ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 48, n° 1 (1988), p. 1-16
<http://www.numdam.org/item?id=AIHPA_1988__48_1_1_0>

Ann. Inst. Henri Poincaré,
Vol. 48, n° 1, 1988, p. 1-16. *Physique théorique*

Magnetofluiddynamic simple waves in special relativity

by

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ABSTRACT. — A thorough study of magnetoacoustic and Alfvén simple waves is performed in the framework of special relativistic magnetofluiddynamics. In some cases all Riemann invariants have been found explicitly. General results concerning the breaking of magnetoacoustic simple waves have been obtained.

RÉSUMÉ. — On étudie les ondes simples, magnétoacoustique et de Alfvén, dans les cas de la magnétofluidodynamique en relativité restreinte.
On détermine les invariants de Riemann dans certains cas. Des résultats généraux sur la rupture des ondes simples magnétoacoustiques sont obtenus.

1. INTRODUCTION

Relativistic Magnetofluiddynamics (RMFD) is of interest in several areas of plasma physics and astrophysics.

Near relativistic shock speeds have been achieved in a laboratory plasma by using an electromagnetically driven shock tube [*1*].

In the field of Astrophysics RMFD can be important amongst other areas, in the theories of gravitational collapse, due to the amplification of frozen magnetic fields. Let R denote the ratio of the magnetic energy density $| b |^2$

to the total fluid energy-density e. Then from the equations of RMFD, it is possible to prove that [2] [3].

$$u^\alpha \nabla_\alpha \mathrm{R} = \frac{1}{e} b^\alpha b^\beta \sigma_{\alpha\beta} + \frac{1}{6} | b |^2 \frac{(e - 3p)}{e^2(e + p)} u^\alpha \nabla_\alpha e$$

u^α is the fluid's 4-velocity, $\sigma_{\alpha\beta}$ the fluid's shear. For an isotropic collapse $\sigma_{\alpha\beta} = 0$ and R increases if $p < e/3$ (a situation which certainly occurs in the early stages of gravitational collapse). Numerical calculations of gravitational collapse in the framework of RMFD, have already been performed [4].

Another interesting example arising from Astrophysics is that of the influence of the primeval magnetic fields on the formation of galaxies. A present intergalatic magnetic field of the order 10^{-9} G, if of primordial origin, could have had a significant effect at the time of recombination in generating critical density fluctuations on the scale of galaxies [5].

Still another noteworthy example is that of neutron stars [6], which have a superconducting core and a surface magnetic field of order 10^{-12} G. RMFD could be important in the study of the structure and the stability of a neutron star [7].

Finally we mention that RFMD effects might be essential ingredients in the physics of jets in extragalactic radiosources [8] as well as in the case of accretion discs around magnetized neutron stars or black holes [9].

In this article we study in detail a class of exact solutions of the equations of RMFD in special relativity.

These solutions represent the non-linear analogue of the plane waves for linear theories and are essential for the understanding of the process of shock formation (a situation likely to occur in many applications). Apart from their own intrinsic relevance, these solutions could be used as benchmark for sophisticated computer codes. In fact, in order to test numerical codes for general relativistic perfect fluid gravitational collapse (without magnetic field) Hawley, Smarr and Wilson have made an extensive use of simple waves solutions [10]. Similarly one could use our simple wave solutions in order to test analogous numerical codes in the magnetofluid-dynamic case.

The plane of the paper is the following: in sec. 2 the field equations of RMFD are written and their general properties briefly revisited. In sec. 3 the general formalism for simple waves is briefly recalled; in sec. 4 we derive exact general results on magnetoacoustic simple waves and in particular the conditions for the breaking of the waves are established.

In sec. 5 some particular exact solutions are found explicitly and in sec. 6 the Alfvén simple waves are completely determined.

2. FIELD EQUATIONS

When one neglects the gravitational field generated by the magnetofluid in comparison with the background gravitational field, the resulting theory is called test-relativistic MFD.

The equations of test RMFD are [*11*] (neglecting the Einstein equations):

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{2.1}$$

$$\nabla_\alpha(\rho u^\alpha) = 0 \tag{2.2}$$

$$\nabla_\alpha(u^\alpha b^\beta - u^\beta b^\alpha) = 0 \tag{2.3}$$

where ∇_α denotes the covariant derivative associated with the metric $g_{\alpha\beta}$ assumed of signature $+\,2$, and the units are such that $c = 1$. The energy-momentum tensor $T^{\alpha\beta}$ for the fluid and magnetic field is:

$$T^{\alpha\beta} = (e + p + |\,b\,|^2)u^\alpha u^\beta + \left(p + \frac{1}{2}|\,b\,|^2\right)g^{\alpha\beta} - b^\alpha b^\beta$$

where u^α is the fluid's 4-velocity, ρ the rest mass density, e the total energy density, p the pressure, and b^α is related to the magnetic field h^α by:

$$b^\alpha = \mu^{1/2}h^\alpha$$

μ being the fluid's constant magnetic permeability.

Also one has the constraints:

$$b_\alpha u^\alpha = 0 \qquad u_\alpha u^\alpha = -1 \tag{2.4}$$

hence b^α is a space-like vector and $|\,b\,|^2 = b_\alpha b^\alpha > 0$.

The fluid quantities ρ, p, e are restricted by the first law of thermodynamics

$$\vartheta ds = d\left(\frac{e}{\rho}\right) + pd\left(\frac{1}{\rho}\right)$$

where s is the specific entropy and ϑ is the absolute temperature: these quantities are related by a state equation of the form:

$$\begin{aligned} e &= e(p, s) \\ \rho &= \rho(p, s)\,. \end{aligned} \tag{2.5}$$

Let us rewrite equations (2.1), (2.2), (2.3) in a suitable form: from equation (2.1), by contracting with u_β and using (2.3) contracted with b_β one obtains the fluid's energy conservation equation

$$u^\alpha\nabla_\alpha e + (e + p)\nabla_\alpha u^\alpha = 0\,. \tag{2.6}$$

From this equation, the conservation of mass (2.2) and the first law of thermodynamics one obtains the adiabaticity relation

$$u^\alpha\nabla_\alpha s = 0\,. \tag{2.7}$$

By contracting (2.1) with b_β it follows:

$$(e+p)\nabla_\alpha b^\alpha + b^\alpha \nabla_\alpha p = 0 . \tag{2.8}$$

Now from equation (2.1) by contracting with $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ and using (2.3), (2.6), (2.7), (2.8), we obtain the conservation of momentum equation

$$(e + p + |b|^2)u^\alpha \nabla_\alpha u^\mu - b^\alpha \nabla_\alpha b^\mu + (h^{\mu\alpha} + u^\mu u^\alpha) b_\nu \nabla_\alpha b^\nu + \\ + \frac{1}{e+p}\{(e+p)h^{\mu\alpha} - e'_p |b|^2 u^\mu u^\alpha + b^\mu b^\alpha\} \nabla_\alpha p = 0 \tag{2.9}$$

where $e'_p = \left(\dfrac{\partial e}{\partial p}\right)_s$ and the thermal gas sound speed (without electromagnetic effects) is $c_s^2 = 1/e'_p \leq 1$.

Let us consider the Maxwell equations: from (2.3) using (2.6), (2.8) we obtain

$$u^\alpha \nabla_\alpha b^\beta - b^\alpha \nabla_\alpha u^\beta + \frac{1}{e+p}(u^\beta b^\alpha - b^\beta u^\alpha e'_p)\nabla_\alpha p = 0 . \tag{2.10}$$

It is well known that the Maxwell equations contain a « constraint » part: this is obtained by contracting (2.3) with u_β (see [*12*])

$$u^\alpha u^\beta \nabla_\alpha b_\beta + \nabla_\alpha b^\alpha = 0 \tag{2.11}$$

and must be taken into account.

Therefore we can take equations (2.6), (2.7), (2.9), (2.10) as the field equations for the field unknown

$$U = (u^\alpha, b^\alpha, p, s)^T$$

these equations can be written in the form of a quasilinear system [*13*]:

$$\mathscr{A}^\alpha \nabla_\alpha U = 0 \tag{2.12}$$

where the field vector U is

$$U = (u^\alpha, b^\alpha, p, s)^T \tag{2.13}$$

and the matrix $\mathscr{A}^\alpha$ is

$$\mathscr{A}^\alpha = \left|\begin{matrix} E u^\alpha \delta^\mu_\nu, & -b^\alpha \delta^\mu_\nu + p^{\mu\alpha} b_\nu, & m^{\nu\alpha}, & 0^{\mu\alpha} \\ b^\alpha \delta^\mu_\nu, & -u^\alpha_\nu \delta^\mu, & f^{\alpha\mu}, & 0^{\alpha\mu} \\ \eta \delta^\alpha_\nu, & 0^\alpha_\nu, & e'_p u^\alpha, & 0^\alpha \\ 0^\alpha_\nu, & 0^\alpha_\nu, & 0^\alpha, & u^\alpha \end{matrix}\right|$$

where $0^{\mu\alpha}$, 0^α, indicate tensors and vectors with vanishing components respectively, $\eta = e + p$, $E = \eta + |b|^2$, $p^{\mu\alpha} = h^{\mu\alpha} + u^\mu u^\alpha$

$$m^{\mu\alpha} = (\eta h^{\mu\alpha} - e'_p |b|^2 u^\mu u^\alpha + b^\mu b^\alpha)/\eta, \qquad f^{\mu\alpha} = (u^\mu b^\alpha e'_p - u^\alpha b^\mu)/\eta .$$

A detailed study of the mathematical structure of system (2.12) has been performed in [*12*]: let us give the main results.

Let Σ be the hypersurface given by

$$\phi(x^\alpha) = 0\,;$$

then the characteristic equation for the system (2.12) reads:

$$\det(\mathscr{A}^\alpha\phi_\alpha) = \mathrm{E}a^2\mathrm{A}^2\mathrm{N}_4 = 0 \tag{2.14}$$

where $a = u^\alpha\phi_\alpha$, $\mathrm{B} = b^\alpha\phi_\alpha$, $\mathrm{G} = \phi_\alpha\phi^\alpha$, $\mathrm{A} = \mathrm{E}a^2 - \mathrm{B}^2$, $\phi_\alpha = \dfrac{\partial\phi}{\partial x^\alpha}$ and

$$\mathrm{N}_4 = \eta(e'_p - 1)a^4 - (\eta + e'_p\,|\,b\,|^2)a^2\mathrm{G} + \mathrm{B}^2\mathrm{G}\,. \tag{2.15}$$

The solution corresponding to $a = 0$, $\mathrm{A} = 0$, $\mathrm{N}_4 = 0$ represent material, Alfvén and magnetoacoustic waves respectively.

In Ref. [*12*] an exhaustive list of the right and left eigenvectors of the matrix $\mathscr{A}^\alpha\phi_\alpha$, corresponding to the various kinds of waves has been given and the hyperbolicity of the system (2.12) has also been studied in detail. In particular it can be proved that if $e'_p \geq 1$ and $\eta \neq (e'_p - 1)\,|\,b\,|^2$, then the system (2.12) is hyperbolic. Another different condition guaranteeing the hyperbolicity of the system (2.12) is that $e'_p \geq 1$ and that $\mathrm{A}=0$ and $\mathrm{N}_4=0$ have a common root.

3. SIMPLE WAVES: GENERAL FORMALISM

Let us consider the following quasi-linear system of partial differential equations in R^4 (with coordinates x^α),

$$\mathscr{A}^\alpha\partial_\alpha\mathrm{U} = 0 \tag{3.1}$$

where $\mathrm{U} = (\mathrm{U}^1, \ldots, \mathrm{U}^\mathrm{N})^\mathrm{T}$ is the field vector and $\mathscr{A}^\alpha = \mathscr{A}^\alpha(\mathrm{U})$ are $\mathrm{N} \times \mathrm{N}$ smooth matrices.

A simple wave for such a system is a smooth solution depending only on one function $\phi = \phi(x^\alpha)$, [*14*] [*15*]

$$\mathrm{U} = \mathrm{U}(\phi)\,. \tag{3.2}$$

Then from (3.1) it follows that

$$\mathscr{A}^\alpha\phi_\alpha d\mathrm{U}/d\phi = 0\,. \tag{3.3}$$

In order to have a non trivial solution $\phi(x^\alpha)$ must satisfy the characteristic equation:

$$\det \mathscr{A}^\alpha\phi_\alpha = 0\,. \tag{3.4}$$

Let $\phi(x^\alpha)$ be a single root of equation (3.4) and R denote the corresponding right eigenvector of the matrix $\mathscr{A}^\alpha\phi_\alpha$. Then $\mathrm{U}(\phi)$ must satisfy the differential system

$$d\mathrm{U}/d\phi = \Pi(\phi)\mathrm{R} \tag{3.5}$$

with $\Pi(\phi)$ a proportionality factor. The solution of this system is equivalent to determining $N-1$ first integrals,

$$J_1(U_1, \ldots, U^N) = \text{const.}, \ldots, J_{N-1}(U^1, \ldots, U^N) = \text{const.}$$

called the Riemann invariants.

In general from (3.4), ϕ will be a function of U, and of x^α e. g.

$$\phi = F(U(\phi), x^\alpha). \tag{3.6}$$

Therefore, in order to obtain explicitly U as function of x^α one must solve (3.6) with respect to ϕ.

In the following we shall restrict to simple waves in one space dimension. It can be shown that this is the most general situation if one wants to avoid caustics [*15*].

4. EULERIAN MAGNETOACOUSTIC SIMPLE WAVES: GENERAL RESULTS

Let $\mathcal{M}$ be a Minkowski spacetime with inertial coordinates (x, y, z, t). For one-dimensional flow the Maxwell equations (2.3) yield,

$$\partial_x(u^1b^0 - u^0b^1) = 0 \tag{4.1}$$

$$\partial_t(u^0b^1 - u^1b^0) = 0 \tag{4.2}$$

whence:

$$J_1 = u^0b^1 - u^1b^0 = \text{const.} \tag{4.3}$$

which is a first integral of the flow. By writing

$$u^\alpha = \Gamma(1, v_x, v_y, v_z)$$

$$b^\alpha = (b^0, b_x, b_y, b_z)$$

where Γ is the Lorentz factor, from $u_\alpha b^\alpha = 0$ it follows

$$J_1 = \Gamma\{(1 - v_x^2)b_x - v_xv_yb_y - v_xv_zb_z\}. \tag{4.4}$$

In the non-relativistic limit,

$$J_1 = b_x$$

which is a well known classical integral of motion [*16*].

For one-dimensional motion in the system (3.1) one writes:

$$\mathcal{A}^0\partial_t U + \mathcal{A}^1\partial_x U = 0. \tag{4.5}$$

It is not restrictive to take ϕ of the following form:

$$\phi(x, t) = x - \lambda(U)t, \tag{4.6}$$

with $\lambda(U)$ satisfying

$$\det(\mathcal{A}^1(U) - \lambda\mathcal{A}^0(U)) = 0; \tag{4.7}$$

λ can be interpreted as the propagation speed with respect to the inertial observer. The ansatz (4.6) for ϕ may become double valued for large t associated with wave breaking at $t = t_b = -1/(d\lambda/d\phi)$ since

$$\partial_t \phi = \frac{-\lambda}{1 + (d\lambda/d\phi)t} \qquad \partial_x \phi = \frac{1}{1 + (d\lambda/d\phi)t}.$$

In this section we shall restrict ourselves to magnetoacoustic waves. We distinguish two cases:

a) $e'_p > 1$.

A material wave can coincide with an Alfvén wave if and only if $a = 0$ is a solution of $A = 0$, which implies, being $E > 0$, $B = 0$.

Similarly a material wave can coincide with a magnetoacoustic wave iff $B^2G = 0$. Now, since $G \neq 0$ for a material and Alfvén waves (under the assumption $e'_p > 1$), it follows that the condition $B^2G = 0$ is equivalent to $B = 0$.

We want to exclude the cases where a material wave can coincide with an Alfvén or a magnetoacoustic wave. Therefore in the following, except when stated otherwise, we shall assume

$$B \neq 0. \tag{4.8}$$

An Alfvén wave can coincide with a magnetoacoustic wave iff $A = 0$ is solution of $N_4 = 0$. We want to exclude also this case, and therefore we shall assume that, except when stated otherwise,

$$\Lambda = \eta a^2 - G\,|\,b\,|^2 \neq 0. \tag{4.9}$$

The conditions (4.8), (4.9) will be imposed at a given initial point. By continuity they will hold in a neighbourhood of the initial point. The extent of such a neighbourhood can be determined only by an (in general numerical) integration of the simple wave equations.

Under these assumptions equation $N_4 = 0$ admits four real and distinct roots for λ (with $|\,\lambda\,| < 1$) (see [*12*]). The corresponding right eigenvectors are:

$$R^\alpha = Ea^2(Bf^\alpha - am^\alpha) + Ea(B^2 - e'_p\,|\,b\,|^2a^2)(\phi^\alpha + 2au^\alpha)/\eta \tag{4.10 a}$$

$$R^{\alpha+4} = R^\alpha B/a + EAaf^\alpha \tag{4.10 b}$$

$$R^8 = Ea^2A \tag{4.10 c}$$

$$R^9 = 0 \tag{4.10 d}$$

where $\qquad f^\alpha = f^{\alpha\mu}\phi_\mu, \qquad m^\alpha = m^{\alpha\mu}\phi_\mu.$

b) $e'_p = 1$.

In this case $N_4 = -AG$, and since we want to exclude $A = 0$, we must have:

$$G = g_{\alpha\beta}\phi^\alpha\phi^\beta = 0. \tag{4.11}$$

The corresponding right eigenvectors are obtained from (4.10 *a*) putting $e'_p = 1$ and substituting for ϕ a solution of (4.11).

The equations (3.5) defining simple waves, in the magnetoacoustic case (both *a*) and *b*)), can be written in the following form, by taking p as independent variable:

$$\begin{aligned} du^\alpha/dp &= \mathrm{R}^\alpha/\mathrm{E}a^2\mathrm{A} \\ db^\alpha/dp &= \mathrm{R}^{\alpha+4}/\mathrm{E}a^2\mathrm{A} \end{aligned} \tag{4.12}$$

together with the obvious Riemann invariant

$$\mathrm{J}_0 = s = \text{const.} \tag{4.13}$$

It is immediate to check that magnetoacoustic simple waves satisfy the constraints (2.4), provided they are satisfied at a given point. Similarly, from equations (4.10) it is easy to check that

$$h_{\alpha\beta}\phi^\alpha db^\alpha/d\phi = 0 \tag{4.14}$$

and therefore the constraint (2.11) is also satisfied.

From equations (4.12) the following results of general character can be obtained.

PROPOSITION 1. — Under assumption (4.8), (4.9) and $e'_p > 1$ the quantity $|b|^2$ is an increasing function of p for fast magnetoacoustic simple waves, whereas it is a decreasing function of p for slow ones.

Proof. — It is easy to show, from equations (4.12) that

$$\frac{1}{2}\frac{d|b|^2}{dp} = e'_p|b|^2/\eta - (\mathrm{B}^2/\eta)a^2 . \tag{4.15}$$

Let v_Σ be the local speed of propagation, defined as [*11*]

$$v_\Sigma^2 = \frac{a^2/\mathrm{G}}{1 + a^2/\mathrm{G}} . \tag{4.16}$$

Then (4.15) can be written as:

$$\frac{1}{2}\frac{d|b|^2}{dp} = (e'_p v_\Sigma^2 - 1)/(1 - v_\Sigma^2) . \tag{4.17}$$

Under the above assumptions, we have $0 \le v_{\Sigma_s} < (e'_p)^{-1/2} < v_{\Sigma_f} < 1$, where v_{Σ_s}, v_{Σ_f} are the slow and fast magnetoacoustic speeds respectively. Hence the statement follows. □

REMARK 1. — For $e'_p = 1$, for the root corresponding to $v_{\Sigma_f} = 1$, we obtain $|b|^2 - 2p = \text{constant}$.

REMARK 2. — We notice that for a fast magnetoacoustic wave $A > 0$, whereas for a slow one $A < 0$. In fact

$$A = E(a^2 + G)(v_\Sigma^2 - v_A^2)$$

where $v_A^2 = b_n^2/E$, $b_n^2 = B^2/(a^2 + G)$ (v_A is the local Alfvén speed, satisfying $v_{\Sigma_s} < v_A < v_{\Sigma_f}$ under our assumptions).

REMARK 3. — Let $q = p + \frac{1}{2}|b|^2$ be the total pressure (gas pressure + magnetic one). Then from (4.17),

$$dq/dp = \frac{v_\Sigma^2(e'_p - 1)}{1 - v_\Sigma^2}$$

and for $e'_p > 1$, $dq/dp > 0$; hence q is always a monotonically increasing function of p.

Now we shall discuss the sign of a and B. From (4.6) we find

$$a = \Gamma(v_x - \lambda), \qquad B = b_x - \lambda b^0, \qquad G = 1 - \lambda^2 \tag{4.18}$$

and under our assumption $a \neq 0$, $B \neq 0$, we shall consider solutions with a stagnation point, where $v_x = v_y = v_z = 0$ at a given pressure p_0.

Hence $a(p_0) = -\lambda(p_0)$, $B(p_0) = b_x(p_0)$, and therefore a and B will maintain their initial signs.

One can always choose the reference frame such that $b_x(p_0) > 0$ and consider only progressive waves, for which $\lambda > 0$. With these choices, one has always $a < 0$, $B > 0$.

Another important results of general nature, concerns the behaviour of the eigenvalue λ along the solution, $\lambda = \lambda(p)$.

By substituting

$$\phi_\alpha = (-\lambda, 1, 0, 0) \tag{4.19}$$

into $N_4 = 0$, we obtain

$$N_4(U, \lambda) = 0 \tag{4.20}$$

which holds identically for a chosen root $\lambda = \lambda(p)$. Hence:

$$\frac{\partial N_4}{\partial U^A}\frac{dU^A}{dp} + \frac{\partial N_4}{\partial \lambda}\frac{d\lambda}{dp} = 0$$

which can be written as

$$\frac{\delta N_4}{\delta p} + \frac{\partial N_4}{\partial \lambda}\frac{d\lambda}{dp} = 0 \tag{4.21}$$

with

$$\frac{\delta N_4}{\delta p} = \left[\frac{\partial N_4}{\partial u^\alpha}R^\alpha + \frac{\partial N_4}{\partial b^\alpha}R^{\alpha+4}\right]\frac{1}{Ea^2A} + \frac{\partial N_4}{\partial p} \tag{4.22}$$

It can be shown that [17]

$$\delta N_4/\delta p = a^2(a^2 K_1 + G K_2) \tag{4.23}$$

where

$$K_1 = \eta e_p'' + (e_p' - 1)(3 - 5e_p') \tag{4.24}$$

$$K_2 = - e_p'' \,|\, b \,|^2 + (e_p' - 1)(3 + 2e_p' \,|\, b \,|^2/\eta)\,. \tag{4.25}$$

Now we need the following results:

LEMMA 1. — Under assumptions (4.8), (4.9), $e_p' > 1$ and the compressibility hypothesis [18]

$$W = - e_p'' + 2e_p'(e_p' - 1)/\eta > 0 \qquad \text{(Weyl condition)} \tag{4.26}$$

one has

$$\delta N_4/\delta p \neq 0\,. \tag{4.27}$$

Proof. — In equation (4.23) we substitute $a^2 = G v_\Sigma^2/(1 - v_\Sigma^2)$, hence

$$\frac{\delta N_4}{\delta p} = \frac{G}{1 - v_\Sigma^2}\left[- v_\Sigma^2 [WE + 3e_p'(e_p' - 1)] + W \,|\, b \,|^2 + 3(e_p' - 1)\right].$$

If $\delta N_4/\delta p = 0$ at some point, then v_Σ^2 must satisfy $N_4 = 0$.
Now

$$N_4 = G^2 P(v_\Sigma^2)/(1 - v_\Sigma^2)$$

where

$$P(v_\Sigma^2) = \eta(e_p' - 1)v_\Sigma^4 - (\eta + e_p' \,|\, b \,|^2 - b_n^2)v_\Sigma^2(1 - v_\Sigma^2) + b^2(1 - v_\Sigma^2) \tag{4.29}$$

The compatibility between $N_4 = 0$ and $\delta N_4/\delta p = 0$ is then the equation:

$$Y = X_1 \,|\, b \,|^4 + X_2 \,|\, b \,|^2 + X_3 = 0$$

with

$$X_1 = - \eta W^2 - 3e_p' W(e_p' - 1)^2$$

$$X_2 = - \eta^2 W^2 - 3\eta W(e_p' - 1) - 9e_p'(e_p' - 1)^3 + b_n^2 [\eta W^2 + 3W(e_p' - 1)]$$

$$X_3 = - 3\eta^2 W(e_p' - 1) + b_n^2 [\eta W + 3e_p'(e_p' - 1)][\eta W + 3(e_p' - 1)^2].$$

From assumption (4.8), (4.9) it follows

$$b_n^2 < |\, b \,|^2$$

and by using this inequality, after some manipulations, we get

$$Y < - 3(e_p' - 1)W[(e_p' - 1)\,|\, b \,|^2 - \eta]^2 < 0\,. \qquad \square$$

PROPOSITION 2. — Under assumption (4.8), (4.9), $e_p' > 1$ and the compressibility hypothesis (4.26), one has for progressive waves

$$d\lambda/dp > 0 \qquad (< 0 \text{ for regressive ones}) \tag{4.25}$$

Proof. — Under our hypothesis the roots of $N_4 = 0$ are all distinct, hence $\partial N_4/\partial \lambda \neq 0$.

It follows that $\partial N_4/\partial\lambda$ has the same sign as at the stagnation point p_0, $v_x = v_y = v_z = 0$. A simple calculation shows that:

$$\left.\frac{\partial N_4}{\partial\lambda}\right|_{p_0} = \pm\, 2\lambda_0[(\eta + e'_p\,|\,b\,|^2 + B^2) - 4e'_p E B^2]^{1/2}_{p_0} \tag{4.26}$$

where

$$\lambda_0^2 = \left.\frac{(\eta + e'_p\,|\,b\,|^2 + B^2) \pm [(\eta + e'_p\,|\,b\,|^2 + B^2) - 4Ee'_p B^2]^{1/2}}{2Ee'_p}\right|_{p_0} \tag{4.27}$$

The choice of the signs $\pm$ corresponds to fast and slow magnetoacoustic waves respectively. Furthermore, since we shall deal with progressive waves, $\lambda_0 > 0$.

It follows that $\partial N_4/\partial\lambda$ is positive for fast magnetoacoustic waves, and negative for slow ones. Because $\delta N_4/\delta p \neq 0$, it follows that $\delta N_4/\delta p$ has the same sign at the stagnation point.

One finds, after lengthy calculations, that

$$\left.\frac{1}{\lambda_0^2}\frac{\partial N_4}{\partial\lambda}\right|_{p_0} = [W\,|\,b\,|^2 + 3(e'_p - 1) - \lambda^2[EW + 3e'_p(e'_p - 1)]]_{p_0}.$$

It can be seen that the sign of $(\delta N_4/\delta p)_{p_0}$ is negative corresponding to the fast magnetoacoustic wave and positive for the slow one. Then the statement follows from

$$d\lambda/dp = -\frac{\dfrac{\partial N_4}{\delta p}}{\dfrac{\partial N_4}{\partial\lambda}} > 0. \quad \square$$

Remark. — For $e'_p = 1$, from (4.23), one has $\delta N_4/\delta p = 0$, which corresponds to the exceptional case.

5. EULERIAN SIMPLE WAVES: DIFFERENTIAL EQUATIONS AND PARTICULAR SOLUTIONS

Equations (4.12) can be written explicitly as follows, in case $e'_p > 1$

$$\frac{d(\Gamma v_x)}{dp} = \alpha_1\Gamma v_x + \alpha_1/a + \alpha_2 b_x \tag{5.1 a}$$

$$\frac{d(\Gamma v_y)}{dp} = \alpha_1\Gamma v_y + \alpha_2 b_y \tag{5.1 b}$$

$$\frac{d(\Gamma v_z)}{dp} = \alpha_1\Gamma v_z + \alpha_2 b_z \tag{5.1 c}$$

$$\frac{db_x}{dp} = \beta_1 \Gamma v_x + \beta_2 b_x - (e'_p - 1)\mathrm{B}a^2/\mathrm{GA} \tag{5.1 d}$$

$$\frac{db_y}{dp} = \beta_1 \Gamma v_y + \beta_2 b_y \tag{5.1 e}$$

$$\frac{db_z}{dp} = \beta_1 \Gamma v_z + \beta_2 b_z . \tag{5.1 f}$$

where

$$\alpha_1 = - a^4(e'_p - 1)/\mathrm{AG}, \qquad \alpha_2 = a\mathrm{B}(e'_p - 1)/\eta A$$
$$\beta_1 = \mathrm{B}(\alpha_1 - 1/\eta)/a, \qquad \beta_2 = e'_p/\eta + \mathrm{B}\alpha_2/a .$$

In the case $e'_p = 1$, under assumption (4.8), (4.9) one has two roots, $\lambda = \pm 1$ and equations (4.12) admit the following invariants besides J_1, s:

$$\begin{aligned} \hat{\mathrm{J}}_2 &= v_y/v_z \\ \hat{\mathrm{J}}_3 &= v_z/(v_x \mp 1) \\ \hat{\mathrm{J}}_4 &= (- \hat{\mathrm{J}}_2 b_z + b_y)p^{-1/2} \\ \hat{\mathrm{J}}_5 &= [b_x \pm (v_x b_x + v_y b_y + v_z b_z)]^2 p \\ \hat{\mathrm{J}}_6 &\doteq \frac{1}{p}\frac{v_x - k1}{v_x - k2} \qquad \text{with} \quad k1,\ k2 \text{ constants}. \end{aligned} \tag{5.2}$$

Now we turn our attention to the case $e'_p > 1$. From equations (5.1) we obtain

$$\frac{d}{dp}\,[\Gamma(v_x b_z - v_z b_y)] = (e'_p - 1)\frac{a^2}{\mathrm{A}}\,[\Gamma(v_x b_z - v_z b_y)] . \tag{5.3}$$

At the stagnation point p_0 we have:

$$[\Gamma(v_x b_z - v_z b_y)]_{p_0} = 0$$

hence, by the uniqueness theorem, (5.3) yields the following invariant:

$$\mathrm{J}_2 = v_x b_z - v_z b_y = 0 . \tag{5.4}$$

Similarly, it can be seen that

$$\frac{d}{dp}\frac{b_y}{b_z} = \frac{\beta_1 \Gamma}{b_2}(v_y - b_y v_z/b_z),$$

whence another invariant,

$$\mathrm{J}_3 = b_y/b_z . \tag{5.5}$$

In general equations (5.1) are too complicated to allow an explicit analytic integration. In the following we shall treat some special cases which can be brought to quadratures.

First of all we treat the case of a longitudinal magnetic field,

$$b_y = b_z = 0. \tag{5.6}$$

From equations (5.1) we have then

$$v_y = v_z = 0.$$

The characteristic equation $N_4 = 0$ admits the following four solutions:

$$\lambda_m = (v_x + \bar{\lambda}_m)/(1 + v_x\bar{\lambda}_m) \qquad m = 1, 2, 3, 4, \tag{5.7}$$

where

$$(\bar{\lambda}_{1,2}) = \pm (e'_p)^{-1/2}, \qquad (\bar{\lambda}_{3,4}) = \pm (|b|^2/\mathrm{E})^{1/2}$$

The roots $\lambda_{3,4}$ coincide with the Alfvén waves and shall not be considered here. The simple waves solutions for the acoustic speed are given by the following invariants:

$$\mathrm{J}_1 = b_x/\Gamma$$
$$\mathrm{J}_\pm = \mathrm{Ln}\left[\frac{1 + v_x}{1 - v_x}\right]^{1/2} \mp \int \frac{(e'_p)^{1/2}}{\eta}\,dp. \tag{5.8}$$

The invariants $\mathrm{J}_\pm$ coincide with those of relativistic fluiddynamics [*19*].

The other case we shall consider is when the fluid's motion is purely longitudinal,

$$v_y = v_z = 0 \tag{5.9}$$

and the magnetic field at the stagnation point is purely transverse

$$b_x(v = 0) = 0. \tag{5.10}$$

Then from equation (4.4) it follows that $b_x = 0$ throughout the flow, hence $b^0 = 0$ and $\mathrm{B} = 0$. In this case we consider the following simple root of $N_4 = 0$, corresponding to the fast magnetoacoustic wave:

$$\lambda = (v_x + \bar{\lambda})/(1 + v_x\bar{\lambda}) \tag{5.11}$$

with

$$\bar{\lambda} = \pm [(\eta + e'_p|b|^2)/\mathrm{E}e'_p]^{1/2}. \tag{5.12}$$

From (5.1 *e*), (5.1 *f*) we obtain the following invariants

$$\mathrm{J}_2 = \mathrm{Ln}\, b_y - \int \frac{e'_p}{\eta}\,dp, \qquad \mathrm{J}_3 = \mathrm{Ln}\, b_z - \int \frac{e'_p}{\eta}\,dp. \tag{5.13}$$

Finally equation (5.1 *a*) gives the invariants:

$$\mathrm{J}_\pm = \frac{1}{2}\mathrm{Ln}\,[(1 + v_x)/(1 - v_x)] \mp \cdot \int \frac{1}{\eta}\,[e'_p(\eta + e'_p|b|^2)/\mathrm{E}]^{1/2}dp. \tag{5.14}$$

For a non barotropic fluid,

$$\int \frac{e'_p}{\eta}\,dp = \mathrm{Ln}\,\rho$$

and (5.13) gives

$$b_y/\rho = \text{const.}, \qquad b_z/\rho = \text{cons.}$$

which are analogous to the corresponding non-relativistic invariants [16].

Another case of interest in when, at the stagnation point, $(b_x)_0 = 0$. It follows that $J_1 = 0$.

For plane polarized waves we can choose $b_y = v_y = 0$, hence from equation (4.4) we obtain

$$b_x = v_x v_z b_z/\delta \qquad \delta = 1 - v_x^2.$$

For the magnetoacoustic waves we have

$$a = \frac{\dfrac{v_x \Delta}{\Gamma} \pm \left[\dfrac{\Delta^2}{\Gamma^2} + \delta\Delta\eta(e'_p - 1)\right]^{1/2}}{\eta(e'_p - 1) + \Delta/\Gamma^2}$$

with

$$\Delta = \eta + e'_p\,|\,b\,|^2 - (b^0)^2/\Gamma^2 .$$

Finally equations (5.1) yield, in this case

$$\frac{dv_x}{dp} = \frac{(e'_p - 1)a^3}{A\Gamma G}(\lambda v_x - 1) \tag{5.15 a}$$

$$\frac{dv_z}{dp} = \frac{(e'_p - 1)a^3\lambda v_z}{A\Gamma G} + \frac{(e'_p - 1)aB}{A\eta\Gamma}(b_z - b^0 v_z) \tag{5.15 b}$$

$$\frac{dv_z}{dp} = -\frac{B\Gamma v_z}{a}\left[\frac{(e'_p - 1)a^4}{AG} + \frac{1}{\eta}\right] + \left[\frac{e'_p}{\eta} + \frac{(e'_p - 1)B^2}{\eta A}\right]b_z \tag{5.15 c}$$

From equation (5.15 *a*) we have that, being $a < 0$ and $A > 0$ for the fast wave, $dv_x/dp > 0$, hence v_x is a monotonically increasing function of p.

Since we require a stagnation point p_0 at which $v_x(p_0) = v_0(p_0) = 0$, from equation (5.15 *b*) and the uniqueness theorem we have $v_z = 0$, and equation (5.15 *c*) reduces to

$$db_z/dp = e'_p b_z/\eta\,. \tag{5.16}$$

Therefore this case reduces to the one previously studied, of a transverse magnetic field.

6. ALFVÉN SIMPLE WAVES

Because Alfvén waves correspond to multiple roots of the characteristic equation, the methods employed in the previous sections are not applicable.

In this case it is convenient to resort to a general method due to Boillat [20]. The system of equations (2.1)-(2.3) can be written in the following conservation form:

$$\partial_\alpha f^{\alpha A} = 0 \qquad A = 0, 1, 2, \ldots, 8 \tag{6.1}$$

with

$$\underset{\sim}{f}^{\alpha} = (T^{\alpha\beta}, \rho u^{\alpha}, u^{\alpha}b^{\beta} - u^{\beta}b^{\alpha}).$$

We look for one-dimensional solutions of the kind:

$$\underset{\sim}{f}^{\alpha} = \underset{\sim}{f}^{\alpha}(\phi)$$

with $\phi = x - \lambda(U)t$.

Then equations (6.1) yield

$$\frac{d}{d\phi}(\phi_{\alpha}f^{\alpha A}) + f^{0A}\frac{d\lambda}{d\phi} = 0.$$

Now, it is well known that the Alfvén waves are exceptional [17],

$$d\lambda/dp = 0.$$

Hence we obtain the following invariants:

$$f^{1A} - \lambda f^{0A} = \text{const.} \qquad A = 0, 1, 2, \ldots, 8. \tag{6.2}$$

From these we obtain, besides the already known invariants J_1, and s

$$J_2^* = |b|^2 \tag{6.3 a}$$

$$J_3^* = \lambda = v_x - \frac{J_1}{\pm \Gamma^2 E^{1/2} - b^0\Gamma} \tag{6.3 b}$$

$$J_4^* = b_y - B\Gamma v_y/a \tag{6.3 c}$$

$$J_5^* = b_z - B\Gamma v_z/a \tag{6.3 d}$$

$$J_6^* = p \tag{6.3 e}$$

$$J_7^* = b_x - \lambda b^0 \tag{6.3 f}$$

$$J_8^* = \Gamma(v_x - \lambda). \tag{6.3 g}$$

ACKNOWLEDGMENTS

Research supported by the Italian Ministry of Education.

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(Manuscrit reçu le 3 avril 1987)
(Version révisée, reçue le 10 juillet 1987)