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Produced representations of Lie algebras and superfields

by

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ABSTRACT. — We use produced representations of super Lie algebras to construct superfields. We compare the representation of the super Poincaré algebra on these fields with the physics literature. The geometry of superspace is defined in terms of superfields without the use of Grassmann numbers or a particular model of supergeometry.

RÉSUMÉ. — On construit des super champs en utilisant les représentations produites des algèbres de Lie. La représentation de la super algèbre de Poincaré dans ces champs est comparée avec la littérature physique. La géométrie du super espace est défini en termes des super champs n'utilisant ni les nombres de Grassmann ni un modèle particulier de la super géométrie.

1. INTRODUCTION

The notion of superfield has proved its use in supersymmetric field theory, and its mathematical content has been the impetus for much interesting mathematical work, in particular the development of several different versions of supergeometry (see [2] for an overview). In one of the first articles on the subject Salam and Strathdee [10] introduced superfields as the natural generalization of functions on Minkowski space being the induced representations of a “super Poincaré group” with formal anti-

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commuting parameters. Oddly enough they did not mention this aspect in their later work [11]. The reluctance to use this picture may be due to the fact that in physical applications we deal with the super *Lie algebra*, which is well defined over the complex numbers, while the exponentiation to a group necessitates the introduction of an extra algebra of Grassmann numbers. Since those numbers have some serious technical and conceptual problems (dimension, topology) we feel, like most physicists, that they are “just bookkeeping devices” and we would rather not use them as a starting point in the definition of superspace and superfield. In this paper we therefore define superfields entirely in terms of the super Lie algebra. We then define superspace in terms of superfields thereby avoiding the choice of a particular model of super geometry.

We shall be using produced representations of (super) Lie algebras. These are the analogues of induced representations of Lie groups, a theory that we shall briefly outline.

Let G be a Lie group with subgroup H and V a representation of the subgroup. The induced representation of V is the set of C^∞ functions $\phi : G \rightarrow V$ with $\phi(hg) = h\phi(g) \forall h \in H$, and G structure defined by $(g \cdot \phi)(g') = (g'g)$. In particular we have the representation $C^\infty(H \backslash G)$ induced from the trivial representation, which consists of scalar functions f on G , constant along left orbits of H . Every induced representation is a module over the trivial induced representation by pointwise multiplication and clearly $(g \cdot f)(g \cdot \phi) = g \cdot (f\phi)$. This property allows us to construct every induced representation from the trivial one $C^\infty(H \backslash G)$ and the H -representation V . Note that an element of the Lie algebra acts as a derivation of this multiplication.

The organization of this paper is as follows: after defining produced representations we introduce the coalgebra as a substitute for the geometric notion of “pointwise”. It is used to define a multiplication between an arbitrary and the trivial produced representation such that the (super) Lie algebra acts by derivations. Special attention is paid to semi-direct sum Lie algebras. In section 2.5 we construct “local coordinates” and show that produced representations are isomorphic to vector valued formal power series. Part 3 is devoted to the application to the super Poincaré algebra. Because of its special structure we can use polynomials (rather than power series) which are more convenient for calculations and can be used in the definition of superspace. We then check that in “local coordinates” the action of the super Poincaré algebra on produced representations corresponds to the action on physicists superfields. Finally we construct superspace as the set of algebra homomorphisms on superfields and find that it has the structure of a complex algebraic graded manifold in the sense of Kostant and Leites [7], [8], with complexified Minkowski space as its associated “body” manifold. There is also an interpretation in terms of Rogers-DeWitt supermanifolds [9], [14].

The convention in this paper is that all objects (algebras, vector spaces, etc.) have a \mathbb{Z}_2 -grading called parity, all homomorphisms are even and all bases are homogeneous unless specified otherwise. If we also want to use odd linear “homomorphisms”, we shall explicitly call them linear maps. The field of definition is the complex numbers \mathbb{C} but except in part 3 we could use any field of characteristic 0.

2. GENERAL THEORY

2.1. Produced modules.

DEFINITION 2.1.1. — Let A be an algebra with unit and V a module of a subalgebra B . The *produced module* is an A -module $V(A/B)$ together with a B -homomorphism $\delta : V(A/B) \rightarrow V$ with the following universal property: for every A -module U with a B -homomorphism $\alpha : U \rightarrow V$ there is a *unique* A -homomorphism $\tilde{\alpha}$ such that the diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\exists! \tilde{\alpha}} & V(A/B) \\
 \alpha \searrow & & \nearrow \delta \\
 & & V
 \end{array} \tag{1}$$

commutes.

As always in such cases $(V(A/B), \delta)$ is unique up to isomorphism provided the module exists.

Given a pair (U, α) , every $u \in U$ defines a homomorphism

$$\begin{aligned}
 \tilde{u} : A &\rightarrow V \\
 a &\mapsto \alpha(au).
 \end{aligned} \tag{2}$$

This suggests the representation ⁽¹⁾

$$V(A/B) = \mathcal{L}_B(A, V) \tag{3}$$

where $\mathcal{L}_B(A, V)$ is the (graded) vector space of B -linear maps $A \rightarrow V$. In what follows it will be more natural to write evaluation of $\phi \in \mathcal{L}_B(A, V)$ in $a \in A$ as $\langle a, \phi \rangle$ (instead of $\phi(a)$) and we will always do so. In this notation B -linearity means

$$\langle ba, \phi \rangle = b \langle a, \phi \rangle \quad \forall b \in B \tag{4}$$

The space $\mathcal{L}_B(A, V)$ carries a natural A -structure given by

$$\langle a_1, a_2 \phi \rangle = \langle a_1 a_2, \phi \rangle$$

⁽¹⁾ The representation is actually an equivalence of functors.

and a \mathbf{B} -homomorphism $\delta = \langle 1_A, \cdot \rangle$. To check the universal property (1) is left as an exercise to the reader. Note that $\mathcal{L}_{\mathbf{B}}(A, V)$ may strictly enclose $\text{Hom}_{\mathbf{B}}(A, V)$ which consists of the *even* \mathbf{B} -linear maps.

The theory of Lie algebras “reduces” to the theory of associative algebras by the introduction of the universal enveloping algebra (see [6, p. 90 f. f.] or [3] for the graded case). We shall not distinguish between \mathfrak{g} -modules (homomorphisms) and $U(\mathfrak{g})$ -modules (homomorphisms). Now let \mathfrak{g} be a Lie algebra with subalgebra \mathfrak{h} . Define the *produced representation* ⁽²⁾ $V(\mathfrak{g}/\mathfrak{h})$ of a \mathfrak{h} -module V by $V(\mathfrak{g}/\mathfrak{h}) \stackrel{\text{def}}{=} V(U(\mathfrak{g})/U(\mathfrak{h}))$.

2.2. The coalgebra and its interpretation.

The theory of Lie produced representations is enriched by the use of the *coalgebra* structure of the universal enveloping algebra. See [13] for a general reference on coalgebras and [4, p. 91] in the context of enveloping algebras. A coalgebra is defined by two homomorphisms, the diagonal Δ and the counit ε . In this case, we use the two algebra homomorphisms

$$\begin{aligned} \Delta : U(\mathfrak{g}) &\rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) \\ X &\mapsto X \otimes 1 + 1 \otimes X \quad X \in \mathfrak{g} \end{aligned} \tag{5}$$

$$\begin{aligned} \varepsilon : U(\mathfrak{g}) &\rightarrow \mathbb{C} \\ X &\mapsto 0 \quad X \in \mathfrak{g} \end{aligned} \tag{6}$$

The diagonal is induced by the Lie diagonal $\mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$, the counit by $\mathfrak{g} \rightarrow 0$. The algebraic properties of a coalgebra, coassociativity: $(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta$ and counitarity: $(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}$ are dual to those of an algebra. Moreover, every enveloping algebra is cocommutative: $\Delta = T\Delta$ with T the twist map which interchanges two factors of a tensor product adding signs if both factors are odd.

Some well known properties of Lie algebras are easy to understand in terms of the coalgebra structure. For example, if V and W are \mathfrak{g} -modules or equivalently $U(\mathfrak{g})$ -modules, then $V \otimes W$ is a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ -module. But since Δ is an algebra homomorphism, $V \otimes W$ is also an $U(\mathfrak{g})$ -module. Explicitly, for $X \in \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ we have

$$\begin{aligned} X(v \otimes w) &= \Delta X(v \otimes w) \\ &= (X \otimes 1 + 1 \otimes X)(v \otimes w) \\ &= Xv \otimes w + (-1)^{|X||v|} v \otimes Xw \end{aligned} \tag{7}$$

and we recognise the ordinary (graded) Leibnitz rule for the action of a

⁽²⁾ The word produced module would lead to a slightly awkward terminology in the sequel.

Lie algebra on a tensor product. The reader is invited to convince himself that the diagonal on the whole of $U(\mathfrak{g})$ expresses the higher order Leibnitz rule.

2.3. The module structure of produced representations.

The field \mathbb{C} is an \mathfrak{h} -module with the null action. Hence we can construct the \mathfrak{g} -module $R = \mathbb{C}(\mathfrak{g}/\mathfrak{h}) = \mathcal{L}_{\mathfrak{h}}(U(\mathfrak{g}), \mathbb{C})$. The space R will play an important role in the sequel because just as in the case of induced representations of Lie groups, all produced representations $V(\mathfrak{g}/\mathfrak{h})$ can be constructed from V and R . The first and most important step in this direction is the next theorem.

THEOREM 2.3.1. — R has the structure of a commutative algebra with unit ε (the counit) and every produced representation $V(\mathfrak{g}/\mathfrak{h})$ is naturally an R -module with \mathfrak{g} acting by derivations.

Proof. — (sketch) A full proof is given in [1]. If the Lie algebra should act by derivations then the whole enveloping algebra should act according to the higher order Leibnitz rule. Reversing the argument, we define the product fg of $f, g \in R$ by

$$\begin{array}{ccc}
 U(\mathfrak{g}) & \xrightarrow{fg} & \mathbb{C} \\
 \Delta \downarrow & & \uparrow \varepsilon \\
 U(\mathfrak{g}) \otimes U(\mathfrak{g}) & \xrightarrow{f \otimes g} & \mathbb{C} \otimes \mathbb{C}
 \end{array} \tag{8}$$

or in other words

$$\langle x, fg \rangle \stackrel{\text{def}}{=} \langle \Delta x, f \otimes g \rangle \quad \forall x \in U(\mathfrak{g}) \tag{9}$$

where we identified $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$. Using coassociativity counitary and cocommutativity, we check that fg is a $U(\mathfrak{h})$ -linear functional, and the properties of a commutative algebra. Actually, the same argument shows that for any coalgebra C and any algebra A the space $\mathcal{L}(C, A)$ is an algebra [13, p. 69]. The Lie algebra acts by derivations because for $X \in \mathfrak{g}$

$$\begin{aligned}
 \langle x, X(fg) \rangle &= \langle \Delta(xX), f \otimes g \rangle \\
 &= \langle (\Delta x), (\Delta X)(f \otimes g) \rangle \\
 &= \langle \Delta x, (X \otimes 1 + 1 \otimes X)(f \otimes g) \rangle \\
 &= \langle x, (Xf)g + (-1)^{|X||f|} f(Xg) \rangle
 \end{aligned} \tag{10}$$

Along the same lines we define a multiplication between R and an arbitrary produces representation. \square

2.4. Semidirect sums.

For the application to the Poincaré algebra we turn to semidirect sum Lie algebras. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{I}$ with \mathfrak{h} the subalgebra and \mathfrak{I} an ideal of \mathfrak{g} . In this special case the produced representation is expressible in terms of \mathfrak{I} .

First note that every $U(\mathfrak{h})$ linear map $\phi \in V(\mathfrak{g}/\mathfrak{h})$ determines an ordinary linear map on $U(\mathfrak{I})$ by restriction to $U(\mathfrak{I}) \hookrightarrow U(\mathfrak{g})$.

$$V(\mathfrak{g}/\mathfrak{h}) \rightarrow V(\mathfrak{I}/\{0\}) \quad (11)$$

in particular

$$\mathbb{C}(\mathfrak{g}/\mathfrak{h}) \rightarrow \mathbb{C}(\mathfrak{I}/\{0\}). \quad (12)$$

It is easy to see that (12) is an algebra homomorphism and more generally that (11) is an homomorphism of both the $\mathbb{C}(\mathfrak{I}/\{0\})$ structure (under the correspondence (12)) and the \mathfrak{I} module structure. In fact

THEOREM 2.4.1. — *The homomorphism (11) is an isomorphism.*

Proof. — As a result of the PBW theorem every $x \in U(\mathfrak{g})$ has a unique representation $x = hi$ $h \in U(\mathfrak{h})$, $i \in U(\mathfrak{I})$. Thus every linear map $\psi \in V(\mathfrak{I}/\{0\})$ on $U(\mathfrak{I})$ has a unique $U(\mathfrak{h})$ linear extension ψ^{ext} to $U(\mathfrak{g})$

$$\langle hi, \psi^{\text{ext}} \rangle = h \langle i, \psi \rangle. \quad (13)$$

□

Using the isomorphism we turn $V(\mathfrak{I}/\{0\})$ into a \mathfrak{g} -module. Define the product $X \cdot \psi$ of $X \in \mathfrak{g}$ and $\psi \in V(\mathfrak{I}/\{0\})$ by

$$\langle i, X \cdot \psi \rangle = \langle iX, \psi^{\text{ext}} \rangle \quad i \in U(\mathfrak{I}). \quad (14)$$

Rewriting iX , using that \mathfrak{I} is an ideal and $U(\mathfrak{h})$ -linearity we find the expressions

$$\langle i, X \cdot \psi \rangle = \begin{cases} \langle i, X\psi \rangle & \text{if } X \in \mathfrak{I} \\ (-1)^{|i||X|} X \langle i, \psi \rangle + \langle [i, X], \psi \rangle & \text{if } X \in \mathfrak{h} \end{cases} \quad (15)$$

2.5. A realization of $V(\mathfrak{g}/\mathfrak{h})$ for $\mathfrak{h} = \{0\}$.

In the previous section we encountered produced representations with $\mathfrak{h} = \{0\}$. We now turn to their structure. For simplicity assume \mathfrak{g} to be finite dimensional.

Let $\tilde{S}(\mathfrak{g})$ be the subspace of symmetric tensors in the full tensor algebra $T(\mathfrak{g})$; $\tilde{\pi}_S$ and $\tilde{\pi}_U$ the restrictions to $\tilde{S}(\mathfrak{g})$ of the canonical projections from $T(\mathfrak{g})$ on $S(\mathfrak{g})$ and $U(\mathfrak{g})$ respectively. By the PBW theorem

$$N : S(\mathfrak{g}) \xrightarrow{\tilde{\pi}_S^{-1}} \tilde{S}(\mathfrak{g}) \xrightarrow{\tilde{\pi}_U} U(\mathfrak{g}) \quad (16)$$

is an isomorphism called the *canonical isomorphism* (see [6, p. 92]). It induces an isomorphism N^* on the spaces of linear maps

$$N^* : V(\mathfrak{g}/\{0\}) = \mathcal{L}(U(\mathfrak{g}), V) \xrightarrow{\sim} \mathcal{L}(S(\mathfrak{g}), V). \tag{17}$$

In particular $R \cong \mathcal{L}(S(\mathfrak{g}), k) = S(\mathfrak{g})^*$.

Now, the symmetric algebra, like $U(\mathfrak{g})$ is a coalgebra (it is the universal enveloping algebra of a Lie algebra with trivial Lie bracket) so $S(\mathfrak{g})^*$ is an algebra.

THEOREM 2.5.1. — *The homomorphism N^* is an algebra homomorphism.*

Proof. — Since

$$N^*fg = N^*fN^*g \quad f, g \in R \tag{18}$$

\Leftrightarrow

$$\langle \Delta NP, f \otimes g \rangle = \langle N \otimes N\Delta P, f \otimes g \rangle \quad \forall P \in S(\mathfrak{g}) \tag{19}$$

it is sufficient to show that $\Delta N = N \otimes N\Delta$ or in other words that N is a coalgebra map. This is proved by induction on the degree of P or a bit of coalgebra theory. \square

For a better understanding of $S(\mathfrak{g})^*$ we choose “local coordinates” and change to a suggestive notation. The *formal power series* $\mathbb{C}[[X_1 \dots X_n]]$ on a graded vector space L with basis $\partial_1 \dots \partial_n$ is the set of formal sums

$$f = \sum_{\alpha} a_{\alpha} X^{\alpha} \quad a_{\alpha} \in k \tag{20}$$

where $X_1 \dots X_n$ are the dual basis vectors, commuting or anti-commuting according to parity, and α a graded multiindex i.e. $\alpha_i = 0, 1, 2, \dots$ if X_i is even, $\alpha_j = 0, 1$ if X_j is odd. We define an action of $P \in S(L)$ on

$f \in \mathbb{C}[[X_1 \dots X_n]]$ by $(Pf)(X_1 \dots X_n) = P\left(\frac{\partial}{\partial X_1} \dots \frac{\partial}{\partial X_n}\right)f(X_1 \dots X_n)$ where

$P\left(\frac{\partial}{\partial X_1} \dots \frac{\partial}{\partial X_n}\right)$ is the differential operator obtained by substituting $\frac{\partial}{\partial X_i}$

for ∂_i in every monomial ∂^{α} in P . We also define a pairing between $S(L)$ and $\mathbb{C}[[X_1 \dots X_n]]$ (denoted by round brackets)

$$(P, f) = (Pf)|_{X_i=0} \tag{21}$$

which induces a homomorphism $\mathbb{C}[[X_1 \dots X_n]] \rightarrow S(L)^*$. The interpretation of Δ as the higher order Leibnitz rule now shows that it is actually an *algebra* homomorphism. In fact if we define more generally the *vector valued formal power series* as formal sums

$$V[[X_1 \dots X_n]] \stackrel{\text{def}}{=} \sum_{\alpha} v_{\alpha} X^{\alpha} \quad v_{\alpha} \in V \tag{22}$$

then

THÉOREM 2.5.2. — *The obvious pairing induces an isomorphism*

$$\mathcal{L}(S(L), V) \cong V[[X_1 \dots X_n]] \quad (23)$$

as modules over $S(L)^ \cong \mathbb{C}[[X_1 \dots X_n]]$ and the action of ∂_i on $\mathcal{L}(S(L), V)$ corresponds to the action of $\frac{\partial}{\partial X_i}$ on $V[[X_1 \dots X_n]]$.*

Proof. — By definition we have

$$\langle \partial^\alpha, \Sigma_\beta v_\beta X^\beta \rangle = c(\alpha) v_\alpha \quad (24)$$

with $c(\alpha)$ a non zero integer, which proves injectivity. It is also surjective because if $\phi \in \mathcal{L}(S(L), V)$ then $\phi = \Sigma_\alpha (\partial^\alpha, \phi) / c(\alpha) X^\alpha$. The last statement is immediate from the definition of the pairing. \square

3. THE SUPER POINCARÉ ALGEBRA

We now apply the preceding methods to the Poincaré algebra and construct both superfields and superspace.

3.1. Definition.

The ($N = 1$) super Poincaré algebra is the semi-direct sum

$$\mathcal{P}_{\text{super}} = \text{so}(n, m) \oplus \mathbb{T} \quad (25)$$

$$\mathbb{T} = \mathbb{T}_{\bar{0}} \oplus \mathbb{T}_{\bar{1}} \quad (26)$$

$$= \mathcal{V} \oplus (\mathcal{S} \oplus \overline{\mathcal{S}}) \quad (27)$$

where \mathcal{V} , \mathcal{S} , $\overline{\mathcal{S}}$ are the vector, spinor and conjugate spinor representation of $\text{so}(n, m)$. The commutation relations are written as

$$\begin{aligned} [M, M'] &= i(MM' - M'M) \\ [M, P] &= iMP & [M, Q] &= MQ & [M, \bar{Q}] &= M\bar{Q} \\ [P, P'] &= 0 & [P, Q] &= 0 & [P, \bar{Q}] &= 0 \\ [Q, Q']_+ &= 0 & [\bar{Q}, \bar{Q}']_+ &= 0 & [Q, \bar{Q}]_+ &= 2\sigma(\bar{Q}, Q). \end{aligned} \quad (28)$$

The i 's are conventional in field theory. Identify a sesquilinear form on a vectorspace with a bilinear form on the space and its complex conjugate. Then the invariant bilinear form

$$\sigma : \overline{\mathcal{S}} \otimes \mathcal{S} \rightarrow \mathcal{V} \quad (29)$$

can be considered as the restriction of the sesquilinear form “ $(\overline{\psi} \gamma^\mu \psi) P_\mu$ ”

to the chiral (Weyl) subspace \mathcal{S} of the Dirac spinors. If we choose a basis $P_\mu, Q_\alpha, \bar{Q}_{\dot{\alpha}}$ ⁽³⁾ of T and a slightly modified dual basis $X^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ defined by

$$\langle P_\mu, X^\mu \rangle = i \quad \text{others zero} \tag{30}$$

$$\langle Q_\alpha, \theta^\alpha \rangle = i \quad \text{others zero} \tag{31}$$

$$\langle \bar{Q}_{\dot{\alpha}}, \bar{\theta}^{\dot{\alpha}} \rangle = i \quad \text{others zero} \tag{32}$$

then the bilinear form expressed in these bases is

$$\sigma = i^{-2} \sigma_{\alpha\dot{\alpha}}^\mu \theta^\alpha \bar{\theta}^{\dot{\alpha}} P_\mu \tag{33}$$

where the $\sigma_{\alpha\dot{\alpha}}^\mu$ are the Pauli matrices in the usual four dimensional case.

3.2. Computation of the produced representation.

We compute the produced representation in terms of differential operators. These operators will then be compared to the expressions in the physics literature.

First use theorems (2.4.1) (2.5.1) and (2.5.2) which give us

$$V(\mathcal{P}_{\text{super}}/\text{so}(n, m)) \cong V(T/\{0\}) \cong V[[X, \theta, \bar{\theta}]] \tag{34}$$

then pull back the action of $\mathcal{P}_{\text{super}}$ on $V(\mathcal{P}_{\text{super}}/\text{so}(n, m))$ to one on $V[[X, \theta, \bar{\theta}]]$ once again denoted by a dot.

For an effective computation of the pulled back representation, it is convenient to use the *vector valued polynomials*

$$\begin{aligned} V[X, \theta, \bar{\theta}] &\stackrel{\text{def}}{=} \{ \phi \in V[[X, \theta, \bar{\theta}]], \phi \text{ breaks off after finitely many terms} \} \\ &= \{ \phi \mid (\partial, \phi) = 0 \quad \forall \partial \in S(T) \quad \text{with} \quad \text{deg}(\partial) \gg 0 \} \\ &= \bigoplus_{\text{deg}=n} \mathcal{L}(S(T)_{\text{deg}=n}, V) \end{aligned} \tag{35}$$

for as $\mathcal{P}_{\text{super}}$ acts as derivations, it is sufficient to determine the images of $X, \theta, \bar{\theta} \in T^*$ and $v1$ for all $v \in V$. We do not lose anything because a derivation on $V[X, \theta, \bar{\theta}]$ has a unique extension to $V[[X, \theta, \bar{\theta}]]$. Unfortunately, for fixed $\phi \in V[X, \theta, \bar{\theta}]$, $A \in \mathcal{P}_{\text{super}}$ there is no *a priori* bound on the degrees of $S(T)$ that pair nontrivially with $A \cdot \phi$, in particular we do not know if $V[X, \theta, \bar{\theta}]$ is invariant under the action of $\mathcal{P}_{\text{super}}$. The problem is that N only preserves the filtration on $S(T)$ for in general no \mathbb{Z} -gradation is defined on an enveloping algebra.

⁽³⁾ The α 's are indices of chiral spinors rather than multiindices.

We take care of this problem by the choice of a $\frac{1}{2}\mathbb{Z}$ -grading on $\mathcal{P}_{\text{super}}$ denoted by Deg and defined by

$$\begin{array}{ccc} \text{so}(n, m) & \mathcal{S} \oplus \overline{\mathcal{P}} & \mathcal{V} \\ & 0 & 1 \\ & \frac{1}{2} & \\ & & 1 \end{array} \quad (36)$$

It induces a $\frac{1}{2}\mathbb{Z}$ -grading Deg on $U(\mathbb{T})$ and $S(\mathbb{T})$ which is preserved by the canonical isomorphism. Hence there is an invariant subspace of $V(\mathbb{T}/\{0\})$, the *polynomial elements*, defined by

$$V(\mathbb{T}/\{0\})_{\text{pol}} = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathcal{L}(U(\mathbb{T})_{\text{Deg}=k}, V). \quad (37)$$

The subspace $V(\mathbb{T}/\{0\})_{\text{pol}}$ has yet another natural $\frac{1}{2}\mathbb{Z}$ -grading once again denoted by Deg . Finally, we endow $V[X, \theta, \bar{\theta}]$ with a $\frac{1}{2}\mathbb{Z}$ grading Deg , defined on generators by

$$\begin{array}{ccc} v1 & \theta, \bar{\theta} & X \\ & 0 & \frac{1}{2} \quad 1. \end{array} \quad (38)$$

LEMMA 3.2.1. — *An element $\phi \in V(\mathbb{T}/\{0\})$ is polynomial if and only if $N^*\phi \in V[X, \theta, \bar{\theta}]$. The homomorphism $N^*V(\mathbb{T}/\{0\})_{\text{pol}} \rightarrow V[X, \theta, \bar{\theta}]$ is Deg preserving.*

COROLLARY 3.2.2. — *If $\phi \in V(\mathbb{T}/\{0\})_{\text{pol}}$ is Deg homogeneous then $N^*\phi$ is killed by all elements $\partial \in S(\mathbb{T})$ with $\text{Deg}(\partial) \neq \text{Deg}(\phi)$.*

Proof. — (Lemma) Note that N is Deg preserving and that $\text{deg}(\partial) \geq \text{Deg}(\partial) \geq \frac{1}{2} \text{deg}(\partial)$ for all $\partial \in S(L)$. We thus have

$$V[X, \theta, \bar{\theta}] \cong \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathcal{L}(S(\mathbb{T})_{\text{Deg}=k}, V) \stackrel{N^*}{\cong} \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathcal{L}(U(\mathbb{T})_{\text{Deg}=k}, V) \cong V(\mathbb{T}/\{0\}) \quad (39)$$

with all the isomorphism Deg preserving. \square

As an example we shall now compute the action of the generator Q . To avoid an unnecessary complicated notation, we write $X, \theta, \bar{\theta}$ for $N^{*-1}(X)$, $N^{*-1}(\theta)$ and $N^{*-1}(\bar{\theta})$. We denote a arbitrary Deg homogeneous element of $S(\mathbb{T})$ by ∂ and in analogy with Quantum field theory, we write $:\partial:$ for $N(\partial)$. We find using corollary (3.2.2).

$$(\partial, Q \cdot X) = \langle : \partial : Q, X \rangle = 0 \quad \text{if} \quad \text{Deg}(\partial) + \text{Deg}(Q) \neq \text{Deg}(X) = 1. \quad (40)$$

The two remaining choices for ∂ give

$$(Q', Q \cdot X) = \langle Q'Q, X \rangle = (Q'Q, X) = 0 \tag{41}$$

$$\begin{aligned} (\bar{Q}, Q \cdot X) &= \langle \bar{Q}Q, X \rangle \\ &= \left\langle \frac{1}{2}(\bar{Q}Q - Q\bar{Q}) + \frac{1}{2}[\bar{Q}, Q], X \right\rangle \end{aligned} \tag{42}$$

$$= \left\langle \frac{1}{2} : \bar{Q}Q : + : \sigma(\bar{Q}, Q) :, X \right\rangle$$

$$= \left(\frac{1}{2} \bar{Q}Q + \sigma(\bar{Q}, Q), X \right)$$

hence we conclude for generators $Q_\alpha X^\mu = \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}}$. Similarly we find

$$Q_\alpha \theta^\beta = i \delta_\alpha^\beta \tag{43}$$

$$Q_\alpha \bar{\theta}^{\dot{\alpha}} = 0 \tag{44}$$

$$Q_\alpha v^1 = 0. \tag{45}$$

This means that Q_α is represented on $V[[X, \theta, \bar{\theta}]]$ by a differential operator

$$Q_\alpha = i \left(\frac{\partial}{\partial \theta^\alpha} - i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial X_\mu} \right) = i D_\alpha. \tag{46}$$

With the same methods

$$P_\mu = i \left(\frac{\partial}{\partial X^\mu} \right) = i D_\mu \tag{47}$$

$$\bar{Q}_{\dot{\alpha}} = i \left(\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \sigma_{\alpha\dot{\alpha}}^\mu \theta^\alpha \frac{\partial}{\partial X_\mu} \right) = -i D_{\dot{\alpha}}. \tag{48}$$

The differential operators D_μ , D_α and $D_{\dot{\alpha}}$ have precisely the form of the left invariant ⁽⁴⁾ chiral super symmetry operators (see [12, p. 111]). The minus sign is a matter of convention.

The action of the subalgebra $so(n, m)$ involves two terms

$$\begin{aligned} (\partial, M \cdot X) &= M \langle : \partial :, X \rangle + \langle [: \partial :, M], X \rangle \\ &= 0 + \langle : [: \partial :, M] :, X \rangle \\ &= \begin{cases} 0 & \text{if } \text{Deg}(\partial) \neq 1 \\ 0 & \text{if } \partial = QQ', \bar{Q}\bar{Q}', Q\bar{Q} \\ -i(MP, X) & \text{if } \partial = P. \end{cases} \end{aligned} \tag{49}$$

⁽⁴⁾ Left invariance is to be expected because the Lie algebra acts from the right on the enveloping algebra, corresponding to a right action of the supergroup on itself. The chiral operators can be interpreted as the projection on superspace, of the infinitesimal (right) flow of the one (odd) parameter subgroups generated by P_μ , Q_α or $\bar{Q}_{\dot{\alpha}}$. Since right action commutes with left action these operators are left invariant.

We conclude that but for a factor i , so (n, m) acts according to the coadjoint action on X and likewise on $\theta, \bar{\theta}$. On generators:

$$M_{\mu\nu}X^\lambda = -i(\eta_{\mu\sigma}\delta_\nu^\lambda - \eta_{\sigma\nu}\delta_\mu^\lambda)X^\sigma \quad (50)$$

$$M_{\mu\nu}\theta^\alpha = -i\left(\frac{1}{2}(\sigma_{\mu\nu})_\beta^\alpha\theta^\beta\right) \quad (51)$$

$$M_{\mu\nu}\bar{\theta}^{\dot{\alpha}} = -i\left(\frac{1}{2}(-\bar{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\right) \quad (52)$$

Here $\frac{1}{2}\sigma_{\mu\nu}\left(-\frac{1}{2}\bar{\sigma}_{\mu\nu}\right)$ is the conventional notation for the spinor (conjugate spinor) representation of $M_{\mu\nu}$. Apart from acting on coordinates, so (n, m) acts by linear transformations of the vector space V

$$\begin{aligned} (\partial, M \cdot v1) &= M \langle \partial, v1 \rangle + \langle [\partial, M], v1 \rangle \\ &= \begin{cases} 0 & \text{if } \text{Deg}(\partial) \neq 0 \\ Mv \stackrel{\text{def}}{=} \rho(M)v & \text{if } \partial = 1 \end{cases} \end{aligned} \quad (53)$$

where ρ is the representation of so (n, m) on V . Hence the generator $M_{\mu\nu}$ is represented by the mixed differential operator/linear operator

$$M_{\mu\nu} = i\left(-X_\mu \frac{\partial}{\partial X^\nu} + X_\nu \frac{\partial}{\partial X^\mu} - \frac{1}{2}(\sigma_{\mu\nu})_\beta^\alpha \theta^\beta \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2}(\bar{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}\right) + \rho(M_{\mu\nu}). \quad (54)$$

Here the relative minus sign with respect to ([12, p. 108]) arises because of the right rather than the left representation that we use.

3.3. Superfields and superspace.

The produced representation reproduces the expressions for the (right) representation of the super Poincaré algebra on superfields after “coordinates have been chosen” with equation (34). It is tempting to identify $V(\mathcal{P}_{\text{super}}/\text{so}(n, m))$ as an invariant model for V -superfields. However the identification (34) also shows that the componentfields (the coefficients in the $\theta, \bar{\theta}$ expansion) are formal powerseries. We shall call such fields *V-superfields with infinitesimal domain*.

In fact if in the spirit of algebraic geometry and functional analysis, we define a superspace $M_{\text{super,inf}}$ as the set of algebra homomorphisms

$$R \cong \mathbb{C}[[X, \theta, \bar{\theta}]] \rightarrow \mathbb{C} \quad (55)$$

then we find that $M_{\text{super,inf}}$ has just one point because R has only one maximal ideal ⁽⁵⁾.

⁽⁵⁾ The space $M_{\text{super,inf}}$ is the closed point of the scheme $\text{spec}(R)$. Since R is a local ring, the structure sheaf at this point is naturally isomorphic to R , which makes it very different from a simple point (see [5]).

The more interesting space is probably $V(\Gamma/\{0\})_{\text{pol}}$ because the component fields are polynomial and thus have an interpretation as functions. Once again, construct a superspace M_{super} as the set of algebra homomorphisms $R_{\text{pol}} \cong \mathbb{C}[X, \theta, \bar{\theta}] \rightarrow \mathbb{C}$. We find that M_{super} is ordinary complexified Minkowski space, but with superfields as the natural ring of "functions" defined on it as follows: the elements θ and $\bar{\theta}$ are nilpotent so they are in the kernel of every $m \in M_{\text{super}}$ (m being an algebra homomorphism).

$$\begin{array}{ccc}
 R_{\text{pol}} \cong \mathbb{C}[X, \theta, \bar{\theta}] & \xrightarrow{m} & \mathbb{C} \\
 \downarrow & \nearrow \tilde{m} & \\
 \mathbb{C}[X = X^\mu] & &
 \end{array} \tag{56}$$

Now $\mathbb{C}[X^\mu]$ is freely generated by $X^1 \dots X^{n+m}$ so the algebra homomorphisms of $\mathbb{C}[X^\mu]$ can be identified with points $(\tilde{m}^1 \dots \tilde{m}^{n+m})$, where $\tilde{m}^i = \tilde{m}(X^i) \stackrel{\text{def}}{=} X^i(\tilde{m})$. Here we introduced the notation $f(m)$ for the evaluation of $m \in M_{\text{super}}$ on $f \in R_{\text{pol}}$ which is more natural if we think of m as a point. Endow M_{super} with the weakest topology such that all $f \in R_{\text{pol}}$ are continuous i. e.

$$U \subset M_{\text{super}} \text{ is open} \Leftrightarrow U = f_1^{-1}(V_1) \cap \dots \cap f_n^{-1}(V_n) \tag{57}$$

where $f_i \in R_{\text{pol}}$ and the V_i are open sets in \mathbb{C} . If we take $f_i = X^i$ and note that R_{pol} is finitely generated then we see that this is just the ordinary topology on complexified Minkowski space. The superstructure is encoded in the structure sheaf \mathcal{O} . It assigns to each open set $U \subset M_{\text{super}}$ the algebra $\mathcal{O}(U)$ of all quotients f/g , $f, g \in R_{\text{pol}}$ with $g(m) \neq 0$ for all $m \in U$ ⁽⁶⁾. Note that g is necessarily even and that in local coordinates we may assume that g depends only on X^μ because we can factor $g(X, \theta, \bar{\theta}) = g'(X)(1 + \text{nilpotent})$. The superfields are local in the sense that they may have poles outside of U .

In the above approach we end up with a complex algebraic version of the Leites-Kostant graded manifold (see [8], [7], [2]). A similar procedure is applicable to construct a Rogers-DeWitt supermanifold [9], [14] by defining superspace as the set of homomorphisms $R \rightarrow B$, where B is a finite or infinite dimensional Grassmann algebra with some appropriate topology.

3.4. Real superspace.

The reader may have wondered why we did not use the reals so as to find a real super Minkowski space. The reason is that spinors are more naturally defined over the complex numbers and realness introduces some technical problems. We shall sketch how to proceed.

⁽⁶⁾ Technically $\mathcal{O}(U)$ is the localization of R_{pol} in the maximal ideals $\{\ker m\}_{m \in U}$.

Endow the Lie algebra with a real structure and induce one on the enveloping algebra. The even part of the super Poincaré algebra has a natural real structure (being a complexified real Lie algebra) whereas for the odd part we can use the Majorana conjugation. This conjugation is the intertwiner with respect to the Clifford algebra between the *Dirac* spinors and their complex conjugate. For most dimensions chirality and the Majorana condition are not compatible so we would have to rewrite everything in terms of Dirac spinors. Next, we dualise the real structure on the enveloping algebra to a real structure on \mathbb{R} and \mathbb{R}_{poi} . The real superspace is the set of algebra homomorphisms that intertwine the real structure on \mathbb{R}_{poi} and \mathbb{C} . These can of course be identified with points in complex Minkowski space with real coordinates. Finally we note that the real structure on \mathbb{R}_{poi} induces one on the structure sheaf \mathcal{O} . Hence there is a natural notion of real superfield.

3.5. Generalizations.

Though convenient for computations, the restriction to semi-direct sum Lie algebras is not really necessary. In general we have $\mathbb{R} \cong \mathbb{C}[[\mathfrak{h}^\perp]]$ as algebras where \mathfrak{h}^\perp is some (graded) complement of $\mathfrak{h} \subset \mathfrak{g}$. However the action of \mathfrak{g} on $\mathbb{C}[[\mathfrak{h}^\perp]]$, is far from canonical. A more fundamental problem is the correct generalization of the right subset of “polynomials” to define the analog of global superspaces. This is a hard problem because in general, we would expect to find algebraic homogeneous spaces (which may be compact), so some kind of patching procedure will be necessary.

One approach to this problem is to begin right from the start with a (graded) group although it is against the spirit of this paper. The precise relationship between the Lie algebra and the (graded) group approach would be interesting to study. We believe that under mild conditions, a produced representation of a real or complex Lie algebra can be obtained by completing a C^∞ induced representation with respect to the maximal ideal of the unit element.

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