true in the limit $y_\alpha \to y$. ■

In general a quasi-*-algebra need not be associative. If, in Definition 2.3, $y \in \mathfrak{A} \backslash \mathfrak{A}_0$, then x and z have to belong to $\mathfrak{A}_0$, and we are in the same situation as in Prop. 2.5. On the other hand, if $y \in \mathfrak{A}_0$, $x, z \in \mathfrak{A} \backslash \mathfrak{A}_0$, it might happen that $x \circ y \in \mathfrak{A}_0$ and $y, z \notin \mathfrak{A}_0$, in which case associativity breaks down. If we take, for instance, the quasi-*-algebra $L^p[a, b]$ introduced above, we get precisely that situation for the following choice. Let $y \in \mathfrak{A}_0$ vanish in a neighborhood of some interior point $t_0 \in (a, b)$, x be continuous except for a simple jump at t_0 and z be discontinuous, with at least a jump in the support of y; then $x \circ y \in \mathfrak{A}_0$ and $y \circ z \notin \mathfrak{A}_0$, so that associativity does not hold.

A natural question is whether (semi-) associativity is preserved upon adding a unit. The answer is twofold.

PROPOSITION 2.6. — The extended partial *-algebra $\mathfrak{A}_e$ is semi-associative iff $\mathfrak{A}$ is semi-associative.

Proof. — For $\tilde{x}, \tilde{y}, \tilde{z} \in \mathfrak{A}_e$, we write $\tilde{x} = x + \alpha e$, $\tilde{y} = y + \beta e$, $\tilde{z} = z + \gamma e$, with $x, y, z \in \mathfrak{A}$. We notice that $\tilde{z} = z + \gamma e \in R\mathfrak{A}_e$ iff $z \in R\mathfrak{A}$.

i) Let $\mathfrak{A}$ be semi-associative. Then $\tilde{y} \in R(\tilde{x})$ is equivalent to $y \in R(x)$. For every $z \in R\mathfrak{A}$, this implies $y \circ z \in R(x)$ and therefore $y \circ z + \beta z + \gamma y \in R(x)$ by linearity, i. e. $\tilde{y} \circ \tilde{z} \in R(\tilde{x})$ for every $\tilde{z} \in R\mathfrak{A}_e$. Since Eq. (2.2) extends trivially from $\mathfrak{A}$ to $\mathfrak{A}_e$, it follows that $\mathfrak{A}_e$ is also semi-associative.

ii) The converse implication follows by restriction from $\mathfrak{A}_e$ to $\mathfrak{A}$ since $R\mathfrak{A}_e = R\mathfrak{A} + \mathbb{C}e$. ■

The result just proved does *not* extend to associativity, since then the argument of *ii*) above breaks down. Indeed, let $\tilde{y} \in L(\tilde{z})$, $\tilde{x} \in L(\tilde{y})$ and $\tilde{x} \circ \tilde{y} \in L(\tilde{z})$. This is equivalent to $y \in L(z)$, $x \in L(y)$ and $x \circ y + \alpha y + \beta x \in L(z)$, and therefore $x \circ y + \beta x \in L(z)$ for the given β. But this need *not* imply

$x \circ y \in \mathrm{L}(z)$ if $x \notin \mathrm{L}(z)$, and so the associativity of $\mathfrak{A}$ does not entail that of $\mathfrak{A}_e$. The latter does follow, however, if in $\mathfrak{A}$, $x \circ y \in \mathrm{L}(z)$ implies $x \in \mathrm{L}(z)$, which is precisely the case for semi-associativity ($z \in \mathrm{R}\mathfrak{A}$).

That $x \circ y + \beta x = x \circ (y + \beta) \in \mathrm{L}(z)$ is not equivalent to $x \circ y \in \mathrm{L}(z)$ is easily seen on the quasi-*-algebra $\mathrm{L}^p[a, b]$ of Example 1. Indeed, let again $t_0 \in (a, b)$ and x a function with a single discontinuity at t_0. Then, if y is continuous and $y(t_0) \neq 0$, the product $x(y - y(t_0))$ is continuous, i. e. belongs to $\mathfrak{A}_0$, whereas xy is discontinuous. Thus, if $z \notin \mathfrak{A}_0$, the former belongs to $\mathrm{L}(z)$ and the latter doesn't. Of course, in that case $1 \in \mathrm{L}^p[a, b]$, but this example shows that the condition may fail even for quasi-*-algebras.

2.C. Commutants.

As indicated already in [*14*], there is a natural notion of commutant in a partial *-algebra, namely for $\mathfrak{N} \subset \mathfrak{A}$:

$$\mathfrak{N}' = \{ x \in \mathfrak{A} \mid x \in \mathrm{M}\mathfrak{N} \equiv \mathrm{L}\mathfrak{N} \cap \mathrm{R}\mathfrak{N} \text{ and } x \circ a = a \circ x, \ \forall a \in \mathfrak{N} \}.$$

If $\mathfrak{N}$ is stable under the involution, $\mathfrak{N}'$ is also, and it is a vector subspace of $\mathfrak{A}$ by distributivity (Def. 2.1 *ii*)). However, in the general case, $\mathfrak{N}'$ need not be a *-subalgebra of $\mathfrak{A}$, even if $\mathfrak{A}$ is associative. Indeed, let $x, y \in \mathfrak{N}'$ and $x \in \mathrm{L}(y)$. Assume $x \circ y \in \mathrm{L}\mathfrak{N}$. Then we may write successively using associativity ($a \in \mathfrak{N}$):

$$(x \circ y) \circ a = x \circ (y \circ a) = x \circ (a \circ y) = (x \circ a) \circ y = (a \circ x) \circ y = a \circ (x \circ y) \tag{2.3}$$

i. e. $x \circ y \in \mathfrak{N}'$. The same result follows if $x \circ y \in \mathrm{R}\mathfrak{N}$, but we need at least one of these conditions. For quasi-*-algebras however, it works always.

PROPOSITION 2.7. — Let $\mathfrak{A}$ be an arbitrary quasi-*-algebra. Then the commutant $\mathfrak{N}'$ of any $^+$-invariant subset $\mathfrak{N} \subset \mathfrak{A}$ is a *-subalgebra of $\mathfrak{A}$.

Proof. — Consider the relation (2.3). If $\mathfrak{N} \subset \mathfrak{A}_0$ and $x \in \mathrm{L}(y)$ then a and one of x or y belong to $\mathfrak{A}_0$. If $\mathfrak{N} \subset \mathfrak{A}_0$, then $\mathfrak{N}' \subset \mathfrak{A}_0$, so that x and y belong to $\mathfrak{A}_0$. In any case, at most *one* of x, y, a is not in $\mathfrak{A}_0$, so that all products are defined, and the equalities follow by continuity as in the proof of Prop. 2.5. ■

2.D. Symmetric partial *-algebras.

Once again a notion familiar for *-algebras extends naturally to partial *-algebras [*9*] [*22*]. First we have to define inverses. Let $\mathfrak{A}$ be a partial *-algebra, with unit e. Given $x \in \mathfrak{A}$, the element $y \in \mathfrak{A}$ is called an *inverse* of x if $y \in \mathrm{M}(x) \equiv \mathrm{L}(x) \cap \mathrm{R}(x)$ and $x \circ y = y \circ x = e$. In general an inverse need not be unique, because of the lack of associativity of $\mathfrak{A}$.

DEFINITION 2.8. — A partial *-algebra $\mathfrak{A}$ with unit e is called *symmetric* if, for every $x \in \mathfrak{A}$, one has $x^+ \in L(x)$ and $(e + x^+ \circ x)$ has an inverse in $\mathfrak{A}$.

We will see in Part II that symmetric partial *-algebras of closed operators have distinctly better properties, exactly as for *-algebras. For quasi-*-algebras, however, the notion of symmetry is not very interesting. Indeed, if a quasi-*-algebra is symmetric, every $x \in \mathfrak{A}$ must belong to $\mathfrak{A}_0$ by the first condition, i. e. $\mathfrak{A} = \mathfrak{A}_0$: the only symmetric quasi-*-algebras are the symmetric *-algebras!

3. PARTIAL *-ALGEBRAS OF MINIMAL CLOSED OPERATORS

We come back to Example 3 of Sec. 2: $\mathscr{H}$ is a Hilbert space, $\mathscr{D}$ is a fixed dense domain in $\mathscr{H}$ and we consider the *-invariant family of closable operators $\mathscr{C}(\mathscr{D}, \mathscr{H})$ defined in (2.1). On the set $\mathscr{C}(\mathscr{D}, \mathscr{H})$, we introduce the following equivalence relation:

$$A_1 \sim A_2 \quad \text{iff} \quad A_1 \upharpoonright \mathscr{D} = A_2 \upharpoonright \mathscr{D}. \tag{3.1}$$

Then the set of equivalence classes is in one-to-one correspondence with the set:

$$\mathscr{C}_0(\mathscr{D}, \mathscr{H}) = \{ A \text{ closable} \mid D(A) = \mathscr{D},\, D(A^*) \supset \mathscr{D} \} \tag{3.2}$$

by the relation: $[A] \Leftrightarrow A_0 \equiv A \upharpoonright \mathscr{D}$ for $A \in \mathscr{C}(\mathscr{D}, \mathscr{H})$. The set $\mathscr{C}_0(\mathscr{D}, \mathscr{H})$ carries a natural involution, namely:

$$A_0 \leftrightarrow A_0^+ \equiv A_0^* \upharpoonright \mathscr{D}. \tag{3.3}$$

In this notation, the maximal Op-*-algebra on $\mathscr{D}$ is the subset $\mathscr{L}^+(\mathscr{D})$ of those $A_0 \in \mathscr{C}_0(\mathscr{D}, \mathscr{H})$ such that $A_0\mathscr{D} \subset \mathscr{D}$ and $A_0^+\mathscr{D} \subset \mathscr{D}$. On $\mathscr{C}_0(\mathscr{D}, \mathscr{H})$ itself, with its involution (3.3), one may introduce several structures of partial *-algebras [*23*]. However, in accordance with the previous works [*13*] [*14*] [*15*], we prefer to take the closure of the elements of $\mathscr{C}_0(\mathscr{D}, \mathscr{H})$, What we get is the set $\mathfrak{C}(\mathscr{D})$ defined in (1.1), which can also be written as follows:

$$\mathfrak{C}(\mathscr{D}) = \{ A \in \mathscr{C}(\mathscr{D}, \mathscr{H}) \mid A = \overline{A \upharpoonright \mathscr{D}} \}. \tag{3.4}$$

The elements of $\mathfrak{C}(\mathscr{D})$ are called $\mathscr{D}$-minimal, i. e. they have $\mathscr{D}$ as a core. Given $A \in \mathscr{C}(\mathscr{D}, \mathscr{H})$, we define the two operators

$$\begin{aligned} A^{\ddagger} &= \overline{A^* \upharpoonright \mathscr{D}} \\ A^{\ddagger\ddagger} &= (A^{\ddagger})^{\ddagger} = \overline{A \upharpoonright \mathscr{D}}. \end{aligned} \tag{3.5}$$

Both belong to $\mathfrak{C}(\mathscr{D})$ and both are independent of the choice of A in its equivalence class mod (3.1), i. e. the operation $A \to A^{\ddagger}$ defines an involution on $\mathfrak{C}(\mathscr{D})$. Thus, through closure, $\mathfrak{C}(\mathscr{D})$ is in one-to-one correspondence with $\mathscr{C}_0(\mathscr{D}, \mathscr{H}) \simeq \mathscr{C}(\mathscr{D}, \mathscr{H})/\sim$, and for each $A \in \mathfrak{C}(\mathscr{D})$, we have the following scheme:

$$\begin{array}{ccccc} A_0 & \subset & A = A^{++} & \subset & A^{+*} \\ \updownarrow {\scriptstyle +} & & \updownarrow {\scriptstyle +} & \times & {\scriptstyle *} \\ A_0^{+} & \subset & A^{+} & \subset & A^{*} \end{array} \tag{3.6}$$

where the operators in the middle belong to $\mathfrak{C}(\mathscr{D})$ and those on the right in general do not (they are called $\mathscr{D}$-maximal). If one of them does, they both do and one gets $A^{+} = A^{*}$. Such an operator is called *standard* [*14*]. This is the case e. g. when A is normal or self-adjoint.

We will now try to define a structure of partial *-algebra on the set $\mathfrak{C}(\mathscr{D})$. The key is the following (trivial) lemma [*13*]:

LEMMA 3.1. — Let A, $B \in \mathscr{C}(\mathscr{D}, \mathscr{H})$ verify the conditions:

i) $$B\mathscr{D} \subset D(A)$$

ii) $$A^*\mathscr{D} \subset D(B^*).$$

Then the operator $A(B \restriction \mathscr{D})$ is closable and the domain of its adjoint contains $\mathscr{D}$. ■

When *i*), *ii*) are satisfied, we have in fact two natural ways of defining a product that belongs to $\mathfrak{C}(\mathscr{D})$, namely:

$$A.B \equiv \overline{A(B \restriction \mathscr{D})} \qquad \text{and} \qquad A \,\square\, B \equiv [B^*(A^* \restriction \mathscr{D})]^{+} \tag{3.7}$$

Hence the set $\mathfrak{C}(\mathscr{D})$ may be given two, *a priori* distinct, structures of partial algebra, that we will call *strong* and *weak*, respectively.

3.A. Strong partial Op*-algebras on $\mathscr{D}$.

On the set $\mathfrak{C}(\mathscr{D})$ we consider the following operations:

. vector space structure:

$$A \hat{+} B \equiv \overline{(A + B) \restriction \mathscr{D}} = (A + B)^{++}$$

$$\lambda A \equiv \overline{(\lambda A) \restriction \mathscr{D}} = (\lambda A)^{++}$$

. involution: $A \mapsto A^{+} \equiv \overline{A^* \restriction \mathscr{D}}$

. (strong) multipliers: $A \in L^s(B)$ or $B \in R^s(A)$ iff

$$\text{(SM1)} \quad \text{Ran}\,(B \restriction \mathscr{D}) \subset D(A)$$

$$\text{(SM2)} \quad \text{Ran}\,(A^{+} \restriction \mathscr{D}) \subset D(B^{+})$$

. partial multiplication: $A.B = \overline{A(B \restriction \mathscr{D})}$ for $A \in L^s(B)$

. unit I, the identity operator.

Then, with these operations, $\mathfrak{C}(\mathscr{D})$ verifies all the requirements of Def. 2.1, except *ii*); namely it might happen that $C \in R(A) \cap R(B)$ and $C \notin R(A \hat{+} B)$, the reason being that $D(A \hat{+} B)$ need *not* contain $D(A) \cap D(B) = D(A+B)$ (1). The general scheme is given by the following diagram, where each arrow denotes a continuous embedding and $\tilde{\mathscr{D}}_+$ denotes the completion of $\mathscr{D}$ in the (projective) topology induced by $D(A) \cap D(B)$ (this topology may be determined, e. g. by the norm $\| f \|_+ = \| Af \| + \| Bf \| + \| f \|$

$$\begin{array}{ccccc} & & D(A) \cap D(B) = D(A+B) & & \\ & \nearrow & & \searrow & \\ \mathscr{D} \rightarrow \tilde{\mathscr{D}}_+ & & & & D(\overline{A+B}) \\ & \searrow & & \nearrow & \\ & & D(A \hat{+} B) & & \end{array} \qquad (3.8)$$

A whole class of pathological examples, generalizing that of Kürsten [*17*], is discussed in the Appendix. The proofs of all these statements may be found in [*14*, Addendum].

Nevertheless it is useful to consider the set $\mathfrak{C}(\mathscr{D})$ equipped with the operations just described. The resulting structure will be denoted $\mathfrak{C}^s(\mathscr{D})$ in the sequel.

Thus we get a first class of partial *-algebras, by considering, as in Def. 2.2, vector subspaces $\mathscr{M}$ of $\mathfrak{C}(\mathscr{D})$ containing I and stable under the . multiplication, with the additional requirement of distributivity, i. e. that L(C) be a vector subspace for every $C \in \mathscr{M}$. These are called *strong partial Op**-*algebras on* $\mathscr{D}$.

The universal strong multipliers of $\mathfrak{C}^s(\mathscr{D})$ are easily characterized [*14*], in terms of the domain $\mathscr{D}(\mathfrak{C}) = \bigcap_{A \in \mathfrak{C}(\mathscr{D})} D(A) \supset \mathscr{D}$:

$$L^s\mathfrak{C}(\mathscr{D}) = \{ A \text{ bounded} \mid A^+\mathscr{D} \subset \mathscr{D}(\mathfrak{C}) \}$$
$$R^s\mathfrak{C}(\mathscr{D}) = \{ A \text{ bounded} \mid A\mathscr{D} \subset \mathscr{D}(\mathfrak{C}) \}.$$

The . multiplication on $\mathfrak{C}^s(\mathscr{D})$ is in general not associative (see [*14*], Proposition 3.2) and counterexamples are easily found, e. g. with differential operators on finite intervals. However, it is always semi-associative, provided that $\mathfrak{C}(\mathscr{D})$ is *fully closed*, by which we mean $\mathscr{D} = \mathscr{D}(\mathfrak{C})$ (see Sec. 4 below).

PROPOSITION 3.2. — Let the set $\mathfrak{C}(\mathscr{D})$ be fully closed. Then the . multiplication on $\mathfrak{C}^s(\mathscr{D})$ is semi-associative.

Proof. — Since $\mathfrak{C} \equiv \mathfrak{C}(\mathscr{D})$ is fully closed, $R^s\mathfrak{C}$ consists of all bounded operators mapping $\mathscr{D}$ into itself. Let $B \in R^s(A)$ and $C \in R^s\mathfrak{C}$. Then $B.C \in R^s(A)$ for we have, $\forall f \in \mathscr{D}$:

$$(B.C)f = BCf \in D(A)$$

(1) This fact, originally overlooked in [*13*] [*14*], was pointed out by K.-D. Kürsten and G. Lassner (private communication and [*17*], where a tricky counterexample is given).

$$A^+ f \in D(B^+) \subset D((B.C)^+).$$

The last inclusion follows from the boundedness of C:

$$\begin{aligned}(B.C)^+ &= C^+.B^+ = \overline{C^*(B^* \upharpoonright \mathscr{D})} = [C^*(B^* \upharpoonright \mathscr{D})]^{**} \\ &= [(B^* \upharpoonright \mathscr{D})^* C]^* \supset C^* B^+\end{aligned}$$

hence

$$D((B.C)^+) \supset D(C^* B^+) = D(B^+).$$

It remains to verify the relation (2.2). For any $f \in \mathscr{D}$, we have $Cf \in \mathscr{D}$ and therefore:

$$\begin{aligned}[(A.B).C]f &= (A.B)Cf = AB(Cf) \\ &= A(BCf) = [A.(B.C)]f\end{aligned}$$

and then Eq. (2.2) follows by taking closures. ■

3.B. **Weak partial Op*-algebras on $\mathscr{D}$.**

As said above, the lack of distributivity of the . multiplication in $\mathfrak{C}(\mathscr{D})$ comes from the fact that $D(A \hat{+} B)$ may be too small. A way of circumventing this defect is to use instead the □ multiplication defined in Eq. (3.7). Given $A, B \in \mathfrak{C}(\mathscr{D})$, define the relation $A \in L^w(B)$ or $B \in R^w(A)$ by the two conditions:

$$\text{(WM1)} \quad \operatorname{Ran}(B \upharpoonright \mathscr{D}) \subset D(A^{+*})$$

$$\text{(WM2)} \quad \operatorname{Ran}(A^+ \upharpoonright \mathscr{D}) \subset D(B^*).$$

It is clear from Eq. (3.6) that (SM1), (SM2) imply (WM1), (WM2), i. e. $L^s(B) \subset L^w(B)$, where the inclusion is strict in general. Then, if $A \in L^w(B)$, we may define the product $A \,\square\, B$.

The following properties are readily verified:

i) For $f \in \mathscr{D}$, $(A \,\square\, B)f = A^{+*}Bf$.

ii) $(A \,\square\, B)^+ = B^+ \,\square\, A^+$.

iii) *Distributivity*: since $D((A + B)^*) \supset D(A^*) \cap D(B^*)$ for arbitrary operators A, B, the set $L^w(C)$ is a vector space for each $C \in \mathfrak{C}(\mathscr{D})$, and similarly for $R^w(C)$ (but we may still have $D(A \hat{+} B) \not\supset D(A) \cap D(B)$!). Then we get, for $f \in \mathscr{D}$ and $C \in R^w(A) \cap R^w(B)$:

$$\begin{aligned}[(A \hat{+} B) \,\square\, C]f &= (A \hat{+} B)^{+*}Cf \\ &= (A^+ \hat{+} B^+)^* Cf \\ &= [(A^* + B^*) \upharpoonright \mathscr{D}]^* Cf \\ &= (A^* \upharpoonright \mathscr{D})^* Cf + (B^* \upharpoonright \mathscr{D})^* Cf, \quad \text{by (WM2)} \\ &= A^{+*}Cf + B^{+*}Cf \\ &= (A \,\square\, C)f + (B \,\square\, C)f,\end{aligned}$$

hence, by closure: $(A \hat{+} B) \,\square\, C = (A \,\square\, C) \hat{+} (B \,\square\, C)$.

Putting all these results together we get:

PROPOSITION 3.3. — Let $\mathfrak{C}^w(\mathscr{D})$ denote the set $\mathfrak{C}(\mathscr{D})$ equipped with the $\hat{+}$ addition, the $+$ involution and the $\square$ multiplication defined on the pairs A, B for which $A \in L^w(B)$. Then $\mathfrak{C}^w(\mathscr{D})$ is a partial Op*-algebra, with unit I. ■

In analogy to the previous case, we may now define a *weak partial Op*-algebra of operators* on $\mathscr{D}$ as a *-subalgebra of $\mathfrak{C}^w(\mathscr{D})$, in the sense of Def. 2.2.

As for $\mathfrak{C}^s(\mathscr{D})$, the universal multipliers of $\mathfrak{C}^w(\mathscr{D})$ are bounded operators:

$$L^w\mathfrak{C}^w(\mathscr{D}) = \{ A \text{ bounded} \mid A^+\mathscr{D} \subset \mathscr{D}_*(\mathfrak{C}) \}$$
$$R^w\mathfrak{C}^w(\mathscr{D}) = \{ A \text{ bounded} \mid A\mathscr{D} \subset \mathscr{D}_*(\mathfrak{C}) \}$$

where we have introduced the domain $\mathscr{D}_*(\mathfrak{C}) = \bigcap_{A \in \mathfrak{C}(\mathscr{D})} D(A^*)$.

Clearly:

$$\mathscr{D} \subset \mathscr{D}(\mathfrak{C}) \subset \mathscr{D}_*(\mathfrak{C}). \tag{3.9}$$

From this follows the analogue of Proposition 3.2:

PROPOSITION 3.4. — Let $\mathscr{D} = \mathscr{D}_*(\mathfrak{C})$. Then the partial Op*-algebra $\mathfrak{C}^w(\mathscr{D})$ is semi-associative.

Proof. — The assumption $\mathscr{D} = \mathscr{D}_*(\mathfrak{C})$ implies that $C \in R^w\mathfrak{C}^w(\mathscr{D})$ is bounded and maps $\mathscr{D}$ into itself. Then, given A, B such that $B \in R^w(A)$, we obtain $B \square C \in R^w(A)$ as in Prop. 3.2, since we have, for any $f \in \mathscr{D}$:

$$(B \square C)f = B^{+*}Cf = BCf \in D(A^{+*})$$
$$A^+ f \in D(B^*) \subset D((B \square C)^*).$$

The last inclusion results from the boundedness of C:

$$(B \square C)^* = [B^{+*}(C \upharpoonright \mathscr{D})]^* = [(BC) \upharpoonright \mathscr{D}]^* \supset C^*B^*.$$

Finally the relation (2.2) is immediate:

$$\begin{aligned}[(A \square B) \square C]f &= (A \square B)^{+*}Cf \\ &= (A \square B)Cf \\ &= A^{+*}BCf \\ &= A^{+*}(B \square C)f \\ &= [A \square (B \square C)]f. \end{aligned}$$ ■

REMARK 3.5. — One may also define on $\mathfrak{C}(\mathscr{D})$ other kinds of multipliers, intermediate between the weak and the strong ones. For instance, we say that B is a *mixed right multiplier* of A if they verify conditions (SM1) and (WM2), i. e. for $f \in \mathcal{D}$:

$$(A \,\square\, B)f = ABf$$
$$(B^{+} \,\square\, A^{+})f = B^{*}A^{+}f. \qquad (3.10)$$

We denote this fact by $B \in R^{M}(A)$ or $A \in L^{M}(B)$.

Similarly, we say that $B \in \tilde{R}^{M}(A)$ or $A \in \tilde{L}^{M}(B)$ if they verify the conditions (WM1) and (SM2), i. e. for $f \in \mathcal{D}$:

$$(A \,\square\, B)f = A^{+*}Bf$$
$$(B^{+} \,\square\, A^{+})f = B^{+}A^{+}f. \qquad (3.11)$$

Thus we get $B \in R^{M}(A)$ iff $A^{+} \in \tilde{L}^{M}(B^{+})$, and $A \in L^{M}(B)$ iff $B^{+} \in \tilde{R}^{M}(A^{+})$. Clearly the operation $+$ is *not* an involution for these types of multipliers, and they do not generate new structures of partial *-algebras on $\mathfrak{C}(\mathcal{D})$. Concerning the problem of distributivity, $\tilde{L}^{M}(C)$ and $R^{M}(C)$ are vector subspaces of $\mathfrak{C}$ for any C, but $L^{M}(C)$ and $\tilde{R}^{M}(C)$ need not be. Similarly the spaces $\tilde{L}^{M}(C)$ and $R^{M}(C)$ have better topological properties, as we will see in Sec. 5 A below. We will come back to these asymmetric multipliers in Part II.

REMARK 3.6. — The notation and terminology used in this paper reflect the presence of the two structures of partial *-algebra on $\mathfrak{C}(\mathcal{D})$, and this has compelled us to diverge from the conventions of [*13*] [*14*]. Indeed conditions (SM1), (SM2) were denoted (M1) and (M2) there, and the operators called simply multipliers: $A \in L(B)$, etc. Similarly (WM1), (WM2) were previously denoted (*M1), (*M2), for *-multipliers: $A \in L^{*}(B)$, etc. Finally, the two conditions *i*), *ii*) of Lemma 3.1 (which coincide with (SM1), (WM2) if $A, B \in \mathfrak{C}(\mathcal{D})$) were called (WM1), (WM2) in [*13*] [*14*], for weak multipliers.

3.C. Strong vs. weak partial Op*-algebras.

In order to make contact with our previous work [*14*] define $\mathfrak{C}^{*}(\mathcal{D})$ as the set of all adjoints of the operators of $\mathfrak{C}(\mathcal{D})$:

$$A \in \mathfrak{C}(\mathcal{D}) \Leftrightarrow A^{*} \in \mathfrak{C}^{*}(\mathcal{D}).$$

It was shown in [*14*] that the set $\mathfrak{C}^{*}(\mathcal{D})$ is a partial *-algebra with respect to the following operations:

. vector space structure: $A \check{+} B = [(A^{*}+B^{*}) \upharpoonright \mathcal{D}]^{*}$, $\lambda A = [\bar{\lambda} A^{*} \upharpoonright \mathcal{D}]^{*}$

. involution: $A \mapsto A^{\dagger} = (A \upharpoonright \mathcal{D})^{*}$

(this implies $A^{\dagger} \upharpoonright \mathcal{D} = A^{+} \upharpoonright \mathcal{D}$ and $A^{\dagger\dagger} = A^{+*}$)

. partial multiplication: $A * B = [B^{\dagger}(A^{\dagger} \upharpoonright \mathcal{D})]^{*}$, defined whenever $A \in L^{w}(B)$, in the sense of conditions (WM1), (WM2).

Thus elements of $\mathfrak{C}^*(\mathscr{D})$ are all $\mathscr{D}$-maximal operators, i. e. those that verify $A = A^{\dagger\dagger} = A^{+*}$.

There is a one-to-one correspondence between the sets $\mathfrak{C}(\mathscr{D})$ and $\mathfrak{C}^*(\mathscr{D})$, given by the linear map $j: \mathfrak{C}(\mathscr{D}) \to \mathfrak{C}^*(\mathscr{D})$ and its inverse:

$$j(A) = A^{\dagger\dagger} \ (= A^{*\dagger} = A^{+*}) \qquad \text{for} \qquad A \in \mathfrak{C}(\mathscr{D})$$
$$j^{-1}(B) = B^{++} \ (= B^{*+} = B^{\dagger *}) \qquad \text{for} \qquad B \in \mathfrak{C}^*(\mathscr{D})$$

Since $A \in L^s(B)$ implies $A \in L^w(B)$, but not the converse in general, it follows that j respects the partial *-algebra structure:

$$\left.\begin{aligned} j(A \,\hat{+}\, B) &= j(A) \,\check{+}\, j(B) = A^{\dagger\dagger} \,\check{+}\, B^{\dagger\dagger} \\ j(A \,.\, B) &= j(A) * j(B) = A^{\dagger\dagger} * B^{\dagger\dagger} \end{aligned}\right. \qquad (A, B \in \mathfrak{C}(\mathscr{D}))$$

but j^{-1} preserves only linear combinations, not products.

The situation is best understood in terms of the notion of *homomorphism*, that we recall from [*14*].

DEFINITION 3.7. — A *homomorphism* of a partial *-algebra $\mathfrak{M}$ into another one $\mathfrak{N}$ is a linear map $\sigma: \mathfrak{M} \to \mathfrak{N}$ such that:

i) $\sigma(x^+) = [\sigma(x)]^+$;

ii) if $x \in L(y)$ in $\mathfrak{M}$, then $\sigma(x) \in L(\sigma(y))$ in $\mathfrak{N}$ and $\sigma(x) \circ \sigma(y) = \sigma(x \circ y)$.

The map σ is an *isomorphism* if it is a bijection and $\sigma^{-1}: \mathfrak{N} \to \mathfrak{M}$ is also an homomorphism.

Notice that $x \in R\mathfrak{M}$, resp. $y \in L\mathfrak{M}$, implies $\sigma(x) \in R\sigma(\mathfrak{M})$, resp. $\sigma(y) \in L\sigma(\mathfrak{M})$. Also if $e \in \mathfrak{M}$ is a unit, $\sigma(e)$ is a unit in $\sigma(\mathfrak{M}) \subset \mathfrak{N}$.

Let now $\mathfrak{M} \subset \mathfrak{C}^s(\mathscr{D})$ be a strong partial Op*-algebra, $\mathfrak{M}^* \equiv j(\mathfrak{M}) \subset \mathfrak{C}^*(\mathscr{D})$ and $\mathfrak{M}^w \equiv j^{-1}(\mathfrak{M}^*) \subset \mathfrak{C}^w(\mathscr{D})$. Then $j: \mathfrak{M} \to \mathfrak{M}^*$ is a homomorphism, the identity $i: \mathfrak{M} \to \mathfrak{M}^w$ is a homomorphism, and $j: \mathfrak{M}^w \to \mathfrak{M}^*$ is an isomorphism. In particular:

$$\left.\begin{aligned} j(A \,\square\, B) &= j(A) * j(B) = A^{\dagger\dagger} * B^{\dagger\dagger} \\ j^{-1}(A^{\dagger\dagger} * B^{\dagger\dagger}) &= A \,\square\, B \end{aligned}\right. \qquad (A, B \in \mathfrak{M})$$

The same relationship exists between $\mathfrak{C}(\mathscr{D})$, $\mathfrak{C}^*(\mathscr{D}) = j(\mathfrak{C}(\mathscr{D}))$ and $\mathfrak{C}^w(\mathscr{D}) = j^{-1}(\mathfrak{C}^*(\mathscr{D}))$ (although, strictly speaking, $\mathfrak{C}(\mathscr{D})$ is not a partial *-algebra, the notion of homomorphism still makes sense).

Although it is very natural, Definition 3.7 does not exclude certain pathologies. For instance the range $\sigma(\mathfrak{M})$ need not be a subalgebra of $\mathfrak{N}$, if σ is not an isomorphism. In that case, indeed, there will be elements $x, y \in \mathfrak{M}$ such that $x \notin L(y)$, but $\sigma(x) \in L(\sigma(y))$ in $\mathfrak{N}$. Then $\sigma(x) \circ \sigma(y)$ is a well-defined element of $\mathfrak{N}$, but it need not belong to $\sigma(\mathfrak{M})$. Of course it may also happen that $\sigma(\mathfrak{M})$ is a subalgebra of $\mathfrak{N}$, e. g. if σ is surjective. But then another pathology may arise. Take for instance the bijection $j : A \mapsto A^{\dagger\dagger}$ from $\mathfrak{C}(\mathscr{D})$ onto $\mathfrak{C}^*(\mathscr{D})$. If $A \notin L^s(B)$, but $A \in L^w(B)$, then

$A^{\dagger\dagger} * B^{\dagger\dagger} = A * B = j(C)$, with $C = (A * B)^{\ddagger\ddagger}$ but C cannot be factorized as A.B, since the latter does not exist!

In that particular case, we have circumvented the difficulty by introducing on $\mathfrak{C}(\mathscr{D})$ the weaker product $\square$, in effect pulling back to $\mathfrak{C}(\mathscr{D})$ the partial *-algebra structure of $\mathfrak{C}^*(\mathscr{D})$: this yields precisely $\mathfrak{C}^w(\mathscr{D})$. For instance, the operator C above *does* factorize in $C = A \square B$.

Exactly the same construction may be performed whenever the homomorphism $\sigma: \mathfrak{M} \to \mathfrak{N}$ is injective, and its range $\sigma(\mathfrak{M})$ is a subalgebra of $\mathfrak{N}$. For every pair $x, y \in \mathfrak{M}$ such that $\sigma(x) \in L(\sigma(y))$ in $\mathfrak{N}$, one defines the new product $x \square y = \sigma^{-1}(\sigma(x) \circ \sigma(y))$. In this way $\mathfrak{M}$ acquires a new, weaker, structure of partial *-algebra, pulled back from $\sigma(\mathfrak{M}) \subset \mathfrak{N}$. Conversely, a linear bijection σ from $\mathfrak{M}$ to a set $\mathfrak{N}$ yields, by isomorphism, a structure of partial *-algebra on the image $\sigma(\mathfrak{M})$: thus the map $j: \mathfrak{C}(\mathscr{D}) \to \mathfrak{C}^*(\mathscr{D})$ provides $\mathfrak{C}^*(\mathscr{D})$ with the structure transported from $\mathfrak{C}^w(\mathscr{D})$.

4. CLOSED AND FULLY CLOSED EXTENSIONS

In the theory of Op*-algebras, it is well-known that the so-called closed and self-adjoint algebras have better properties. What is the corresponding situation for partial Op*-algebras?

Let $\mathfrak{M}$ be a subset of $\mathfrak{C}(\mathscr{D})$. It defines on $\mathscr{D}$ the so-called $t_{\mathfrak{M}}$ topology, given by the seminorms $\| f \|_A = \| Af \|$, $A \in \mathfrak{M}$. For this (projective) topology, the natural domain $\mathscr{D}(\mathfrak{M}) = \bigcap_{A \in \mathfrak{M}} D(A)$ is complete, but need not be the completion of $\mathscr{D}[t_{\mathfrak{M}}]$, denoted $\tilde{\mathscr{D}} \equiv \tilde{\mathscr{D}}[t_{\mathfrak{M}}]$. Similarly for $\mathfrak{M}^* = j(\mathfrak{M}) \subset \mathfrak{C}^*(\mathscr{D})$ (notice that $t_{\mathfrak{M}} = t_{\mathfrak{M}^*}$ on $\mathscr{D}$). Hence one has in general:

$$\mathscr{D} \subset \tilde{\mathscr{D}}[t_{\mathfrak{M}}] \subset \mathscr{D}(\mathfrak{M}) \subset \mathscr{D}_*(\mathfrak{M}) \equiv \mathscr{D}(\mathfrak{M}^*) \tag{4.1}$$

We say that $\mathfrak{M}$ is *closed* if $\mathscr{D} = \tilde{\mathscr{D}}[t_{\mathfrak{M}}]$, *fully closed* if $\mathscr{D} = \mathscr{D}(\mathfrak{M})$, *essentially self-adjoint* if $\mathscr{D}(\mathfrak{M}) = \mathscr{D}_*(\mathfrak{M})$ and *self-adjoint* if $\mathscr{D} = \mathscr{D}_*(\mathfrak{M})$. For Op*-algebras $\tilde{\mathscr{D}}[t_{\mathfrak{M}}] = \mathscr{D}(\mathfrak{M})$, so that closed and fully closed are synonymous, and in that case essentially self-adjoint has its usual meaning $\tilde{\mathscr{D}}[t_{\mathfrak{M}}] = \mathscr{D}_*(\mathfrak{M})$.

An interesting consequence is that, for a fully closed subset $\mathfrak{M} \subset \mathfrak{C}(\mathscr{D})$, its universal strong right multipliers, $R^s\mathfrak{M}$, map $\mathscr{D}$ into itself. The same is true for $R^w\mathfrak{M}$ if $\mathfrak{M}$ is self-adjoint.

As is well-known, an Op*-algebra $\mathfrak{A}$ may always be extended by continuity to a (fully) closed Op*-algebra $\overline{\mathfrak{A}}$ on the domain $\mathscr{D}(\mathfrak{A})$, isomorphic to $\mathfrak{A}$. Here the situation is more complicated, for three reasons:

a) we have two different possible extensions, one to $\tilde{\mathscr{D}}(t_{\mathfrak{M}}]$, the other one to $\mathscr{D}(\mathfrak{M})$;

b) we have to distinguish between extensions in the strong and in the weak structure;

c) we are dealing with equivalence classes of operators, as discussed in Sec. 2.

Point *c*) is not a problem, however, as the next lemma shows.

LEMMA 4.1. — Let $\mathfrak{M}$ be a subset of $\mathfrak{C}(\mathcal{D})$. Then the following inclusions hold:

$$\mathfrak{C}(\mathcal{D}(\mathfrak{M})) \subset \mathfrak{C}(\tilde{\mathcal{D}}[t_{\mathfrak{M}}]) \subset \mathfrak{C}(\mathcal{D}) \tag{4.2}$$

Furthermore all three sets in (4.2) contain the same bounded operators.

Proof. — Starting with closable operators, we have the obvious inclusions:

$$\mathscr{C}(\mathcal{D}(\mathfrak{M}), \mathscr{H}) \subset \mathscr{C}(\tilde{\mathcal{D}}[t_{\mathfrak{M}}], \mathscr{H}) \subset \mathscr{C}(\mathcal{D}, \mathscr{H}).$$

On each space, we may consider the equivalence relation corresponding to the domain in question, as in Eq. (3.1). But in fact the three equivalence relations are the same. Indeed, if $A_1, A_2 \in \mathscr{C}(\mathcal{D}(\mathfrak{M}), \mathscr{H})$ coincide on $\mathcal{D}(\mathfrak{M})$, they coincide *a fortiori* on $\mathcal{D}$. Conversely, if $A_1 \upharpoonright \mathcal{D} = A_2 \upharpoonright \mathcal{D}$, then $(A_1 - A_2) \upharpoonright \mathcal{D} = 0$, which is a bounded operator. Hence $\overline{(A_1 - A_2) \upharpoonright \mathcal{D}} = 0$ on $\mathscr{H}$, and, by restriction, $(A_1 - A_2) \upharpoonright \mathcal{D}(\mathfrak{M}) = 0$ as well. The argument is identical for $\tilde{\mathcal{D}}[t_{\mathfrak{M}}]$ and also for an arbitrary non-zero bounded operator. Thus the equivalence relation on $\mathscr{C}(\mathcal{D}(\mathfrak{M}), \mathscr{H})$, resp. $\mathscr{C}(\tilde{\mathcal{D}}[t_{\mathfrak{M}}], \mathscr{H})$, is simply the restriction of the one defined on $\mathscr{C}(\mathcal{D}, \mathscr{H})$, and therefore

$$\mathscr{C}_0(\mathcal{D}(\mathfrak{M}), \mathscr{H}) \subset \mathscr{C}_0(\tilde{\mathcal{D}}[t_{\mathfrak{M}}], \mathscr{H}) \subset \mathscr{C}_0(\mathcal{D}, \mathscr{H}).$$

Taking closures (which is a one-to-one operation), we get Eq. (4.2). ∎

Under additional assumptions, we may get more than the inclusions (4.2). Indeed:

LEMMA 4.2. — Let $\mathcal{D}[t_{\mathfrak{M}}]$ be barrelled. Then $\mathfrak{C}(\tilde{\mathcal{D}}[t_{\mathfrak{M}}]) = \mathfrak{C}(\mathcal{D})$.

Proof. — Since $\mathcal{D}[t_{\mathfrak{M}}]$ is barrelled, every element $X \in \mathfrak{C}(\mathcal{D})$ is continuous from $\mathcal{D}[t_{\mathfrak{M}}]$ into $\mathscr{H}$, by the closed graph theorem. It follows that $t_{\mathfrak{M}} = t_{\mathfrak{C}(\mathcal{D})}$ on $\mathcal{D}$. Hence, given any $X \in \mathfrak{C}(\mathcal{D})$, $\tilde{\mathcal{D}}[t_{\mathfrak{M}}]$ is contained in D(X) and is a core for X. This means that $X \in \mathfrak{C}(\tilde{\mathcal{D}}[t_{\mathfrak{M}}])$, which proves the assertion. ∎

Of course the argument works if $\mathcal{D}[t_{\mathfrak{M}}]$ is a Fréchet space, but then $\mathcal{D} = \tilde{\mathcal{D}}[t_{\mathfrak{M}}]$ and there is nothing to prove!

Concerning Point *b*) above, it has been shown in [*14*] that every strong partial Op*-algebra $\mathfrak{M}$ on $\mathcal{D}$ admits a unique closure $\overline{\mathfrak{M}}$, which is a closed partial Op*-algebra on $\tilde{\mathcal{D}}$, isomorphic to $\mathfrak{M}$. The set $\overline{\mathfrak{M}}$ consists of the same operators as $\mathfrak{M}$, but considered as closures of their restriction to $\tilde{\mathcal{D}}$, and the relation $A \in L(B)$ holds in $\overline{\mathfrak{M}}$ iff it holds in $\mathfrak{M}$.

Since $\mathcal{D} \subset \mathcal{D}(\mathfrak{M}) \subset D(A)$ for every $A \in \mathfrak{M}$, $\mathcal{D}(\mathfrak{M})$ is a core for A. Denote by $\hat{\mathfrak{M}}$ the set of all operators from $\mathfrak{M}$, but considered as closures of their restriction to $\mathscr{D}(\mathfrak{M})$, i. e. as elements of $\mathfrak{C}(\mathscr{D}(\mathfrak{M}))$; obviously $\hat{\mathfrak{M}}$ is fully closed. Then, as was shown in [14], the identity map gives $\hat{\mathfrak{M}}$ a structure of strong partial Op*-algebra over $\mathscr{D}(\mathfrak{M})$, denoted $\hat{\mathfrak{M}}^s$, and the identity $i: \hat{\mathfrak{M}}^s \to \mathfrak{M}$ is a homomorphism, but not necessarily an isomorphism. Indeed, there might be less multipliers in $\hat{\mathfrak{M}}^s$ than in $\mathfrak{M}$, since $B\mathscr{D} \subset D(A)$ need not imply $B\mathscr{D}(\mathfrak{M}) \subset D(A)$.

However, this phenomenon does not take place for weak partial Op*-algebras. Indeed:

PROPOSITION 4.3. — Let $\mathfrak{M} \equiv \mathfrak{M}^w$ be a weak partial Op*-algebra over $\mathscr{D}$. Denote by $\hat{\mathfrak{M}}$ the same set of operators, but viewed as closures of their restriction to $\mathscr{D}(\mathfrak{M})$. Then $\hat{\mathfrak{M}}$ is a fully closed weak partial Op*-algebra on $\mathscr{D}(\mathfrak{M})$, denoted $\hat{\mathfrak{M}}^w$, and the identity $i: \mathfrak{M}^w \to \hat{\mathfrak{M}}^w$ is an isomorphism.

Proof. — Take first $A, B \in \mathfrak{M}$ such that $A \in L^w(B)$ in $\mathfrak{C}^w(\mathscr{D})$. Then for any $g \in D(B) \cap D(A \,\square\, B)$ and $f_0 \in \mathscr{D}$, we have:

$$\langle (A \,\square\, B)g, f_0 \rangle = \langle Bg, A^+ f_0 \rangle = \langle g, (B^+ \,\square\, A^+) f_0 \rangle .$$

Since $\mathscr{D}$ is a core for every element of $\mathfrak{M}$, any vector $f \in D(A^+) \cap D(B^+ \,\square\, A^+)$ may be approximated in the graph norm of A^+ by a sequence $f_0^{(n)} \in \mathscr{D}$: $f_0^{(n)} \to f$, $A^+ f_0^{(n)} \to A^+ f$. Hence we get:

$$\begin{aligned} \langle (A \,\square\, B)g, f \rangle = \langle Bg, A^+ f \rangle &= \lim_n \langle g, (B^+ \,\square\, A^+) f_0^{(n)} \rangle \\ &= \langle g, (B^+ \,\square\, A^+) f \rangle \end{aligned} \tag{4.3}$$

since $f \in D(B^+ \,\square\, A^+)$. Of course this does *not* imply that

$$(B^+ \,\square\, A^+) f_0^{(n)} \to (B^+ \,\square\, A^+) f$$

strongly. Also $(B^+ \,\square\, A^+) f_0^{(n)} = B^+(A^+ f_0^{(n)})$, but $(B^+ \,\square\, A^+) f$ cannot necessarily be factorized in the same way, since $A^+ f$ need not belong to $D(B^*)$.

The relation (4.3) is true, in particular, for every $f, g \in \mathscr{D}(\mathfrak{M})$. But this means that $Bg \in D((A^+ \upharpoonright \mathscr{D}(\mathfrak{M}))^*)$, and $A^+ f \in D((B \upharpoonright \mathscr{D}(\mathfrak{M}))^*) = D(B^*)$, i. e. $A \in L^w(B)$ in $\mathfrak{C}^w(\mathscr{D}(\mathfrak{M}))$.

The converse implication being evident, the relation $A \in L^w(B)$ holds in $\mathfrak{C}^w(\mathscr{D})$ iff it holds in $\mathfrak{C}^w(\mathscr{D}(\mathfrak{M}))$. Furthermore, the involution $+$ and the multiplication $\square$, which depend on the domain, are in fact the same for every domain $\mathscr{D}_1$ such that $\mathscr{D} \subseteq \mathscr{D}_1 \subseteq \mathscr{D}(\mathfrak{M})$, as can be checked readily. This proves the assertion. ■

Combining Proposition 4.3 with the results obtained in [14], one gets finally the following extension theorem.

THEOREM 4.4. — Every subset $\mathfrak{M}$ of $\mathfrak{C}(\mathscr{D})$ determines a closed subset $\overline{\mathfrak{M}}$ of $\mathfrak{C}(\tilde{\mathscr{D}}[t_{\mathfrak{M}}])$ and a fully closed subset $\hat{\mathfrak{M}}$ of $\mathfrak{C}(\mathscr{D}(\mathfrak{M}))$, where all three sets consists of the same closed operators. Then:

i) if $\mathfrak{M} \equiv \mathfrak{M}^s$ is a *strong* partial Op*-algebra, it determines two strong partial Op*-algebras $\overline{\mathfrak{M}}^s$, $\hat{\mathfrak{M}}^s$ and three weak ones $\mathfrak{M}^w$, $\overline{\mathfrak{M}}^w$, $\hat{\mathfrak{M}}^w$; the relationship between these six partial Op*-algebras is given by the following diagram, where $\rightarrow$ denotes a homomorphism and $\Leftrightarrow$ an isomorphism:

$$\begin{array}{ccccc} \mathfrak{M}^s & \Leftrightarrow & \overline{\mathfrak{M}}^s & \leftarrow & \hat{\mathfrak{M}}^s \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{M}^w & \Leftrightarrow & \overline{\mathfrak{M}}^w & \Leftrightarrow & \hat{\mathfrak{M}}^w \end{array} \tag{4.4}$$

ii) if $\mathfrak{M} \equiv \mathfrak{M}^w$ is a *weak* partial Op*-algebra, it has weak extensions $\overline{\mathfrak{M}}^w$ and $\hat{\mathfrak{M}}^w$, both isomorphic to $\mathfrak{M}^w$. ■

The conclusion of the whole analysis is that, without loss of generality, a strong partial Op*-algebra may always be assumed to be closed, but not fully closed, whereas a weak partial Op*-algebra may always be taken to be fully closed. This applies in particular to $\mathfrak{C}^s(\mathscr{D})$ and $\mathfrak{C}^w(\mathscr{D})$, respectively.

5. TOPOLOGIES ON PARTIAL Op*-ALGEBRAS

As for Op*-algebras [*8*], one may introduce on partial *-algebras of operators two classes of natural topologies, called respectively quasi-uniform and $\mathscr{K}$-weak. They generalize the familiar topologies of von Neumann algebras [*1*].

5.A. Quasi-uniform topologies.

These topologies are specific to subsets $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$ where they are defined and they depend explicitly of the (left or right) multipliers of $\mathfrak{N}$: hence they depend on the product which is chosen, . or $\square$.

The domain $\mathscr{D}$ itself will carry its projective topology $t_{\mathfrak{C}}$, defined by the seminorms $f \mapsto \| Af \|$, $A \in \mathfrak{C}(\mathscr{D})$. We will denote by $\mathbb{B}(\mathscr{D})$ the class of $t_{\mathfrak{C}}$-bounded subsets of $\mathscr{D}$, i. e. $\mathscr{M} \in \mathbb{B}(\mathscr{D})$ iff $\sup_{f \in \mathscr{M}} \| Af \| < \infty$ for all $A \in \mathfrak{C}(\mathscr{D})$.

Given a subset $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$, containing I, consider a subset $\mathscr{L} \subset L^w\mathfrak{N}$, containing I. Then one may consider on $\mathfrak{N}$ a *quasi-uniform* topology $\tau_*^l(\mathscr{L})$, defined by the seminorms:

$$\| B \|_l^{A, \mathscr{M}} = \sup_{f \in \mathscr{M}} \{ \| (A \square B)f \| + \| (B^+ \square A^+)f \| \} \tag{5.1}$$

where $A \in \mathscr{L}$ and $\mathscr{M} \in \mathbb{B}(\mathscr{D})$ is a bounded set of $\mathscr{D}[t_{\mathfrak{C}}]$.

Similarly, every subset $\mathscr{R} \subset R^w\mathfrak{N}$, containing I, determines another quasi-uniform topology on $\mathfrak{N}$, $\tau_*^r(\mathscr{R})$, with seminorms:

$$\| B \|_r^{C, \mathscr{M}} = \| B^+ \|_l^{C, \mathscr{M}} \tag{5.2}$$

(remember that $C \in \mathscr{R}$ iff $C^+ \in \mathscr{R}^+ \subset L^w\mathfrak{N}^+$).

Clearly $\mathscr{L}_1 \subset \mathscr{L}_2$ implies $\tau_*^l(\mathscr{L}_1) \prec \tau_*^l(\mathscr{L}_2)$ on $\mathfrak{N}$ ($\prec$ means « weaker than ») and $\mathscr{R}_1 \subset \mathscr{R}_2$ implies $\tau_*^r(\mathscr{R}_1) \prec \tau_*^r(\mathscr{R}_2)$. As particular cases, one has:

i) $\mathscr{L} = \mathrm{L}^w\mathfrak{N}$ or $\mathscr{R} = \mathrm{R}^w\mathfrak{N}$, which in each case gives strongest left, resp. right, quasi-uniform topology on $\mathfrak{N}$.

ii) $\mathscr{L} \subset \mathrm{L}^s\mathfrak{N}$ or $\mathscr{R} \subset \mathrm{R}^s\mathfrak{N}$, in which case one may replace $\square$ products by . products in (5.1), (5.2): these are the quasi-uniform topologies to use for strong partial Op*-algebras.

In particular, given a subset $\mathfrak{N} = \mathfrak{N}^+$, its spaces of multipliers carry natural quasi-uniform topologies: $\tau_*^l(\mathfrak{N})$ on $\mathrm{R}^w\mathfrak{N}$, $\tau_*^r(\mathfrak{N})$ on $\mathrm{L}^w\mathfrak{N}$, and $\tau_*(\mathfrak{N}) \equiv \tau_*^l(\mathfrak{N}) \cap \tau_*^r(\mathfrak{N})$ on $\mathrm{M}^w\mathfrak{N} \equiv \mathrm{L}^w\mathfrak{N} \cap \mathrm{R}^w\mathfrak{N}$ (i. e. $\tau_*(\mathfrak{N})$ is defined by all seminorms $\| \cdot \|_l^{\mathrm{A},\mathscr{M}}$, $\| \cdot \|_r^{\mathrm{A},\mathscr{M}}$ ($\mathrm{A} \in \mathfrak{N}$) simultaneously).

In the case of the whole space $\mathfrak{C}(\mathscr{D})$ the situation gets simpler. The quasi-uniform topologies on $\mathfrak{C}(\mathscr{D})$ are intermediate between the extreme ones:

. the strongest, $\tau_*^r(\mathrm{R}^w\mathfrak{C})$ or $\tau_*^l(\mathrm{L}^w\mathfrak{C})$

. the weakest, $\tau_* \equiv \tau_*^r(\mathrm{I}) = \tau_*^l(\mathrm{I})$ corresponding to $\mathscr{L} = \mathscr{R} = \{\mathrm{I}\}$ and given by the seminorms:

$$\| \mathrm{B} \|^{\mathscr{M}} = \sup_{f \in \mathscr{M}} \{ \| \mathrm{B}f \| + \| \mathrm{B}^+ f \| \}, \qquad \mathscr{M} \in \mathbb{B}(\mathscr{D}). \tag{5.3}$$

However it follows from [*14*] that the three topologies τ_*, $\tau_*^r(\mathrm{R}^w\mathfrak{C})$ and $\tau_*^l(\mathrm{L}^w\mathfrak{C})$ are equivalent whenever $\mathfrak{C}$ is self-adjoint, $\mathscr{D} = \mathscr{D}_*(\mathfrak{C})$. Then all quasi-uniform topologies on $\mathfrak{C}(\mathscr{D})$ coincide. Similarly, if $\mathfrak{N} = \mathfrak{N}^+$, the topologies $\tau_*^l(\mathfrak{N})$ and $\tau_*^r(\mathfrak{N})$ coincide on the space of multipliers $\mathrm{R}^w\mathfrak{N} \cap \mathrm{L}^w\mathfrak{N}$; their restriction will be denoted simply $\tau_*(\mathfrak{N})$.

Every quasi-uniform topology may be weakened by restricting the bounded sets $\mathscr{M} \in \mathbb{B}(\mathscr{D})$ to a particular subclass, in particular the finite subsets of $\mathscr{D}$. In that particular case, we will denote the resulting topology by $\tau_f^l(\mathscr{L})$, resp. $\tau_f^r(\mathscr{R})$, $\tau_f(\mathfrak{N})$. Corresponding to τ_*, we recover in this way the familiar *strong*-topology* (s^*), with seminorms [*1*] [*16*]:

$$\| \mathrm{A} \|_f^* = \| \mathrm{A}f \| + \| \mathrm{A}^+ f \|, \qquad f \in \mathscr{D}. \tag{5.4}$$

Other familiar topologies on $\mathfrak{C}(\mathscr{D})$ [*8*] [*24*] are:

. the *strong* topology (s), with seminorms $\| \mathrm{A} \|_f = \| \mathrm{A}f \|$, $f \in \mathscr{D}$

. the *weak* topology (w), with seminorms $p_{f,g}(\mathrm{A}) = | \langle g, \mathrm{A}f \rangle |$, $f, g \in \mathscr{D}$.

Those four topologies on $\mathfrak{C}(\mathscr{D})$ are ordered as follows:

$$w \prec s \prec s^* \prec \tau_* \tag{5.5}$$

The results obtained in [*14*] about quasi-uniform topologies may be resumed as follows:

i) $\mathfrak{C}(\mathscr{D})$ is complete in τ_* and *a fortiori* in s^* (this was shown independently in [*16*]).

ii) Spaces of multipliers in $\mathfrak{C}^w(\mathscr{D})$ are complete in their natural topologies: $R^w\mathfrak{N}$ in $\tau_f^l(\mathfrak{N})$ and $\tau_*^l(\mathfrak{N})$, $L^w\mathfrak{N}$ in $\tau_f^r(\mathfrak{N})$ and $\tau_*^r(\mathfrak{N})$, and, for $\mathfrak{N}=\mathfrak{N}^+$, $M^w\mathfrak{N}$ in $\tau_f(\mathfrak{N})$ and $\tau_*(\mathfrak{N})$.

iii) Spaces of left or right multipliers in $\mathfrak{C}^s(\mathscr{D})$ need *not* be complete in the corresponding topologies $\tau_{f,*}^{l,r}(\mathfrak{N})$. However, it follows from the proof of [*14*, Prop. 5.7] that the spaces $R^M\mathfrak{N}$ and $\tilde{L}^M\mathfrak{N}$ of *mixed* multipliers of $\mathfrak{N}$ are complete in $\tau_{f,*}^l(\mathfrak{N})$, resp. $\tau_{f,*}^r(\mathfrak{N})$, but the other ones $\tilde{R}^M\mathfrak{N}$ and $L^M\mathfrak{N}$ need not be. These facts will be useful in Part II.

iv) The partial multiplication $(A, B) \mapsto A \,\square\, B$ is separately continuous from $L^w(\mathfrak{N})[\tau_*^r(\mathfrak{N})] \times \mathfrak{N}[\tau_*^l(L^w\mathfrak{N})]$ into $\mathfrak{C}^w(\mathscr{D})[\tau_*]$. The corresponding result is true for $(A, B) \mapsto A.B$ in $\mathfrak{C}^s(\mathscr{D})$ and also for the topologies τ_f.

REMARK 5.1. — All these topologies may be defined on $\mathscr{C}(\mathscr{D}, \mathscr{H})$ as well, as was done for s^* in [*16*], but then they are *not* Hausdorff. Taking the quotient of $\mathscr{C}(\mathscr{D}, \mathscr{H})$ modulo $\sim$, as explained in Sec. 3, corresponds precisely to the construction of Hausdorff spaces associated to $\mathscr{C}(\mathscr{D}, \mathscr{H})$, i. e. $\mathscr{C}_0(\mathscr{D}, \mathscr{H})$ with the appropriate topology.

5.B. Weak topologies.

The quasi-uniform topologies introduced in 5.A on subsets $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$ are very natural, but they have two drawbacks: they are specifically related to a given subset and they are very unwieldy in practice. For that reason we shall introduce another class of (locally convex) topologies, called generically *weak* topologies. They generalize the one introduced under the same name in [*24*] and another one used in [*16*]. They will be used systematically in Part II.

Let $\mathscr{K}$ be a dense subset of $\mathscr{H}$. Then we define the following topologies on $\mathfrak{C}(\mathscr{D})$:

. the $\mathscr{K}$*-weak topology* ($\mathscr{K}$-*w*), with seminorms:

$$p_{k,f}(A) = |\langle k, Af \rangle|, \qquad k \in \mathscr{K}, \qquad f \in \mathscr{D} \tag{5.6}$$

. the $\mathscr{K}$*-weak *-topology* ($\mathscr{K}$-*w**) with seminorms:

$$p_{k,f}^*(A) = |\langle k, Af \rangle| + |\langle k, A^+ f \rangle|, \qquad k \in \mathscr{K}, \qquad f \in \mathscr{D} \tag{5.7}$$

Among these one finds familiar cases:

. the *weak* topology (*w*), also called $\mathscr{D}$-weak by [*11*]:

$$w \equiv \mathscr{D}\text{-}w = \mathscr{D}\text{-}w^*$$

. the *quasi-weak* topology: $qw \equiv \mathscr{H}\text{-}w$ [*24*]

. the *quasi-weak* *-topology: $qw^* \equiv \mathscr{H}\text{-}w^*$.

Several relationships between these topologies are immediate:

i) $\mathscr{K}_1 \subset \mathscr{K}_2$ implies $\mathscr{K}_1\text{-}w \prec \mathscr{K}_2\text{-}w$, $\mathscr{K}_1\text{-}w^* \prec \mathscr{K}_2\text{-}w^*$.

ii) In particular, for $\mathscr{D} \subset \mathscr{K} \subset \mathscr{H}$:

$$w \prec \mathscr{K}\text{-}w \begin{array}{c} \nearrow\, qw \,\searrow \\ \searrow\, \mathscr{K}\text{-}w^* \,\nearrow \end{array} qw^* \prec s^* . \tag{5.8}$$

Among those topologies we will use in particular the so-called $\mathfrak{N}$-weak topologies, denoted $\mathfrak{N}$-w or $\mathfrak{N}$-w^*, corresponding to subsets of the form:

$$\mathscr{K} = \mathfrak{N}\mathscr{D} = \{ \mathrm{A}f \mid \mathrm{A} \in \mathfrak{N},\ f \in \mathscr{D} \}$$

where $\mathfrak{N}$ is a subset of $\mathfrak{C}(\mathscr{D})$ containing I (hence $\mathscr{K} \supset \mathscr{D}$ and thus $w \prec \mathfrak{N}\text{-}w \prec qw$). The commutant topology of [*16*] corresponds to $\mathfrak{N}=\mathfrak{A}'_\sigma$, the weak unbounded commutant of a subset $\mathfrak{A} \subset \mathfrak{C}(\mathscr{D})$. Such topologies will be used in Part II to prove statements of the type: 'the bicommutant $\mathfrak{A}''$ is the closure of $\mathfrak{A}$ in some topology τ' (since there are many types of unbounded bicommutants, many different topologies will be needed). Thus we postpone all results of this sort to Part II.

To conclude, we note the following relation among all topologies we have introduced. On a given set $\mathfrak{N} \subset \mathfrak{C}(\mathscr{D})$, for any $\mathscr{K}(\mathscr{D} \subset \mathscr{K} \subset \mathscr{H})$ and any $\mathscr{L} \subset \mathrm{L}^w\mathfrak{N}$, $\mathscr{R} \subset \mathrm{R}^w\mathfrak{N}$, one has:

$$w \begin{array}{c} \nearrow\, \mathscr{K}\text{-}w \prec \mathscr{K}\text{-}w^* \,\searrow \\ \searrow\, \quad s \quad \nearrow \end{array} s^* \prec \tau_* \begin{array}{c} \nearrow\, \tau^l_*(\mathscr{L}) \\ \searrow\, \tau^r_*(\mathscr{R}) \end{array} \tag{5.9}$$

REMARK 5.3. — Other types of topologies may be introduced on $\mathfrak{C}(\mathscr{D})$. For instance one may consider $\mathfrak{C}(\mathscr{D})$ as a set of continuous linear maps from $\mathscr{D}$ or $\mathscr{D}(\mathfrak{C})$ into $\mathscr{H}$, with the (strong) topology inherited from $\mathscr{L}(\mathscr{D}, \mathscr{H})$, and similarly for $\mathfrak{N}$. More precisely, one may describe the continuity properties of the . multiplication in $\mathfrak{C}^s(\mathscr{D})$ by making the following identifications:

$$\mathfrak{N} \subset \mathscr{L}(\mathscr{D}(\mathfrak{N}), \mathscr{H})$$
$$\mathrm{R}^s\mathfrak{N} \subset \mathscr{L}(\mathscr{D}, \mathscr{D}(\mathfrak{N}))$$
$$\mathfrak{N}\,.\,\mathrm{R}^s\mathfrak{N} \subset \mathfrak{C}(\mathscr{D}) \subset \mathscr{L}(\mathscr{D}, \mathscr{H})$$

where each space of continuous maps on the r. h. s. is given its strong topology. Then it was proven in [*13*] that the . multiplication $\mathfrak{N} \times \mathrm{R}^s\mathfrak{N} \to \mathfrak{N}\,.\,\mathrm{R}^s\mathfrak{N}$ is jointly sequentially continuous and separately continuous. The same result holds true in $\mathfrak{C}^w(\mathscr{D})$ for the $\square$ multiplication $\mathfrak{N} \times \mathrm{R}^w\mathfrak{N} \to \mathfrak{N} \,\square\, \mathrm{R}^w\mathfrak{N}$, but here the space used in the definition of the topologies is $\mathscr{D}_*(\mathfrak{N})$, not $\mathscr{D}(\mathfrak{N})$, since $\mathrm{B} \in \mathrm{R}^w(\mathrm{A})$ means $\mathrm{B}\mathscr{D} \subset \mathrm{D}(\mathrm{A}^{+*}) \subset \mathscr{D}_*(\mathfrak{N})$.

For Op*-algebras, Arnal and Jurzak [*25*], and Inoue *et al.* [*24*] have defined several kinds of topologies, which generalize in a more direct fashion the familiar topologies on von Neumann algebras [*1*]. In particular, [*24*] contains an exhaustive comparison of all of these. Clearly these new topologies have straightforward generalizations to partial Op*-algebras, but we will not pursue that point, since those extensions will not be needed for the study of (bi)-commutants developed in Part II.

APPENDIX

NON-DISTRIBUTIVITY OF $\mathfrak{C}^s(\mathscr{D})$

Kürsten has given in [17] a counterexample to the distributivity property of $\mathfrak{C}^s(\mathscr{D})$, namely three operators A, B, $C \in \mathfrak{C}(\mathscr{D})$ such that $C \in R^s(A) \cap R^s(B)$, but $C \notin R^s(A \hat{+} B)$. In this appendix we generalize that example substantially, so as to make it essentially generic.

As in [17], we work with sequence spaces: $\mathscr{H} = \ell^2$, with $\{ e_{(n)} \}$ the canonical unit vector basis, and $\mathscr{D} = s$, the Schwartz space of rapidly decreasing sequences. However this last choice is essentially irrelevant: almost *any* dense subspace of ℓ^2 would give the same result (see below). We take an arbitrary vector $f \in \ell^2 \backslash s$, with all components non-zero:

$$f = \sum_{n=1}^{\infty} f_n e_{(n)}, \qquad f_n \neq 0 \qquad \forall n \in \mathbb{N}.$$

and, for each $n = 1, 2, \ldots$, define a new vector χ_n:

$$\chi_n = (f_n)^{-1} e_{(n)} - (f_{n+1})^{-1} e_{(n+1)}. \tag{A.1}$$

We have immediately:

$$\| \chi_n \|^2 = | f_n |^{-2} + | f_{n+1} |^{-2} \tag{A.2}$$

and, for $k \neq l$:

$$\langle \chi_{2k}, \chi_{2l} \rangle = \langle \chi_{2k+1}, \chi_{2l+1} \rangle = 0. \tag{A.3}$$

With help of those vectors, we define two operators A_0, B_0 on s:

$$A_0 = \sum_{k=1}^{\infty} \langle \chi_{2k}, \cdot \rangle \chi_{2k} \tag{A.4a}$$

$$B_0 = \sum_{k=1}^{\infty} \langle \chi_{2k-1}, \cdot \rangle \chi_{2k-1}. \tag{A.4b}$$

The following properties are immediate:

i)
$$\langle \chi_n, f \rangle = 0, \qquad \text{for all} \quad n \in \mathbb{N}. \tag{A.5}$$

ii) for all $m \in \mathbb{N}$, one has:

$$A_0^m = \sum_{k=1}^{\infty} \| \chi_{2k} \|^{2m-2} \langle \chi_{2k}, \cdot \rangle \chi_{2k} \tag{A.6a}$$

$$B_0^m = \sum_{k=1}^{\infty} \| \chi_{2k-1} \|^{2m-2} \langle \chi_{2k-1}, \cdot \rangle \chi_{2k-1}. \tag{A.6b}$$

iii) A_0 and B_0 leave s invariant and are essentially self-adjoint on s. Defining $A = \overline{A}_0$,

$B = \overline{B}_0$, we obtain two self-adjoint operators, both belonging to $\mathfrak{C}(s)$. We observe that, for all $m, n \in \mathbb{N}$, $f \in D(A^m) \cap D(B^n)$ and $A^m f = B^n f = 0$.

Indeed, consider truncated sequences $f^{(J)} = \sum_{n=1}^{J} f_n e_{(n)}$.

Then for $J = 2L + 1$, $A^m f^{(2L+1)} = 0$, $\forall m \in \mathbb{N}$. Hence, $\{ f^{(2L+1)}, L = 0, 1, 2 \dots \}$ is a Cauchy sequence for the graph norm of A^m and $f^{(2L+1)} \to f$, hence $f \in D(A^m)$ and $A^m f = 0$. Similarly for B^n, with the sequence $\{ f^{(2L)} \}$.

Finally, we consider the rank one operator $C = \langle e_{(1)}, . \rangle f$. Its adjoint C^* has also rank one, and so both belong to $\mathfrak{C}(s)$. Furthermore, C is a right multiplier of both A^m and B^n (in fact, $A^m C y = B^n C y = 0$, $\forall y \in s$). We claim that $C \notin R^s(A^{2m} \hat{+} B^{2m})$. Clearly it is enough to show that $f \notin D(A^{2m} \hat{+} B^{2m})$, $\forall m \in \mathbb{N}$.

For that purpose we define, following [17], two auxiliary operators:

$$Ry = \left[y_1/f_1 + \frac{1}{2} \langle \tilde{f}, Ay \rangle \right] e_{(1)}, \qquad y \in D(A) \tag{A.7a}$$

$$Sz = \frac{1}{2} \langle \tilde{f}, Bz \rangle e_{(1)}, \qquad z \in D(B) \tag{A.7b}$$

where $\tilde{f} = \sum_{n=1}^{\infty} (-1)^{n+1} f_n e_{(n)}$. Then a straightforward estimate shows that:

$$\| Ry \|^2 \leqslant K(\| y \|^2 + \| Ay \|^2) = K \| y \|_A^2$$

where $K = \max (| f_1 |^{-2}, \| f \|^2)$ and $\| . \|_A$ is the graph norm of A. More generally (using the self-adjointness of A), we get:

$$\| Ry \| \leqslant \text{const} \ \| y \|_{A^m}, \qquad \text{for all} \quad m \in \mathbb{N} . \tag{A.8}$$

In the same way:

$$\| Sz \| \leqslant \text{const} \ \| z \|_{B^n}, \qquad \text{for all} \quad n \in \mathbb{N} . \tag{A.9}$$

The operators R, S do not belong to $\mathfrak{C}(s)$. In fact they are not even closable. Indeed $\tilde{f} \notin D(A^*) = D(A)$ since

$$\| A\tilde{f} \|^2 = \sum_{n=1}^{\infty} | f_n |^{-2} = \infty ,$$

and also $\tilde{f} \notin D(B^*) = D(B)$. Therefore $h \in D(R^*)$ or $D(S^*)$ iff $\langle h, e_{(1)} \rangle = 0$, i.e.

$$D(R^*) = D(S^*) = \{ e_{(1)} \}^{\perp} ,$$

a subspace of codimension one, certainly not dense in ℓ^2. On the other hand, R and S coincide on s, yet $R \neq S$. In particular:

$$Rf = e_{(1)}$$
$$Sf = 0 \tag{A.10}$$

Assume now that $f \in D(A^{2m} \hat{+} B^{2m})$. This means, there is a sequence $\{ h_n \} \in s$ such that $h_n \to f$ and $(A^{2m} + B^{2m}) h_n \to (A^{2m} \hat{+} B^{2m}) f$. But this implies that $\{ h_n \}$ is a Cauchy sequence in the graph norms of A^m and of B^m, and therefore $f \in D(A^m) \cap D(B^m)$ and $A^m h_n \to A^m f = 0$, $B^m h_n \to B^m f = 0$. Using now (A.8-9) with $y = z = h_n - f$, we get:

$$Rf = \lim_n Rh_n = \lim_n Sh_n = Sf$$

(since $Rh_n = Sh_n$), but this contradicts the relations (A.10). The conclusion is that

$f \notin D(A^{2m} \hat{+} B^{2m})$ and therefore C is not a right multiplier for $A^{2m} \hat{+} B^{2m}$, although it is one for A^{2m} and B^{2m} separately. Thus $\mathfrak{C}^s(s)$ is not distributive!

Now a careful reading of the proof shows that the specific properties of s have not been used, except in the fact that $s \subset D(A_0^m) \cap D(B_0^m)$, for all $m \in \mathbb{N}$. In fact the method may be adapted to almost any domain $\mathcal{D} \subset \ell^2$ containing the finite sequences. Given such a $\mathcal{D}$, we choose a vector $f \in \ell^2 \backslash \mathcal{D}$ and construct operators A_0, B_0 as above, Eqs. (A.4 *a-b*). A rough estimate shows that:

$$\| A_0^2 y \|^2 + \| B_0^2 y \|^2 \leqslant \text{const} \sum_{n=1}^{\infty} | \hat{f}_n |^{-8} | y_n |^2 \qquad \text{(A.11)}$$

where $\hat{f}_1 = \min(f_1, f_2)$, $\hat{f}_n = \min(f_{n-1}, f_n, f_{n+1})$ for $n \geqslant 2$. Thus $\mathcal{D} \subset D(A_0^2) \cap D(B_0^2)$ if one has

$$\sum_{n=1}^{\infty} | \hat{f}_n |^{-8} | y_n |^2 < \infty, \qquad \forall y \in \mathcal{D} \qquad \text{(A.12)}$$

Conversely, given $f \in \ell^2$, the argument above works for any domain $\mathcal{D}$ which verifies the condition (A.12). Notice that f itself cannot satisfy it, so that $f \notin \mathcal{D}$. For such a domain $\mathcal{D}$, the whole discussion may be repeated verbatim, with the conclusion that $\mathfrak{C}^s(\mathcal{D})$ violates distributivity.

For a general Hilbert space $\mathcal{H}$, with dense domain $\mathcal{D}$, one may perform the same analysis by choosing an orthonormal basis $\{ e_{(n)} \}$ contained in $\mathcal{D}$. Again, given $f \in \mathcal{H} \backslash \mathcal{D}$, the argument works if $\mathcal{D}$ satisfies the condition (A.12), where $y_n = \langle e_{(n)}, y \rangle$, etc., and then distributivity again breaks down for $\mathfrak{C}^s(\mathcal{D})$. Clearly this is the generic situation.

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