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Conformal invariance and time decay for non linear wave equations. II

by

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ABSTRACT. — We continue the study of the non linear wave equation

$$\Box\varphi + f(\varphi) = 0 \tag{*}$$

and of the time decay of its solutions initiated in a previous paper [4]. Using the approximate conformal invariance of the equation (*) we prove in particular that for $n = 3, 4$ and for $f(\varphi) = \varphi\,|\,\varphi\,|^{p-1}$ with $p_2(n) < p < 1 + 4/(n-2)$, $p_2(3) \simeq 2.686$, $p_2(4) \simeq 2.165$, all solutions of (*) with finite energy and finite conformal charge at time zero decay in time according to

$$\|\,\varphi(t)\,\|_l \leqslant C\,|\,t\,|^{-(n-1)(1/2-1/l)}$$

for $2 \leqslant l \leqslant 2n/(n-2)$.

RÉSUMÉ. — On continue l'étude de l'équation d'onde non linéaire

$$\Box\varphi + f(\varphi) = 0 \tag{*}$$

et de la décroissance en temps de ses solutions commencée dans un article précédent [4]. En utilisant l'invariance conforme approchée de l'équation (*), on démontre en particulier que pour $n = 3, 4$ et pour

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$f(\varphi)=\varphi|\varphi|^{p-1}$ avec $p_2(n)<p<1+4/(n-2)$, $p_2(3)\simeq 2.686$, $p_2(4)\simeq 2.165$, toutes les solutions de (*) d'énergie finie et de charge conforme finie au temps zéro décroissent en temps comme

$$\|\varphi(t)\|_l \leqslant C\,|t|^{-(n-1)(1/2-1/l)}$$

pour $2 \leqslant l \leqslant 2n/(n-2)$.

1. INTRODUCTION

This short note is a sequel to a previous paper with the same title ([*4*], hereafter referred to as I) where we studied the implications of the approximate conformal invariance of the non linear wave equation

$$\square\varphi + f(\varphi) = 0 \qquad (1.1)$$

where φ is a complex function defined in space-time $\mathbb{R}^{n+1}$, $\square \equiv \partial^2/\partial t^2 - \Delta$, Δ is the Laplace operator in $\mathbb{R}^n$, and f is a non linear complex valued function, a typical form of which is the sum of two powers

$$f(\varphi) = \lambda_1\varphi\,|\,\varphi\,|^{p_1-1} + \lambda_2\varphi\,|\,\varphi\,|^{p_2-1} \qquad (1.2)$$

with

$$1 \leqslant p_1 \leqslant p_2 \begin{cases} < 1 + 4/(n-2) & \text{if} \quad n \geqslant 3 \\ < \infty & \text{if} \quad n \leqslant 2 . \end{cases} \qquad (1.3)$$

We refer to I for a general introduction and a comprehensive bibliography. We shall use freely the notation and results of I. Item q (equation, proposition, etc.) in Section p of I will be quoted as (I.p.q) and reference [q] of I as [I.q]. It is known [I.6] [I.7] that the Cauchy problem for the equation (1.1) with initial data $(\varphi(t_0), \dot\varphi(t_0)) = (\varphi_0, \psi_0)$ at time t_0 in the energy space $X_e \equiv H^1 \oplus L^2$ has a unique solution $(\varphi, \dot\varphi) \in \mathscr{C}(\mathbb{R}, X_e)$, under assumptions on f which reduce to $\lambda_2 \geqslant 0$ and to (1.3) in the special case (1.2). In I we proved that, under an additional mild assumption on f (see assumption (I.A.3), especially (I.2.27)) the solutions associated with initial data (φ_0, ψ_0) in a smaller space $\Sigma = \Sigma_1 \oplus \Sigma_0$ (see (I.2.10)-(I.2.13)) satisfy the approximate conservation law associated with the approximate conformal invariance of the equation (1.1) (see Proposition I.2.3, especially (I.2.32)). This led us to define the conformal charge (I.2.7), which is the sum of a free or kinetic part (I.2.8) and of an interaction or potential part (I.2.9). Under additional assumptions on the interaction, the conservation law implies that the kinetic part of the conformal charge is uniformly bounded in time (or logarithmically bounded in time in certain cases with $n = 2$) for solutions with initial data in Σ (see Propositions I.4.1, I.4.2 and I.4.3). When combined with direct estimates involving the kinetic part of the conformal charge (see Propositions I.3.2, I.3.3 and I.3.4), that result implies fairly strong time decay properties of the corresponding solutions. The assumptions required on f to ensure those results include a repulsivity condition (see assumption (I.A.4), especially (I.4.1)) which reduces to $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$ in the special case (1.2), and a condition which amounts to a lower bound on p_1. The results follow readily from the conservation law in the favourable case where

$$p_1 \geqslant 1 + 4/(n-1)$$

(see Proposition I.4.1). For smaller values of p_1, the situation is more complicated, but we devised a method which should allow us to extend those results to the range $p_1 > p_2(n)$ for $n \geqslant 2$, where $p_2(n)$ is the larger root of the quadratic equation (I.1.9) and satisfies

$$1 + 4/n < p_2(n) < 1 + 4/(n-1)$$

for all $n \geqslant 2$. We applied that method to derive the expected results in the range $p_1 > p_2(n)$ in space dimension $n = 2$ (see Proposition I.4.2). For higher dimensions, we gave only a preliminary application of that method and derived the expected results for $n = 3$ in the case of a single power interaction with p_1 satisfying a lower bound strictly stronger than $p_2(3)$.

The purpose of this paper is to extend the latter results to space dimensions $n = 3$ and $n = 4$ under assumptions on f that reduce to the repulsivity condition $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$ and to

$$p_2(n) < p_1 \leqslant p_2 \leqslant 1 + 4n/[(n-2)(n+1)] \tag{1.4}$$

in the special case (1.2). Of course, in the case of one single power $p_1 = p_2 = p$, that result together with Proposition I.4.1 allows us to cover the whole interval

$$p_2(n) < p < 1 + 4/(n-2).$$

The method is basically the same as that used in I, but its implementation requires more elaborate estimates involving homogeneous Besov spaces. We expect the remaining restriction to $n \leqslant 4$ to be of a technical nature, as well as the unnatural upper limit in (1.4) which replaces the natural upper limit in (1.3). Actually the present version of the method yields some results for $n \geqslant 5$, but the increase in technical complexity seems to be out of proportion with their present interest. As in the case $n = 2$ treated in I, the method consists in first extracting some time decay of the L^{p_1+1}-norm of the solutions from the conservation laws (see Lemma I.4.1, especially (I.4.8)). One then substitutes that decay into the integral equation (I.4.19) associated with the equation (1.1) and one derives additional (stronger) time decay therefrom. One rewrites the free conformal charge as the sum of the squared L^2-norms of the functions $\Phi_A = A\varphi$ (A = L, M or D) (see (I.4.12)-(I.4.14)), one derives integral equations for those functions (see (I.4.17) and (I.4.18)), and one estimates those functions by substituting the decay previously obtained for the solution of (1.1) into those equations. That program will be carried out in Section 2.

In addition to the tools and notation used in I, we shall use the homogeneous Besov spaces of arbitrary order and the associated Sobolev inequalities, for which we refer to [1] [8] and to the appendix of [I.6]. We use the notation $\dot{B}_l^\rho \equiv \dot{B}_{l,2}^\rho(\mathbb{R}^n)$ for those spaces. For any interval I and any Banach space B, for any q, $1 \leqslant q \leqslant \infty$, we denote by $L^q(I, B)$ the space of measurable functions φ from I to B such that $\| \varphi(.) ; B \| \in L^q(I)$. For any l, $1 \leqslant l \leqslant \infty$, we denote by $\bar{l}$ the conjugate exponent: $1/l + 1/\bar{l} = 1$, and we define the variables $\alpha(l)$, $\beta(l)$, $\gamma(l)$ and $\delta(l)$ by

$$\alpha(l) = 2\beta(l)/(n+1) = \gamma(l)/(n-1) = \delta(l)/n = 1/2 - 1/l .$$

2. BOUNDEDNESS OF THE CONFORMAL CHARGE AND TIME DECAY

We begin our derivation with the extraction of some time decay of the solutions of the equation (1.1). We restrict our attention to space dimension $n \geqslant 3$ since the case $n = 2$ is adequately covered by Proposition I.4.2. We need a number of assumptions on the interaction f, which we list below. Most of them already appeared in I. The seemingly erratic numbering of the assumptions results from the requirement that identical assumptions have the same number in both papers.

(A.1′) $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$, $f(0) = 0$ and f satisfies the estimate

$$| f'(z) | \leqslant C(| z |^{p_1 - 1} + | z |^{p_2 - 1}) \tag{2.1}$$

for some p_1, p_2 with $1 \leqslant p_1 \leqslant p_2 < 1 + 4/(n-2)$ and for all $z \in \mathbb{C}$.

(A.2.a) There exists a function $V \in \mathscr{C}^1(\mathbb{C}, \mathbb{R})$ such that $V(0) = 0$, $V(z) = V(|z|)$ for all $z \in \mathbb{C}$, and $f(z) = \partial V/\partial \bar{z}$.

(A.4′) The function V defined in the assumption (A.2.a) satisfies

$$C | z |^{p_1 + 1} \leqslant (p_1 + 1)V(z) \leqslant 2 \operatorname{Re} \bar{z} f(z) \tag{2.2}$$

for some $p_1 \geqslant 1$, some $C > 0$ and for all $z \in \mathbb{C}$.

We take the same p_1 in (A.1′) and (A.4′) for simplicity. The assumption (A.4′) combines the assumption (I.A.4) and the condition (I.4.7) (see (I.4.9)). We are interested only in the case where

$$p_1 \leqslant 1 + 4/(n-1) \tag{2.3}$$

since the simpler opposite case is adequately covered by Proposition I.4.1.

We recall that under the assumptions (A.1′), (A.2.a), (A.4′) and (2.3), all solutions of the equation (1.1) with initial data in Σ satisfy the conclusions of Lemma I.4.1 and in particular the estimates (I.4.5), (I.4.6) and (I.4.8). In order to obtain an improved time decay for those solutions, we need the following elementary variation of Gronwall's inequality.

LEMMA 2.1. — Let $0 \leqslant \gamma < 1$, $\gamma + \delta > 1$, and let $g \in \mathscr{C}(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$g(t) \leqslant (1 + t)^{-\gamma} + a \int_0^t d\tau \,|\, t - \tau \,|^{-\gamma}(1 + \tau)^{-\delta} g(\tau)\,. \tag{2.4}$$

Then g satisfies the estimate

$$g(t) \leqslant C(1 + t)^{-\gamma} \tag{2.5}$$

for some C (depending only on a, γ, δ) and all $t \in \mathbb{R}^+$.

Proof. — The function $h(t) \equiv g(t)(1 + t)^{\gamma}$ satisfies the inequality

$$h(t) \leqslant 1 + a(1 + t)^{\gamma} \int_0^t d\tau \,|\, t - \tau \,|^{-\gamma}(1 + \tau)^{-(\gamma+\delta)} h(\tau)\,. \tag{2.6}$$

Let $0 < \varepsilon < 1$. We split the region of integration in (2.6) into the subregions $(1+\tau) \leqslant (1-\varepsilon)(1+t)$, or equivalently $(t-\tau) \geqslant \varepsilon(1+t)$, and $(1+\tau) \geqslant (1-\varepsilon)(1+t)$ and obtain

$$\begin{aligned} h(t) \leqslant 1 &+ a\varepsilon^{-\gamma} \int_0^t d\tau (1 + \tau)^{-(\gamma+\delta)} h(\tau) \\ &+ a(1 - \varepsilon)^{-(\gamma+\delta)}(1 + t)^{-\delta} \int_0^{\varepsilon(1+t)} d\sigma \sigma^{-\gamma} h(t - \sigma) \end{aligned}$$

so that the functions $h(t)$ and

$$\bar{h}(t) = \operatorname*{Sup}_{0 \leqslant \tau \leqslant t} h(\tau)$$

satisfy

$$\begin{aligned} h(t) \leqslant 1 &+ a\varepsilon^{-\gamma} \int_0^t d\tau (1 + \tau)^{-(\gamma+\delta)} \bar{h}(\tau) \\ &+ a(1 - \gamma)^{-1} \varepsilon^{1-\gamma}(1 - \varepsilon)^{-(\gamma+\delta)}(1 + t)^{-(\gamma+\delta-1)} \bar{h}(t) \end{aligned}$$

and therefore, for ε sufficiently small

$$\bar{h}(t) \leqslant c + ca\varepsilon^{-\gamma} \int_0^t d\tau (1 + \tau)^{-(\gamma+\delta)} \bar{h}(\tau) \tag{2.7}$$

with

$$c^{-1} = 1 - a(1 - \gamma)^{-1} \varepsilon^{1-\gamma}(1 - \varepsilon)^{-(\gamma+\delta)}\,.$$

The result follows from (2.7) as in the standard proof of Gronwall's inequality. Q. E. D.

We also recall the basic estimate [2] [7]

$$\| \exp(i\omega t)\varphi\,;\, \dot{\mathrm{B}}_l^{-\beta(l)} \| \leqslant \mathrm{C} \,|\, t \,|^{-\gamma(l)} \| \varphi\,;\, \dot{\mathrm{B}}_{\bar{l}}^{\beta(l)} \| \tag{2.8}$$

which holds for all $t \in \mathbb{R}\backslash\{0\}$, $2 \leqslant l \leqslant \infty$ and $\varphi \in \dot{B}_l^{\beta(l)}$. We can now derive the following improved decay estimate of the solutions of the equation (1.1). We define $l_S = 2(n+1)/(n-1)$ and we denote by $\alpha_S, \beta_S, \gamma_S, \delta_S$ the associated quantities $\alpha(.)$, ... In particular $\alpha_S = 1/(n+1)$ and $\beta_S = 1/2$.

LEMMA 2.2. — Let $n \geqslant 3$. Let f satisfy the assumptions (A.1′), (A.2.a) and (A.4′) with p_1, p_2 satisfying (1.4). Let $0 \leqslant \rho \leqslant \alpha_S$. Let $t_0 \in \mathbb{R}$, let $(\varphi_0, \psi_0) \in \Sigma$ and let $(\varphi, \dot{\varphi})$ be the solution of the equation (1.1) in $\mathscr{C}(\mathbb{R}, \Sigma)$ with initial data (φ_0, ψ_0) at time t_0, as described in Proposition I.2.3. Then φ satisfies the following decay estimate

$$\| \varphi(t) ; \dot{B}^{\rho}_{l_S} \| \leqslant C(1 + |t|)^{-\gamma_S}. \tag{2.9}$$

Proof. — Without loss of generality, we take $t_0 = 0$ and we assume that p_1 and p_2 satisfy (2.3) and

$$p_2 - 1 = 4n/[(n-2)(n+1)]. \tag{2.10}$$

We estimate φ through the use of the integral equation (I.4.19), restricting our attention to positive times. The free term

$$\varphi^{(0)}(t) = \dot{K}(t)\varphi_0 + K(t)\psi_0$$

is estimated through the conservation of the energy and of the conformal charge (Proposition I.2.3 with $f = 0$) for the linear wave equation $\square\varphi = 0$. In fact, from energy conservation

$$\| \nabla\varphi^{(0)}(t) \|_2^2 \leqslant \| \nabla\varphi_0 \|_2^2 + \| \psi_0 \|_2^2 = E_0, \tag{2.11}$$

while from the conservation of the conformal charge

$$\| \varphi^{(0)}(t) \|_2^2 \leqslant 4(n-2)^{-2} Q_0 \tag{2.12}$$

by (I.3.27), with

$$\begin{aligned} Q_0 &= Q_0(t, \varphi^{(0)}(t), \dot{\varphi}^{(0)}(t)) = Q_0(0, \varphi_0, \psi_0) \\ &= \| r\psi_0 \|_2^2 + \| r\nabla\varphi_0 \|_2^2 - (n-1)\| \varphi_0 \|_2^2, \end{aligned}$$

and more generally for all l, $2 \leqslant l \leqslant 2^*$

$$\| \varphi^{(0)}(t) \|_l \leqslant C\,|t|^{-\gamma(l)} Q_0^{(1-\alpha(l))/2} E_0^{\alpha(l)/2} \tag{2.13}$$

by Proposition I.3.3. It follows now from the Sobolev inequalities that

$$\| \varphi^{(0)}(t) ; \dot{B}^{\rho}_{l_S} \| \leqslant C \| \varphi^{(0)}(t) \|_2^{1-\nu} \| \nabla\varphi^{(0)}(t) \|_2^{\nu} \tag{2.14}$$

with $\nu = \rho + \delta_S$, which satisfies $\nu \leqslant 1$ since $\rho \leqslant \alpha_S$, so that by (2.11) and (2.12) the left-hand side of (2.14) is uniformly bounded in time. On the other hand, it follows by interpolation from (2.11), (2.12), (2.13) with $l = 2^*$ and Lemma A.1 in the appendix that

$$\| \varphi^{(0)}(t) ; \dot{B}^{\rho}_{l_S} \| \leqslant C\,|t|^{-\gamma_S} \tag{2.15}$$

for $0 \leqslant \rho \leqslant \alpha_s$ and all $t \in \mathbb{R} \backslash \{ 0 \}$. From (2.15) and the previous remark, it follows that $\varphi^{(0)}$ satisfies the estimate

$$\| \varphi^{(0)}(t) ; \dot{B}^{\rho}_{l_s} \| \leqslant C(1 + |t|)^{-\gamma_s} \tag{2.16}$$

for $0 \leqslant \rho \leqslant \alpha_s$ and all $t \in \mathbb{R}$.

We next estimate the integral in (I.4.19). Using (2.8) with $l = l_s$, we obtain

$$\| K(t - \tau) f(\varphi) ; \dot{B}^{\rho}_{l_s} \| \leqslant C | t - \tau |^{-\gamma_s} \| f(\varphi) ; \dot{B}^{\rho}_{\bar{l}_s} \| . \tag{2.17}$$

The last norm is estimated by the use of the assumption (A.1′) and of Lemma 3.2 in [I.6] as

$$\| f(\varphi) ; \dot{B}^{\rho}_{\bar{l}_s} \| \leqslant C \| \varphi ; \dot{B}^{\rho}_{l_s} \| \sum_{i=1,2} \| \varphi \|_{l_i}^{p_i - 1} \tag{2.18}$$

with

$$l_i = (p_i - 1)(n + 1)/2$$

so that $2 < l_i \leqslant 2^*$ under the assumption (1.4), $l_2 = 2^*$ in the special case (2.10), and $l_i \gtreqless p_i + 1$ according to whether $p_i - 1 \gtreqless 4/(n - 1)$. We then estimate the last norms in (2.18) by Lemma I.4.1 and more precisely by interpolation between (I.4.8) and (I.4.10), so that

$$\| f(\varphi) ; \dot{B}^{\rho}_{\bar{l}_s} \| \leqslant C \| \varphi ; \dot{B}^{\rho}_{l_s} \| (1 + \tau)^{-\delta} \tag{2.19}$$

where $\delta = \mathrm{Min}(\delta_1, \delta_2)$, $\delta_i = (p_i - 1)\nu(l_i)$, and $\nu(l)$ is the decay exponent available from Lemma I.4.1 for the L^l-norm of φ. In the simpler case $l = l_2 = 2^*$, we obtain from (I.4.10) with $l = 2^*$

$$\begin{aligned} \delta_2 &= (p_2 - 1)(1 - \mu)(n - 1)/n \\ &= 4(n - 1)[(n - 2)(n + 1)]^{-1}((p_1 - 1)(n - 1)/2 - 1) . \end{aligned}$$

Since $p_1 > p_2(n) > 1 + 4/n$, this yields

$$\delta_2 > 4(n - 1)/[n(n + 1)]$$

so that

$$\gamma_s + \delta_2 - 1 > 2(n - 2)/[n(n + 1)] > 0 . \tag{2.20}$$

In the more delicate case $l = l_1$, we estimate φ in L^l by interpolation between (I.4.8) and (I.4.10) with $l = 2$, so that

$$\delta_1 = (p_1 - 1)\{ (\delta(l_1) - \delta(p_1 + 1))\mu + \delta(l_1)(2\delta(p_1 + 1) - 1) \} \delta(p_1 + 1)^{-1} .$$

Now the condition $p_1 > p_2(n)$ can be written equivalently as

$$\eta \equiv 1 - \mu - \delta(p_1 + 1) \equiv (1/2)(p_1(2\delta(p_1 + 1) - 1) - 1) > 0 . \tag{2.21}$$

Omitting the index 1 on p and l, we obtain

$$\begin{aligned} \delta_1 &= (p - 1)(\delta(p + 1) - \delta(l))(\eta\delta(p + 1)^{-1} - 1) \\ &\quad + (p - 1)(2\delta(p + 1) - 1) . \end{aligned}$$

Now

$$(p-1)(\delta(p+1)-\delta(l)) = 2(\delta_s - \delta(p+1))$$

so that, by using (2.21)

$$\delta_1 = 2\eta\delta_s\delta(p+1)^{-1} + 2 - 2\delta_s$$

and therefore, for $p = p_1 > p_2(n)$,

$$\gamma_s + \delta_1 - 1 = 2\eta\delta_s\delta(p_1+1)^{-1} > 0\,. \tag{2.22}$$

Collecting (2.16), (2.17) and (2.19) we obtain (for positive time)

$$\|\varphi(t)\,;\dot{\mathrm{B}}^{\rho}_{l_s}\| \leqslant \mathrm{C}(1+t)^{-\gamma_s} + \mathrm{C}\int_0^t d\tau\,|t-\tau|^{-\gamma_s}(1+\tau)^{-\delta}\|\varphi(\tau)\,;\dot{\mathrm{B}}^{\rho}_{l_s}\| \tag{2.23}$$

with $\gamma_s + \delta > 1$, by (2.20), (2.22). Lemma 2.2 now follows from (2.23) through Lemma 2.1. Q. E. D.

The second step in our derivation of the boundedness of the conformal charge consists in proving that the functions $\Phi_A = A\varphi$ (A = L, M or D) are uniformly bounded in L^2, by substituting the decay just obtained for φ into the integral equations (I.4.17) and (I.4.18). For that purpose, we need some additional information on the solutions of the linear wave equation with initial data in L^2 and on the propagator K.

LEMMA 2.3. — (1) For any (r, q) with $0 \leqslant 2/q = \gamma(r) < 1$, the operator $h \to e^{i\omega \cdot} h$ is bounded from L^2 to $L^q(\mathbb{R}, \dot{B}_r^{-\beta(r)})$.

(2) For any (r, q) and (r', q') with $0 \leqslant 2/q = \gamma(r) < 1$ and $0 \leqslant 2/q' = \gamma(r') < 1$, for any interval $I \subset \mathbb{R}$, for any $t_0 \in I$, the operator

$$h \to \int_{t_0}^t d\tau \mathrm{K}(t-\tau)h(\tau) \tag{2.24}$$

is bounded from $L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta(r)-1})$ to $L^{q'}(I, B_{r'}^{-\beta(r')})$ with norm uniformly bounded with respect to I and t_0.

Proof. — Part (1) follows from the basic estimate (2.8) by an argument which is well known by now (see for instance Lemma 2.3 in [*3*]) and the use of the Hardy-Littlewood-Sobolev inequality ([*5*], p. 117). Part (2) follows from Part (1) by the same argument as in the proof of the analogous result for the Schrödinger equation [*6*] [*9*]: let $t_0 = 0$ for simplicity. The result follows by interpolation from the fact that for any (r, q) with $0 \leqslant 2/q = \gamma(r) < 1$, the operator

$$h \to \int_0^t d\tau e^{i\omega(t-\tau)}h(\tau)$$

is bounded between the following pairs of spaces (with $\beta = \beta(r)$)

(1) from $L^1(I, L^2)$ to $L^\infty(I, L^2)$

(2) from $L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta})$ to $L^q(I, \dot{B}_r^{-\beta})$

(3) from $L^1(I, L^2)$ to $L^q(I, \dot{B}_r^{-\beta})$

(4) from $L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta})$ to $L^\infty(I, L^2)$.

Now the first case follows from the unitarity of $e^{i\omega\cdot}$ in L^2, the second case follows from the estimate (2.8) and the Hardy-Littlewood-Sobolev inequality, and the fourth case follows from the third one by duality. In order to prove the third case, we note that

$$\left\| \int_0^t d\tau e^{i\omega(t-\tau)}h(\tau) ; L^q(I, \dot{B}_r^{-\beta}) \right\| \leqslant \int_I d\tau \, \| e^{i\omega(\cdot-\tau)}h(\tau) ; L^q(I, \dot{B}_r^{-\beta}) \|$$

$$\leqslant C \int_I d\tau \, \| h(\tau) \|_2 = C \| h ; L^1(I, L^2) \|$$

by Part (1) of the lemma. Q. E. D.

The task of estimating the functions Φ_A can be performed in a reasonably simple way in space dimensions $n \leqslant 4$, one of the reasons being that for $n \leqslant 4$ the relevant values of p are larger than 2, namely

$$p_2(n) > 1 + 4/n \geqslant 2 .$$

An additional assumption is needed, which reinforces (A.1′) and which we state as follows

(A.1″) $f \in \mathscr{C}^2(\mathbb{C}, \mathbb{C})$, $f(0) = f'(0) = 0$, and f satisfies the estimate

$$| f''(z) | \equiv \mathrm{Max}\left(\left| \frac{\partial^2 f}{\partial z^2} \right|, \left| \frac{\partial^2 f}{\partial z \partial \bar{z}} \right|, \left| \frac{\partial^2 f}{\partial \bar{z}^2} \right| \right) \leqslant C(| z |^{p_1 - 2} + | z |^{p_2 - 2}) \quad (2.25)$$

for some p_1, p_2 with $2 \leqslant p_1 \leqslant p_2 < 1 + 4/(n-2)$ and for all $z \in \mathbb{C}$.

We can now state our main result.

PROPOSITION 2.1. — Let $n = 3$ or 4. Let f satisfy the assumptions (A.1″), (A.2.a) and (A.4′) with p_1 and p_2 satisfying (1.4). Let $t_0 \in \mathbb{R}$, let $(\varphi_0, \psi_0) \in \Sigma$ and let $(\varphi, \dot{\varphi})$ be the solution of the equation (1.1) in $\mathscr{C}(\mathbb{R}, \Sigma)$ with initial data (φ_0, ψ_0) at time t_0, as described in Proposition I.2.3. Then

(1) For any (r, q) with $0 \leqslant 2/q = \gamma(r) < 1$, the functions $\Phi_A = A\varphi$ ($A = L$, M or D) belong to $L^q(\mathbb{R}, \dot{B}_r^{-\beta(r)})$.

(2) $Q_0(t, \varphi, \dot{\varphi})$ is bounded uniformly in time and φ satisfies the estimate (I.4.2).

Proof. — Part (2) is an immediate consequence of Part (1) with $r = 2$, of (I.4.12) and of Proposition I.3.3. We concentrate our attention on Part (1). We estimate Φ_A through the use of the integral equations (I.4.17) and (I.4.18). We consider only the equation (I.4.17) which covers the cases A = L and A = M. The additional term in (I.4.18) for A = D can be estimated by similar methods. We take $t_0 = 0$ for simplicity. From the discussion preceding Lemma I.4.2, from that lemma and from Lemma 2.3, Part (1), it follows that the free term

$$\Phi_A^{(0)}(t) = \dot{K}(t)\Phi_A^{(0)} + K(t)\dot{\Phi}_A^{(0)}$$

in the equations (I.4.17) and (I.4.18) belongs to $L^q(\mathbb{R}, \dot{B}_r^{-\beta(r)})$ for the relevant values of (r, q).

We next consider the integral in (I.4.17) and we estimate the integrand. Without loss of generality, we assume that p_1 satisfies

$$p_1 - 1 < 3(n+1)/[n(n-1)], \tag{2.26}$$

a condition which is compatible with $p_1 > p_2(n)$ for $n = 3, 4$, and that p_2 satisfies (2.10). We decompose f as $f = f_1 + f_2$, with $|f_i''(z)| \leqslant C|z|^{p_i - 2}$, $i = 1, 2$, and we estimate the contributions of f_1 and f_2 to the integral in (I.4.17) separately, omitting the subscript i for simplicity. In order to be able to exploit Lemma 2.3, we pick r and r' (both depending on p) such that $0 \leqslant \gamma \equiv \gamma(r)$, $\gamma' \equiv \gamma(r') < 1$, and we estimate the norm of $f'\Phi_A$ in $\dot{B}_{\bar{r}}^{\beta-1}$ in terms of the norm of Φ_A in $\dot{B}_{r'}^{-\beta'}$ (where $\beta = \beta(r)$, $\beta' = \beta(r')$). This requires that the operator of multiplication by f' be bounded from $\dot{B}_{r'}^{-\beta'}$ to $\dot{B}_{\bar{r}}^{\beta-1}$, or equivalently, by duality, from $\dot{B}_r^{1-\beta}$ to $\dot{B}_{\bar{r}'}^{\beta'}$. By a straightforward extension of Lemma 3.2 of [I.6], we can estimate (for sufficiently regular u)

$$\| f'u ; \dot{B}_{\bar{r}'}^{\beta'} \| \leqslant C(\| u ; \dot{B}_m^{\beta'} \| \, \| f' \|_k + \| u \|_{m'} \| f' ; \dot{B}_{k'}^{\beta'} \|) \tag{2.27}$$

provided $0 \leqslant \beta' < 1$ and

$$1/\bar{r}' = 1/m + 1/k = 1/m' + 1/k' .$$

Estimating the last norm in (2.27) by Lemma 3.2 of [I.6] and using the Sobolev inequalities several times, we obtain from (2.27):

$$\| f'u ; \dot{B}_{\bar{r}'}^{\beta'} \| \leqslant C \| u ; \dot{B}_r^{1-\beta} \| \, \| \varphi ; \dot{B}_l^{\beta'} \|^{p-1}$$

provided $\beta + \beta' \leqslant 1$ and

$$(p-1)(n/l - \beta') = 1 + (\gamma + \gamma')/2 , \tag{2.28}$$

and therefore by duality

$$\| f'\Phi_A ; \dot{B}_{\bar{r}}^{\beta-1} \| \leqslant C \| \Phi_A ; \dot{B}_{r'}^{-\beta'} \| \, \| \varphi ; \dot{B}_l^{\beta'} \|^{p-1} . \tag{2.29}$$

We take $l = l_S$ and estimate the last norm in (2.29) by Lemma 2.2, thereby continuing (2.29) as

$$\ldots \leqslant C \| \Phi_A ; \dot{B}_{r'}^{-\beta'} \| (1 + |\tau|)^{-(p-1)\gamma_S} \tag{2.30}$$

provided $0 \leqslant \beta' \leqslant \alpha_S$. We choose r and r', depending on p, as follows: for $p = p_2$, we take $\beta' = \alpha_S$, $\beta + \beta' = 1$, which is compatible with $\gamma < 1$

for $n = 3, 4$, and for which (2.28) reduces to (2.10). For $p = p_1$ we take $\beta' = 0$, which is compatible with $\gamma < 1$ under the assumption (2.26). We next choose $\varepsilon > 0$ sufficiently small, we define r'' by $\beta'' \equiv \beta(r'') = \beta'/(1 - \varepsilon)$ (in particular $r' = r'' = 2$ for $p = p_1$) and we estimate the first norm in (2.30) by interpolation and by the use of (I.4.5), (I.4.12) as

$$\begin{aligned} \| \Phi_A ; \dot{B}_{r'}^{-\beta'} \| &\leqslant \| \Phi_A \|_2^{\varepsilon} \| \Phi_A ; \dot{B}_{r''}^{-\beta''} \|^{1-\varepsilon} \\ &\leqslant C \| \Phi_A ; \dot{B}_{r''}^{-\beta''} \|^{1-\varepsilon} (1 + |\tau|)^{\mu\varepsilon} . \end{aligned} \quad (2.31)$$

Let now I be an interval, let $2/q = \gamma(r)$ and $2/q'' = \gamma(r'')$. Using (2.30), (2.31), we can estimate

$$\begin{aligned} \| f'\Phi_A ; L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta-1}) \| \leqslant C \| \Phi_A ; L^{q''}(I, \dot{B}_{r''}^{-\beta''}) \|^{1-\varepsilon} \\ \times \| (1 + |\,.\,|)^{\mu\varepsilon - (p-1)\gamma_S} ; L^s(I) \| \end{aligned} \quad (2.32)$$

by Hölder's inequality, with

$$1/s = 1/\bar{q} - (1 - \varepsilon)/q'' = 1 - (\gamma + \gamma')/2 .$$

The last norm in (2.32) is bounded uniformly with respect to I provided

$$(p - 1)\gamma_S > \mu\varepsilon + 1 - (\gamma + \gamma')/2$$

or equivalently, by using (2.28) with $l = l_S$

$$(p - 1)(n/2 - \alpha_S - \beta') > 2 + \mu\varepsilon . \quad (2.33)$$

For $p = p_2$ given by (2.10) and correspondingly $\beta' = \alpha_S$, (2.33) reduces to

$$2n(n^2 + n - 4)/[(n + 1)^2(n - 2)] > 2 + \mu\varepsilon ,$$

easily seen to hold for $n = 3, 4$ (actually for all n) and ε sufficiently small. For $p = p_1$ and correspondingly $\beta' = 0$, (2.33) reduces to

$$p_1 - 1 > 2(2 + \mu\varepsilon)(n + 1)/[(n - 1)(n + 2)] ,$$

easily seen to hold for $n = 3, 4$, for $p_1 > p_2(n)$ and ε sufficiently small. In both cases, we obtain therefore

$$\| f'\Phi_A ; L^{\bar{q}}(I, \dot{B}_{\bar{r}}^{\beta-1}) \| \leqslant C \| \Phi_A ; L^{q''}(I, \dot{B}_{r''}^{-\beta''}) \|^{1-\varepsilon} \quad (2.34)$$

with a constant C independent of I.

We are now in a position to complete the proof of the proposition. Let $I = [-T, T]$, let (r, q) satisfy $0 \leqslant \gamma(r) = 2/q < 1$, $\gamma(r)$ close to 1, and define the space

$$X(I) = L^\infty(I, L^2) \cap L^q(I, \dot{B}_r^{-\beta(r)}) .$$

By interpolation, X(I) satisfies the continuous embedding

$$X(I) \subset L^{q'}(I, \dot{B}_{r'}^{-\beta(r')})$$

for all r' with $2 \leqslant r' \leqslant r$. By Lemma 2.2, $\Phi_A^{(0)} \in X(\mathbb{R})$ and the operator (2.24) is bounded from $L^{\bar{q}'}(I, \dot{B}_{\bar{r}'}^{\beta(r')-1})$ to X(I) for any (r', q') with $0 \leqslant \gamma(r') = 2/q' < 1$, with norm uniformly bounded with respect to I. From that property and from the estimates (2.34) corresponding to $p = p_1$ and $p = p_2$, we obtain the estimate

$$\| \Phi_A ; X(I) \| \leqslant \| \Phi_A^{(0)} ; X(\mathbb{R}) \| + C \| \Phi_A ; X(I) \|^{1-\varepsilon} \tag{2.35}$$

with a constant C independent of I. The final result is an immediate consequence of (2.35). Q. E. D.

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APPENDIX

In order to derive the Besov space estimate (2.15) by interpolation from the estimates (2.11), (2.12) and (2.13), we need first to replace the L^l-norm estimate (2.13) by a similar estimate for the $\dot{B}^0_l$-norm of φ. This is done through the following lemma.

LEMMA A.1. — Let $n \geqslant 3$, let $l \geqslant 2$, and assume that the following estimate holds

$$\| \exp(i\omega t)\varphi \|_l \leqslant C \, | t |^{-\gamma(l)} \| \nabla\varphi \|_2^{\alpha(l)} \| x\nabla\varphi \|_2^{1-\alpha(l)} \tag{A.1}$$

for all $\varphi \in \Sigma_1$. Then also the following estimate holds

$$\| \exp(i\omega t)\varphi \, ; \dot{B}^0_l \| \leqslant C \, | t |^{-\gamma(l)} \| \nabla\varphi \|_2^{\alpha(l)} \| x\nabla\varphi \|_2^{1-\alpha(l)} \tag{A.2}$$

for all $\varphi \in \Sigma_1$ (possibly with a different C).

Proof. — We use the definition of Besov spaces through a dyadic decomposition performed by using a (doubly infinite) sequence of functions, denoted $\{ \varphi_j \}$ in the Appendix of [I.6] and which we denote $\{ \theta_j \}$ here to avoid confusion with φ. In particular

$$\operatorname{Supp} \hat{\theta}_j \subset \{ \xi : 2^{j-1} \leqslant | \xi | \leqslant 2^{j+1} \} \tag{A.3}$$

and

$$\sum_{j \in \mathbb{Z}} \hat{\theta}_j(\xi) = 1$$

for all $\xi \neq 0$, where $\hat{}$ denotes the Fourier transform.

Let $\gamma = \gamma(l)$, $\alpha = \alpha(l)$ and let $\varphi \in \Sigma_1$. We apply (A.1) to the function $\theta_j * \varphi$ and use elementary commutation properties to obtain

$$\begin{aligned} \| \theta_j * \exp(i\omega t)\varphi \|_l &\leqslant C \, | t |^{-\gamma} \| \theta_j * \nabla\varphi \|_2^{\alpha} \| x(\theta_j * \nabla\varphi) \|_2^{1-\alpha} \\ &\leqslant C \, | t |^{-\gamma} \| \theta_j * \nabla\varphi \|_2^{\alpha} (\| \theta_j * x\nabla\varphi \|_2^{1-\alpha} + \| (x\theta_j) * \nabla\varphi \|_2^{1-\alpha}) \end{aligned} \tag{A.4}$$

Taking the l^2-norm of both members of (A.4), applying Hölder's inequality and using the fact that $\dot{B}^0_2 = L^2$, we obtain

$$\| \exp(i\omega t)\varphi \, ; \dot{B}^0_l \| \leqslant C \, | t |^{-\gamma} \| \nabla\varphi \|_2^{\alpha} (\| x\nabla\varphi \|_2^{1-\alpha} + \| h \, ; l^2 \|^{1-\alpha}) \tag{A.5}$$

where $h = \{ h_j \}$ is the sequence defined by

$$h_j = \| (x\theta_j) * \nabla\varphi \|_2 .$$

Now $\widehat{x\theta_j} = i\nabla_\xi \hat{\theta}_j$ so that $\widehat{x\theta_j}$ satisfies the same support property (A.3) as $\hat{\theta}_j$. In particular $x\theta_j = x\theta_j * \tilde{\theta}_j$ where $\tilde{\theta}_j = \theta_{j-1} + \theta_j + \theta_{j+1}$ and therefore

$$h_j = \| (x\theta_j) * \tilde{\theta}_j * \nabla\varphi \|_2 \leqslant \| x\theta_j \|_1 \| \tilde{\theta}_j * \nabla\varphi \|_2$$

by the Young identity. By homogeneity

$$\| x\theta_j \|_1 = C 2^{-j}$$

and therefore

$$\| h \, ; l^2 \| \leqslant C \| \nabla\varphi \, ; \dot{B}_2^{-1} \| \leqslant C \| \varphi \|_2 \leqslant C \| x\nabla\varphi \|_2 \tag{A.6}$$

where the last inequality is the Hardy inequality. Substituting (A.6) into (A.5) yields (A.2).

Q. E. D.

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