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PETER D. HISLOP
SHU NAKAMURA

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Semiclassical resolvent estimates

by

Peter D. HISLOP (1) **and Shu NAKAMURA** (2)
Centre de Physique Théorique, C.N.R.S.,
Luminy Case 907, 13288 Marseille Cedex 9, France

ABSTRACT. — We prove estimates in the semiclassical regime of small h on the boundary values of the resolvent of the Schrödinger operator: $H(h) = -h^2 \Delta + V$ in a neighborhood of a non-trapping energy E. The potential V is bounded, but not necessarily decaying with derivatives decaying at infinity. The method also applies to potentials with local singularities and to a family of Stark Hamiltonians. The proof is based on Mourre theory and decay estimates for wave packets in the classically forbidden region.

RÉSUMÉ. — Dans le régime semi-classique (petit h), nous estimons les valeurs au bord de la résolvante de l'opérateur de Schrödinger $H(h) = -h^2 \Delta + V$ dans un voisinage d'une énergie non liante E. Le potentiel V est borné mais n'est pas nécessairement décroissant mais ou avec des dérivées décroissantes à l'infini. La méthode s'applique aussi à des potentiels avec des singularités locales et à une famille d'Hamiltoniens de Stark. La preuve repose sur la théorie de Mourre et des estimations de décroissance des paquets d'ondes dans la zone classiquement interdite.

(1) Also PHYMAT, Département de Mathématiques, Université de Toulon et du Var, 83130 La Garde, France; Mathematics Department, University of Kentucky, Lexington, KY 40506.
(2) Department of Pure and Applied Sciences, College of Arts and Sciences, University of Tokyo 3-8-1, Komaba, Meguro-ku, Tokyo 153, Japan.

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1. INTRODUCTION

Semiclassical estimates on the resolvent of Schrödinger operators are an important technical tool for studying the behavior of observables like the scattering matrix and the total cross-section ([RT-1], [RT-2], [Y], *see* also [N-1] for an application to the shape resonances). In this note, we give a simple proof of these estimates for a large class of potentials. We give the details for reasonably smooth potentials and discuss the generalization in Section 4. We consider the following conditions:

CONDITION (A). — V is a real valued function such that $V=V_1+V_2$ with $V_i \in C^i(\mathbb{R}^n)$, $i=1, 2$ and

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha V_1(x)\right| \leqq C\langle x\rangle^{-1-|\alpha|} \quad \text{for} \quad |\alpha|=0, 1,$$

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha V_2(x)\right| \leqq C\langle x\rangle^{-|\alpha|} \quad \text{for} \quad |\alpha|=0, 1, 2$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$, $\alpha=(\alpha_1, \ldots, \alpha_n)$, $|\alpha|=\sum \alpha_i$.

We will consider fixed energy $E \in \mathbb{R}$ and let $G(E) : =\{x \in \mathbb{R}^n \mid V(x)-E>0\}$.

CONDITION (B). — There are constants δ, $\varepsilon_0>0$ and a C^3-vector field $f: \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) $$\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right| \leqq C\langle x\rangle^{1-|\alpha|} \quad \text{for} \quad |\alpha| \leqq 2$$

and $|\Delta(\nabla . f)(x)| \leqq C$;

(ii) $$2\left(\inf_{\xi \in \mathbb{R}^n} |\xi|^{-2}\langle \xi, J_f(x)\xi\rangle\right)(E-V(x))-f(x).\nabla V(x) \geqq \varepsilon_0 \tag{1.1}$$

for any $x \in G_c(E+\delta) : =\mathbb{R}^n \setminus G(E+\delta)$, where J_f is the Jacobian of f and $\langle .,. \rangle$ denotes the Euclidean inner product.

Condition (A) implies $H(h)$ is self-adjoint on $H^2(\mathbb{R}^n)$ (the Sobolev space of order 2). Our main result is:

THEOREM. — *Let* $H(h) : =-h^2\Delta+V$ *and suppose that* V *satisfies Conditions* (A) *and* (B) *at energy* E. *Then there is an open interval* $I \ni E$ *such that for any* $\alpha>1/2$ *and* $\lambda \in I$,

$$\lim_{\varepsilon \to 0} \langle x\rangle^{-\alpha}(H-\lambda \pm i\varepsilon)^{-1}\langle x\rangle^{-\alpha}$$

exists, and

$$\|\langle x\rangle^{-\alpha}(H-\lambda \pm i0)^{-1}\langle x\rangle^{-\alpha}\| \leqq Ch^{-1} \quad (\lambda \in I) \tag{1.2}$$

if h is sufficiently small.

This result is a key ingredient in the estimation of the semiclassical behavior of the scattering cross-section $\sigma_h(E, \omega)$, $E>0$, $\omega \in S^{n-1}$. For potentials $V(x)=O(\langle x \rangle^{-\alpha})$, $\alpha > \frac{n+1}{2}$, and energies E such that (1.1) holds on $\mathbb{R}^n$ with $f(x)=x$, the leading behavior of $\sigma_h(E, \omega)$ is $O(h^{-\nu})$, where $\nu \equiv (n-1)(\alpha-1)^{-1}$ (*cf.* [RT-2], [Y]). Using the above theorem, it should be possible to extend [RT-2] to the more general situation where V satisfies Condition (A) (with possible local singularities, *see* Section 4) and is non-trapping in the sense of Condition (B). A similar result may hold for Stark Hamiltonians discussed in Section 4. We also remark that our methods apply to generalized N-body Schrödinger operators, although the potential V does not satisfy Condition (A). The potential $V=\sum_{j=1}^{N} V_j \circ \pi_j$, where $\{\pi_j\}_{j=1}^{N}$ is a set of mutually orthogonal projections in $\mathbb{R}^n$. We assume that each V_j satisfies Condition (A) on $\pi_j(\mathbb{R}^n)$. Then, if we take $f(x)=x$ in Condition (B) and consider energies E for which the resulting nontrapping condition (1.1) holds on $\mathbb{R}^n$, the analog of the above theorem holds for $H=-h^2\Delta+V$. To see this, we simply note that all the remainder terms in (3.3)-(3.5) vanish except for $(x.\nabla)V$ and $(x.\nabla)^2 V$ because $\partial^2 f_i/(\partial x_j \partial x_k)=0$. (Jensen [J] has recently obtained similar results in this case).

Our proof of this theorem is given in Sections 2-3. In Section 4, we discuss generalizations to potentials with singularities and to Stark Hamiltonians. Our method of proof utilizes the local positive commutator approach of E. Mourre ([M], [CFKS]) to obtain estimates in the nontrapping region $\mathbb{R}^n \setminus G(E+\delta)$ and semiclassical decay estimates on wave packets localized to $G(E+\delta)$ (*cf.* the Appendix).

Some results on semiclassical resolvent estimates are known. These first appeared in a paper by Robert and Tamura [RT-1] who consider nontrapping potential $V \in C_0^\infty(\mathbb{R}^n)$. Later, in [RT-2] they obtained semiclassical resolvent estimates at (classically) nontrapping energy E for smooth potentials decaying at infinity as $\langle x \rangle^{-\rho}$, $\rho>0$, using both Mourre theory and Fourier integral methods. We note that Condition (B) implies the classical condition of [RT-1], [RT-2]. A shorter proof of their result was given by Gérard and Martinez [GM] who constructed an escape function $a(x, p)$ such that the Poisson bracket $\{h, a\}$ is globally positive. Yafaev [Y] also used Mourre theory to obtain semiclassical resolvent estimates in the high energy regime for potentials C^2 in the $|x|$-variable and satisfying $|x|\left|\left(\frac{\partial}{\partial |x|}\right)^k V(x)\right| \leqq C(k=0, 1, 2)$. A method of Lavine [L] was also applied to prove estimate (1.2) for decaying potentials under nontrapping condition (1.1) with $f(x)=x$ [N-1].

We note that the semiclassical resolvent estimate is closely related to the absence of resonances near the real axis in the semiclassical limit. Our nontrapping condition (1.2) first appeared in a proof of the absence of resonance in [N-2] (*see* also [BCD-1], [DeBH], [HeSj], [K], [S-1]).

2. SEMICLASSICAL MOURRE ESTIMATES

We restate the standard assumptions of the Mourre theory for a self-adjoint operator H and a skew-operator A in an h-dependent manner. For $s \geqq 0$, let $\mathscr{H}_s := D((|H|+1)^{s/2})$ with the norm $\|\psi\|_s := \|(|H|+1)^{s/2}\psi\|$, and $\mathscr{H}_{-s} := \mathscr{H}_s^*$. $\|\cdot\|_{s,t}$ denotes the norm of the maps from $\mathscr{H}_s$ to $\mathscr{H}_t$. We let C denote a h-independent constant whose value may change from line to line.

(M1) $D(A) \cap \mathscr{H}_2$ is dense in $\mathscr{H}_2$.

(M2) The form [H, A] defined on $D(A) \cap \mathscr{H}_2$ extends to a bounded operator from $\mathscr{H}_2$ to $\mathscr{H}_{-1}$ and $\|[H, A]\|_{2,-1} \leqq Ch$.

(M3) There exists a self-adjoint operator H_0 with $D(H_0) = D(H)$ such that $[H_0, A]$ extends to a bounded operator from $\mathscr{H}_2$ to $\mathscr{H}_0$, $\|[H_0, A](H_0+i)^{-1}\| \leqq C$, $\|H(H_0+i)^{-1}\| \leqq C$ and $D(A) \cap D(H_0 A)$ is a core for H_0.

(M4) The form [[H, A], A] where [H, A] is as in (M2) extends from $D(A) \cap D(HA)$ to a bounded operator from $\mathscr{H}_2$ to $\mathscr{H}_{-2}$ and $\|[[H, A], A]\|_{2,-2} \leqq Ch$.

DEFINITION (The semiclassical Mourre estimate). — Let g be a function such that $g \in C_0^\infty(\mathbb{R})$, $0 \leqq g(x) \leqq 1$ and $g=1$ on a neighborhood of an interval I. We say that the semiclassical Mourre estimate holds on I if there exist such a $g \in C_0^\infty(\mathbb{R})$, an operator $K(h)$ from $\mathscr{H}_2$ to $\mathscr{H}_{-2}$ with $\|K(h)\|_{2,-2} \to 0$ as $h \to 0$ and $\alpha_0 > 0$ such that

$$M^2 := g(H)[H, A]g(H) \geqq \alpha_0 h g(H)^2 + h g(H) K(h) g(H). \qquad (2.1)$$

PROPOSITION 2.1. — *Let* $H(h)$ *be a self-adjoint operator and* $A(h)$ *a skew-adjoint operator satisfying* (M1)-(M4), *and suppose the Mourre estimate* (2.1) *holds on* $I \subset \mathbb{R}$. *Then there exist* $h_0 > 0$ *such that for any* $\alpha > 1/2$, $h \in (0, h_0)$ *and* $E \in I$, $\lim_{\varepsilon \to 0} \langle A \rangle^{-\alpha}(H-E \pm i\varepsilon)^{-1}\langle A \rangle^{-\alpha}$ *exists and*

$$\|\langle A \rangle^{-\alpha}(H-E \pm i0)^{-1}\langle A \rangle^{-\alpha}\| \leqq Ch^{-1}. \qquad (2.2)$$

Proof. — (1) We retrace the proof of Mourre as presented in [CFKS] and [PSS] keeping track of the h-dependence, and we refer Section 4.3 of [CFKS] for details. At first we remark that if h is sufficiently small, the second term of the RHS of (2.1) is dominated by the first term, and hence it can be omitted.

For $\varepsilon>0$ let $G_\varepsilon(z):=(H-i\varepsilon M^2-z)^{-1}$ which is analytic in z for $\operatorname{Re} z\in I$ and $\operatorname{Im} z>0$. Then we obtain the following estimates (*cf.* Lemma 4.14 of [CFKS]):

$$\|g(H)G_\varepsilon(z)\varphi\|\leqq(2\varepsilon\alpha_0 h)^{-1/2}|(\varphi, G_\varepsilon(z)\varphi)|^{1/2}, \tag{2.3}$$

$$\|(1-g(H))G_\varepsilon(z)\|\leqq C(1+\varepsilon h\|G_\varepsilon(z)\|), \tag{2.4}$$

$$\|G_\varepsilon(z)\|\leqq C(\varepsilon\alpha_0 h)^{-1}, \tag{2.5}$$

if ε is sufficiently small. It follows in the same way as in [CFKS] that the bounds (2.3), (2.4) and (2.5) hold with $\|.\|_{0,2}$ replacing $\|.\|$.

(2) Let $D_\varepsilon:=(1+|A|)^{-\alpha}(\varepsilon|A|+1)^{\alpha-1}$ for $\alpha\in(1/2, 1]$, $\varepsilon>0$ and let $F_\varepsilon(z):=D_\varepsilon G_\varepsilon(z)D_\varepsilon$ for $z:\operatorname{Re} z\in I$, $\operatorname{Im} z>0$. By (2.5) and the definition of $F_\varepsilon(z)$,

$$\|F_\varepsilon(z)\|\leqq\|D_\varepsilon\|^2.\|G_\varepsilon(z)\|\leqq C(\varepsilon\alpha_0 h)^{-1}. \tag{2.6}$$

From (2.3) and (2.4) with $\varphi=D_\varepsilon\psi$, we have

$$\|G_\varepsilon D_\varepsilon\|\leqq C((\alpha_0\varepsilon h)^{-1/2}\|F_\varepsilon\|^{1/2}+1).$$

The derivative of $F_\varepsilon(z)$ in ε is estimated using (2.3)-(2.6) (*cf.* [CFKS], Lemma 4.15), and we obtain

$$\left\|\frac{dF_\varepsilon}{d\varepsilon}\right\|\leqq C\varepsilon^{\alpha-1}(1+(\alpha_0\varepsilon h)^{-1/2}\|F_\varepsilon\|^{1/2}+\|F_\varepsilon\|). \tag{2.8}$$

It follows from (2.6) and (2.8) that there exists $C>0$ such that

$$\overline{\lim_{\varepsilon\downarrow 0}}\sup_{\lambda\in I}\|\langle A\rangle^{-\alpha}(H-\lambda\pm i\varepsilon)\langle A\rangle^{-\alpha}\|\leqq Ch^{-1} \tag{2.9}$$

after integrating a finite number of times ([CFKS], Proposition 4.11).

(3) By differentiating $F_\varepsilon(z)$ in z, we have

$$\|F_\varepsilon(z)-F_\varepsilon(z')\|\leqq|z-z'|\sup_z\|D_\varepsilon G_\varepsilon(z)^2 D_\varepsilon\|\leqq C\varepsilon^{-1}|z-z'| \tag{2.10}$$

for sufficiently small fixed h. Here we used estimates (2.7) and $\|F_\varepsilon\|\leqq C$. (2.8) and (2.9) imply

$$\begin{aligned}\|F_0(z)-F_0(z')\|&\leqq\|F_0(z)-F_\varepsilon(z)\|\\&\quad+\|F_\varepsilon(z)-F_\varepsilon(z')\|+\|F_\varepsilon(z')-F_0(z')\|\\&\leqq C\varepsilon^{\alpha-1/2}+\varepsilon^{-1}|z-z'|.\end{aligned} \tag{2.11}$$

If we set $\varepsilon=|z-z'|^\beta$ with $\beta=(\alpha-1/2)^{-1}$, then we obtain the Hölder continuity of order $(\alpha-1/2)/(\alpha+1/2)$ for $F_0(z)$. The existence of the limit of $F_0(z)$ as $\operatorname{Im} z\to 0$, $\operatorname{Re} z\in I$ follows from this. Consequently, (2.2) follows from (2.9). ■

Remark 2.2. — It follows from (M2), (M4) and Lemma 4.12 of [CFKS], *i.e.* that $\|[A, g(H)]\|_{-1,1}\leqq C$ in our situation, that for any $g\in C_0^\infty(\mathbb{R})$, $[g(H)[H, A]g(H), A]$ extends to a bounded operator and is

$O(h)$. As an alternative to (M4) we can take

(M4′) for any $g \in C_0^\infty(\mathbb{R})$, $\| [g(H)[H, A]g(H), A] \| \leqq C h$.

3. PROOF OF THEOREM

In this section, we prove that Conditions (A) and (B) imply that $H(h)$ satisfies (M1)-(M4) and the semiclassical Mourre estimate for supp g sufficiently small and containing the nontrapping energy E. The conjugate operator is

$$A := \frac{h}{2}[\nabla . f(x) + f(x) . \nabla] \tag{3.1}$$

where f is the vector field of Condition B.

LEMMA 3.1. — *Let* $H(h) := -h^2 \Delta + V$ *where* V *satisfies Conditions* (A) *and* (B). *Let* $g \in C_0^\infty(I)$, $I \subset \mathbb{R}$ *compact and* $E \in I$. *Then*

(i) A *and* H *satisfy* (M1)-(M4) *with* $H_0 := H$ *in* (M3).

(ii) *There exist* $\alpha_0 > 0$ *and a bounded operator* $K(h)$ *with* $\| K(h) \| \to 0$ *as* $h \to 0$ *such that for* $|I|$ *sufficiently small,*

$$g(H)[H, A]g(H) \geqq \alpha_0 h g(H)^2 + h g(H) K(h) g(H). \tag{3.2}$$

The operator $K(h)$ *is given explicitly in* (3.8) *below.*

In the proof of this lemma, we use a decay result for wave packets in the classically forbidden region $G(E)$. This result, in its optimal form due to [BCD-2], is discussed in the Appendix.

Let δ be as in Condition (B). The function: $K(x) := \inf_{\xi \in \mathbb{R}^n} \{ |\xi|^{-2} \langle \xi, J_f(x) \xi \rangle \}$ is easily seen to be uniformly Lipschitz continuous, and let c_0 be the Lipschitz constant.

LEMMA 3.2. — *Let* $K(x)$ *be as above and* ε_0 *be as in* (1.1). *Then there exists* $\tilde{K}(x) \in C^\infty(\mathbb{R}^n)$ *such that*

(i) $\tilde{K}(x) \leqq K(x)$, $x \in \mathbb{R}^n$;

(*ii*) $2\tilde{K}(x)(V(x) - E) - f(x) . \nabla V(x) \geqq \varepsilon_0/2$, $x \in G_c(E + \delta)$.

Proof. — Let $c_\varkappa$ be a mollifier: $c_\varkappa \in C_0^\infty(\{ |x| \leqq \kappa \})$, $\int c_\varkappa(x)\, dx = 1$. Let $K_\varkappa := c_\varkappa * K$, so $K_\varkappa \in C^\infty$. Since K is uniformly Lipschitz, it follows that

$$K(x) - c_0 \kappa \leqq K_\varkappa(x) < K(x) + c_0 \kappa$$

for $x \in G_c(E + \delta)$. Set $\tilde{K}(x) := K_\varkappa(x) - c_0 \kappa$, then this proves (i). For (ii),

$$2\tilde{K}(x)(V(x) - E) - f(x) . \nabla V(x) \geqq \varepsilon_0 - 2 c_0 \kappa (V(x) - E)$$

for $x \in G_c(E+\delta)$. If $\kappa < \varepsilon_0 (4 c_0 \sup |V(x)-E|)^{-1}$, (ii) holds. ■

Proof of lemma 3.1. — (1) Since $C_0^\infty(\mathbb{R}^n)$ is a common core for D(H), D(A), etc., it is sufficient to prove the estimates. By a simple calculation, as a quadratic form on $C_0^\infty(\mathbb{R}^n)$:

$$[H, A] = h\left\{2h^2 p J_f p - f.\nabla V - \frac{h^2}{2}\Delta(\nabla.f)\right\} \tag{3.3}$$

where $J_f = (\partial f_i/\partial x_j)$ is the Jacobian matrix of f and $p = -i\nabla$. By Conditions (A) and (B), $\| [H, A] \|_{2,0} \leq ch$, hence (M1)-(M3) are satisfied. As for (M4), [[H, A], A] as a quadratic from on $C_0^\infty(\mathbb{R}^n)$ has the form:

$$\begin{aligned}[[H, A], A] = h^2 [h^2 \{ &-2p_i J_{ij,\,k} f_k p_j + 2p_k J_{ki} J_{ij} p_j + 2 p_i J_{ij} J_{kj} p_k \\ &+ i(f_{k,\,ik} J_{ij} p_j - p_i J_{ij} f_{k,\,jk})\} \\ &- [(f.\nabla V), f.\nabla] + \frac{h^2}{2}[(f.\nabla), \Delta(\nabla.f)]]\end{aligned} \tag{3.4}$$

where $J_{ij,k} := \frac{\partial}{\partial x_k}(J_f)_{ij}$, $f_{i,k} := (\partial f_i)/(\partial x_k)$, etc. The term $h^2\{\ldots\}$ is clearly uniformly bounded by H, and the last is also uniformly bounded by H. The second term is

$$\begin{aligned}-h^2 [(f.\nabla V), f.\nabla] &= h^2 \{ f.\nabla(f.\nabla V_2) \\ &\quad + [\nabla_j, f_j(f.\nabla V_1)] - (\nabla.f).(f.\nabla V_1)\} \\ &= h^2\{f.\nabla(f.\nabla V_2) - (\nabla.f)(f.\nabla V_1)\} \\ &\quad - h[(h\nabla_j), f_j(f.\nabla V_1)] = I_1 + I_2.\end{aligned} \tag{3.5}$$

Clearly, I_1 is $O(h^2)$, and $(H+i)^{-1} I_2 (H+i)^{-1}$ is $O(h)$ since $h.\nabla_j$ is uniformly H-bounded. Thus $\| [[H, A], A] \|_{2,\,-2} = O(h)$.

(2) In the sense of quadratic forms, it follows from Lemma 3.2 that $p J_f p \geq p K p \geq p \tilde{K} p$ and $2.p\tilde{K}p = \tilde{K}p^2 + p^2\tilde{K} + \Delta\tilde{K}$. We obtain from (3.3):

$$\begin{aligned}[H, A] \geq h\Big[&\tilde{K}(H-E) + (H-E)\tilde{K} + 2\tilde{K}(E-V) - f.\nabla V \\ &+ h^2\left\{\Delta\tilde{K} - \frac{1}{2}\Delta(\nabla.f)\right\}\Big].\end{aligned} \tag{3.6}$$

Let $g \in C_0^\infty(I)$, $E \in I$ and let χ be the characteristic function of $G_c(E+\delta)$. By Lemma 3.2,

$$(2\tilde{K}(E-V) - f.\nabla V)\chi \geq (\varepsilon_0/2)\chi.$$

Let $\beta := \sup_{x \in G(E+\delta)} |2\tilde{K}(E-V) - f.\nabla V|$ and $\gamma = \sup_{x \in \mathbb{R}^n} |\tilde{K}(x)|$. Then for $|I|$ sufficiently small,

$$g(H)[H, A]g(H) \geqq h\left(\frac{\varepsilon_0}{2} - 2\gamma|I|\right)g(H)^2$$
$$-hg(H)\left[(1-\chi)\left(\beta + \frac{\varepsilon_0}{2}\right) + h^2\Delta(\nabla.f) - h^2\Delta\tilde{K}\right]g(H)$$
$$\geqq \frac{h\varepsilon_0}{4}g(H)^2 + hg(H)K(h)g(H) \quad (3.7)$$

where

$$K(h) := \left(\beta + \frac{\varepsilon_0}{2}\right)E_I(H)(\chi - 1)E_I(H) - \frac{h^2}{2}\Delta(\nabla.f) + h^2\Delta\tilde{K} \quad (3.8)$$

and $E_I(H)$ is the spectral projection for H and I. By Lemma A.2, $\|E_I(H)(\chi-1)\| = O(h^N)$ for any N, so we have $\|K(h)\| = O(h^2)$. This completes the proof. ■

Proof of Theorem. – By Lemma 3.1, the hypothesis of Proposition 2.1 are satisfied, so the resolvent of $H(h)$ satifies (2.2). To pass to (1.3), we use the fact there exists a constant C independent of h such that

$$\|\langle x\rangle^{-\alpha}(H+i)^{-1}\langle A\rangle^{\alpha}\| \leqq C \quad (3.9)$$

for $\alpha \in [0,1]$ (*cf.* Lemma 8.2 of [PSS]). Estimate (3.9) is proved directly for $\alpha = 1$ using the fact $|\langle x\rangle^{-1} f(x)| \leqq C$ which follows from Condition (B), and extended by complex interpolation. ■

Remark 3.3. – In certain cases, a more precise propagation estimate results from (2.2) if we replace $\langle A\rangle^{-\alpha}$ by $\langle f\rangle^{-\alpha}$. This is the case when f vanishes on some unbounded set.

Remark 3.4. – Instead of Lemma A.2, we can also apply the cut and paste technique (or so-called geometric method) to isolate the classically forbidden region. In fact, if the semiclassical resolvent estimate is proved for H on $L^2(G_c(E+\delta))$, the estimate on $L^2(\mathbb{R}^n)$ follows (*cf.* (A.5) or [BCD-2]). Since nontrapping inequality (1.1) holds globally on $G_c(E+\delta)$, the semiclassical resolvent estimate on $L^2(G_c(E+\delta))$ can be proved by the above argument.

4. GENERALIZATIONS

A. Stark Hamiltonians

The methods developed here can be extended to a class of Stark Hamiltonians as we now indicate. In place of Condition (A) we assume $V \in C^2(\mathbb{R}^n)$, $|V(x)| \leqq C\langle x \rangle$ and $\left|\left(\frac{\partial}{\partial x}\right)^\alpha V(x)\right| \leqq C$, $|\alpha|=1,2$. The vector field in Condition (B) must satisfy $f \in C^4(\mathbb{R}^n)$, $|f(x)| \leqq C$ and

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right| \leqq C\langle x \rangle^{-1} \quad \text{for} \quad |\alpha|=1, 2, 3, 4.$$

The nontrapping condition is as in (1.1). Note that the proof of Lemma 3.2 must be improved to show that $|K(x)-\tilde{K}(x)| \leqq \kappa \langle x \rangle^{-1}$ with small $\kappa>0$ using the fact that $K(x)=0(\langle x \rangle^{-1})$. We also need the following lemma:

LEMMA 4.1. — *Let* $V \in C(\mathbb{R}^n)$ *and suppose that* $|V(x)| < C\langle x \rangle^\gamma$ *for some* $\gamma : 0 \leqq \gamma \leqq 2$. *Then* $-h^2\langle x \rangle^{-\gamma}\Delta$ *is relatively* $H(h)$*-bounded uniformly in* h.

It follows from the assumptions and this lemma that $\|[H, A](H+i)^{-1}\| = O(h)$ and $\|[[H, A], A](H+i)^{-1}\| = O(h^2)$. With these modifications, one proves (M1)-(M4) and the semiclassical Mourre estimate (2.1). As a consequence, we obtain the semiclassical resolvent estimate

$$\|(H-E \pm i0)^{-1}\|_{B(H^1, H^{-1})} \leqq C h^{-1}$$

where $H^1(\mathbb{R}^n)$ is the usual Sobolev space with norm $\|\varphi\|^2_{H^1} := \|\varphi\|^2 + h^2\|\nabla\varphi\|^2$. Here we used the fact that $D(A) \to H^1(\mathbb{R}^n)$, and the inclusion map is bounded uniformly in h.

B. Local Singularities

The results of Section 3 apply if V is singular in the classically forbidden region for an interval of nontrapping energies around E. In this case, we require $V \in L^p(G(E+\delta))$ for δ as in Condition (B), with $p=2$ for $n \leqq 3$ and $p>n/2$ for $n \geqq 4$. As is easily seen from the proof, we only need V to be bounded away from $G(E+\delta)$ so the decay estimate $\|(1-\chi) E_I(H)\| = O(h^N)$ holds for this class of potentials.

C. Exploding potentials

We can also treat potentials of the type

$$V \in C^2(\mathbb{R}^n), \left|\left(\frac{\partial}{\partial x}\right)^\alpha V(x)\right| \leqq C\langle x\rangle^{2-|\alpha|}, |\alpha| \leqq 2, \quad \text{and} \quad V(x) \to -\infty$$

as $|x| \to \infty$. Again, we must take vector fields f such that $f \in C^4(\mathbb{R}^n)$ and $\left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right| \leqq C\langle x\rangle^{-1-|\alpha|}$, $|\alpha| \leqq 4$. Following modifications similar to those described in Part A above, we obtain a semiclassical Mourre estimate and the result that $\|(H-E \pm iO)^{-1}\|_{B(H^1, H^{-1})} \leqq C h^{-1}$.

APPENDIX

Decay of wave packets

The purpose of this section is to prove Lemma A.2 the result of which is used in equation (3.8). We use a perturbation idea of [BCD-2] and a simple iteration argument on the localized resolvent. Although Lemma A.2 is sufficient for our purposes, we mention a result of [BCD-2] which states that there exists $\sigma > 0$, where σ is described in terms of a distance in the Agmon metric, such that $\|(1-\chi) E_I(H)\| = O(e^{-\sigma/h})$.

LEMMA A.1. – *Suppose* $F > E$ *and* $\sup|\nabla V| = C < \infty$. *Then*

$$\operatorname{dist}(G(F), G_c(E)) \geqq C^{-1}(F-E) \tag{A.1}$$

where $G_c(E) := \mathbb{R}^n \setminus G(E)$ *and* dist (.,.) *is the Euclidean distance.*

Proof. – Let $x \in G(F)$, $y \in G_c(E)$, then

$$F-E \leqq V(x)-V(y) = \int_0^1 \frac{d}{dt} V(\gamma(t))\, dt \tag{A.2}$$

for the path $\gamma : \gamma(t) = tx + (1-t)y$. By the assumption,

$$[\text{the RHS of } (A.2)] = \int_0^1 \frac{d\gamma}{dt} \cdot (\nabla V)(\gamma(t))\, dt$$

$$\leqq C \int_0^1 \left|\frac{d\gamma}{dt}\right| dt = C \operatorname{dist}(x, y). \tag{A.3}$$

This proves the lemma. ■

We note that the assumption $\sup|\nabla V| < \infty$ is necessary only on the convex hull of $G(E)$ in order to apply the method to exploding potentials [Section 4 (C)].

LEMMA A.2. — *Suppose* $\sup |\nabla V| < \infty$. *Let* χ *be the characteristic function of* $G_c(F)$ *and* $I = [D, E]$ *with* $D < E < F$. *Then*

$$\| (1-\chi) E_I(H) \| \leqq C_N . h^N \tag{A.4}$$

for any N, *where* $E_I(H)$ *is the spectral projection of* H.

Proof. — Let $\varepsilon := (F-E)/(2N+4)$. By virtue of Lemma A.1, there exist C^∞-functions $\{J_j\}_{j=1, \ldots, N}$ such that (i) $0 \leqq J_j(x) \leqq 1$; (ii) $\sup |\nabla J_j(x)| < \infty$; (iii) $J_j(x) = 1$ if $x \in G(F-2j\varepsilon)$ and $=0$ if $x \in G_c(F-(2j+1)\varepsilon)$. Let $V_0(x) := \max\{V(x), E+2\varepsilon\}$, and let $H_0 = -h^2 \Delta + V_0(x)$. Then $\sigma(H_0) \subset [E+2\varepsilon, \infty)$. We have the geometric resolvent equation:

$$J_N R(z) = R_0(z) J_N + R_0(z) M_N R(z) \tag{A.5}$$

where

$$R(z) = (H-z)^{-1},$$
$$R_0(z) = (H_0 - z)^{-1} \quad \text{and} \quad M_j = -h^2 \{ \nabla (\nabla J_j) + (\nabla J_j) \nabla \}.$$

It is easy to see $\operatorname{supp} M_j \subset G(F-(2j+1)\varepsilon) \cap G_c(F-2j\varepsilon)$, and hence $M_{j+1} J_j = 0$. Using this identity, we obtain

$$\begin{aligned}(1-\chi) R_0(z) M_N R(z) &= (1-\chi) J_{N-1} R_0(z) M_N R(z)\\ &= (1-\chi) [J_{N-1}, R_0(z)] M_N R(z)\\ &= (1-\chi) R_0(z) M_{N-1} R_0(z) M_N R(z)\\ &= (1-\chi) R_0(z) M_1 R_0(z) M_2 \ldots R_0(z) M_N R(z). \end{aligned} \tag{A.6}$$

Let Γ be a positively oriented, simple closed around I, and away from $[E+2\varepsilon, \infty)$. Then, as the first term of the RHS of (A.5) is analytic in Γ, we conclude

$$\begin{aligned}(1-\chi) E_I(H) &= -\frac{1}{2\pi i} \int_\Gamma (1-\chi) J_N R(z) E_I(H) \, dz\\ &= -\frac{1}{2\pi i} \int (1-\chi) R_0(z) M_1 \ldots R_0(z) M_N R(z) E_I(H) \, dz. \end{aligned} \tag{A.7}$$

Now, since $\| M_j R_0(z) \| = O(h)$ and $\| R(z) E_I(H) \| \leqq C$ on Γ, it follows immediately from (A.7) that $\| (1-\chi) E_I(H) \| = O(h^N)$. ■

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