ANNALES DE L'I. H. P., SECTION A

NAKAO HAYASHI
SABUROU SAITOH

Analyticity and smoothing effect for the Schrödinger equation

Annales de l'I. H. P., section A, tome 52, nº 2 (1990), p. 163-173
<http://www.numdam.org/item?id=AIHPA_1990__52_2_163_0>

Ann. Inst. Henri Poincaré,

Vol. 52, n° 2, 1990, p. 163-173. *Physique théorique*

Analyticity and smoothing effect for the Schrödinger equation

by

Nakao HAYASHI and Saburou SAITOH

Department of Mathematics, Faculty of Engineering,
Gunma University, Kiryu 376, Japan

ABSTRACT. – In this paper we give two identities in a certain class of analytic functions. By using these identities we obtain isometrical identities and *a priori* estimates of solutions for the Schrödinger equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R},$$
$$u(0, x) = \varphi(x), \qquad x \in \mathbb{R},$$

which yields the analyticity of solutions of the equation for $t \neq 0$, if the initial functions decay exponentially as $|x| \to \infty$. Furthermore, we apply our result to the Schrödinger equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \mathrm{V}u$$

with $u(0, x) = \varphi(x)$ under a certain analytical condition on V.

RÉSUMÉ. – Dans cet article nous montrons deux identités pour certaines classes de fonctions analytiques. Nous en déduisons des identités isométriques et des estimations *a priori* des solutions de l'équation de Schrödinger

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R},$$
$$u(0, x) = \varphi(x), \qquad x \in \mathbb{R},$$

qui conduisent à l'analyticité des solutions de l'équation pour $t \neq 0$, si la condition initiale décroît exponentiellement vite si $|x| \to \infty$. De plus nous

appliquons notre résultat à l'équation de Schrödinger $i\partial_t u + \frac{1}{2}\partial_x^2 u = \mathrm{V}u$ avec $u(0, x) = \varphi(x)$ sous une condition d'analyticité sur V.

---

1. INTRODUCTION AND RESULTS

In this paper we shall obtain two indentities in a certain class of analytic functions, which are useful to investigate the analyticity of solutions for the Schrödinger equations

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

$$u(0, x) = \varphi(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

and

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \mathrm{V}u, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.3}$$

$$u(0, x) = \varphi(x), \qquad x \in \mathbb{R}. \tag{1.4}$$

NOTATION. — For each $r > 0$ we denote by $\mathrm{S}(r)$ the strip $\{-r < \operatorname{Im} z < r\}$ in the complex $z = x + iy$ plane. We let $\mathrm{A}_1(r) = \Big\{$ the set of all analytic functions $f(z)$ on $\mathrm{S}(r)$ such that

$$\frac{1}{2r}\int_{-r}^{r}\int_{\mathbb{R}} |f(z)|^2\, dx\, dy < \infty$$

for each $r > 0\Big\}$, $\mathrm{A}(\infty, \alpha) = \Big\{$ the set of all analytic functions $f(z)$ on $\mathrm{S}(\infty)$ such that

$$\frac{1}{\sqrt{\alpha\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(y^2/\alpha)} |f(z)|^2\, dx\, dy < \infty,$$

for each $\alpha > 0\Big\}$, $\mathrm{A}_2(r) = \{$ the set of all analytic functions $f(z)$ on $\mathrm{S}(r)$ such that $\sup_{\mathrm{S}(r)} |f(z)| < \infty$ for each $r > 0\}$. For $x \in \mathbb{R}$, if a complex valued function $f(x)$ has an analytic continuation onto $\mathrm{S}(r)$, then we denote this by the same letter $f(z)$, and if $g(z)$ is an analytic function on $S(r)$, then we denote the restriction of $g(z)$ to the real axis by $g(x)$. Let L^2 denote the usual Hilbert space $L^2(\mathbb{R})$ with the norm

$$|f|_2^2 = |f(x)|_2^2 = \int_{\mathbb{R}} |f(x)|^2\, dx$$

and with the L^2-scalar product $(.,.)$.

We use the following assumptions,

(B.1) $\varphi(x)$ satisfies $\int_{\mathbb{R}} e^{x^2} |\varphi(x)|^2\, dx < \infty$.

(B.2) $\varphi(x)$ satisfies $\int_{\mathbb{R}} \frac{\sinh 2x}{2x} |\varphi(x)|^2\, dx < \infty$.

(B.3) $V(x)$ is a real-valued function.

(B.4) $V(x)$ has an analytic continuation to $S(r)$ and $V(z) \in A_2(r)$.

We now state our results in this paper.

THEOREM 1. — (I) *We assume* (B.1). *Then, there exists a unique solution* $u(t, x)$ *of* (1.1)-(1.2) *such that* $e^{-(ix^2/2t)} u(t, x)$ *has an analytic continuation to* $S(\infty)$ *and*

$$e^{-(iz^2/2t)} u(t, z) \in A(\infty, t^2), \qquad \textit{for} \quad t \in \mathbb{R} \setminus \{0\}. \tag{1.5}$$

Furthermore, $u(t, x)$ *satisfies*

$$|u(t, x)|_2 = |\varphi|_2, \qquad \textit{for} \quad t \in \mathbb{R}, \tag{1.6}$$

and

$$\frac{1}{|t|\sqrt{\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(y^2/t^2)} |e^{-(iz^2/2t)} u(t, z)|^2\, dx\, dy = \int_{\mathbb{R}} e^{x^2} |\varphi(x)|^2\, dx \qquad \textit{for} \quad t \in \mathbb{R} \setminus \{0\}. \tag{1.7}$$

(II) *We assume* (B.2). *Then, there exists a unique solution* $u(t, x)$ *of* (1.1)-(1.2) *such that* $e^{-(ix^2/2t)} u(t, x)$ *has an analytic continuation to* $S(|t|)$ *and*

$$e^{-(iz^2/2t)} u(t, z) \in A_1(|t|) \qquad \textit{for} \quad t \in \mathbb{R} \setminus \{0\}. \tag{1.8}$$

Furthermore, $u(t, x)$ *satisfies* (1.6) *and*

$$\frac{1}{2|t|} \int_{-|t|}^{|t|} \int_{\mathbb{R}} |e^{-(iz^2/2t)} u(t, x)|^2\, dx\, dy = \int_{\mathbb{R}} \frac{\sinh 2x}{2x} |\varphi(x)|^2\, dx \qquad \textit{for} \quad t \in \mathbb{R} \setminus \{0\}. \tag{1.9}$$

THEOREM 2. — *We assume* (B.2), (B.3) *and* (B.4). *Then, there exists a unique solution* $u(t, x)$ *of* (1.3)-(1.4) *satisfying* (1.6) *for* $t \in \mathbb{R}$ *and* (1.8) *for* $0 < |t| \leqq r$.

Remark 1. — The property (1.8) in Theorems 1-2 implies that if the initial functions decay exponentially as $|x| \to \infty$, then the solutions of (1.1)-(1.2) or (1.3)-(1.4) are analytic for $t \neq 0$ in the sense of $A_1(|t|)$. In [3], A. Jensen has proved that if the initial functions decay sufficiently rapidly as $|x| \to \infty$ and the potential V is sufficiently smooth, then the solutions of (1.3)-(1.4) are sufficiently smooth for $t \neq 0$, but he has not mentioned the analyticity of solutions of (1.3)-(1.4).

Remark 2. — We give some examples of $V(x)$ satisfying the condition (B.4).

(i) If $V(x) = e^{-x^2}$, then $V(x)$ satisfies (B.4) for any $r > 0$.

Theorem 2 implies that $e^{-(iz^2/2t)} u(t, z) \in A_1(|t|)$ for any $t \in \mathbb{R} \setminus \{0\}$, where $u(t, x)$ is the solution of (1.3)-(1.4) with $V(x) = e^{-x^2}$.

(ii) If $V(x) = \dfrac{1}{1+x^2}$, then $V(x)$ satisfies (B.4) for $0 < r < 1$.

Theorem 2 implies that $e^{-(iz^2/2t)} u(t, z) \in A_1(|t|)$ for $0 < |t| < 1$, where $u(t, x)$ is the solution of (1.3)-(1.4) with $V(x) = \dfrac{1}{1+x^2}$.

The above examples show that the width of the analytical strip of the solutions $u(t, z)$ containing the real axis depends heavily on the condition on $V(x)$.

Remark 3. — The identity (1.7) has been obtained by S. Saitoh [5] by using the theory of reproducing kernels.

The above theorems are derived from the following

LEMMA 1. — (A) *Let* $f(z) \in A_1(r)$. *Then, the formal power series*

$$\sum_{j=0}^{\infty} \frac{(2r)^{2j}}{(2j+1)!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 \, dx$$

converges and we have the following identity

$$\frac{1}{2r} \int_{-r}^{r} \int_{\mathbb{R}} |f(z)|^2 \, dx \, dy = \sum_{j=0}^{\infty} \frac{(2r)^{2j}}{(2j+1)!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 \, dx. \qquad (1.10)$$

Conversely, for a function $f(x)$ *with a finite integral* (1.10) *on the real line, there exists an analytic extension* $f(z)$ *such that* $f(z) \in A_1(r)$ *satisfying the identity* (1.10).

(B) *Let* $f(z) \in A(\infty, \alpha)$. *Then, the formal power series*

$$\sum_{j=0}^{\infty} \frac{\alpha_j}{j!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 \, dx$$

converges and we have the following identity

$$\frac{1}{\sqrt{\alpha\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(y^2/\alpha)}|f(z)|^2\,dx\,dy=\sum_{j=0}^{\infty}\frac{\alpha^j}{j!}\int_{\mathbb{R}}|\partial_x^j f(x)|^2\,dx. \qquad (1.11)$$

Conversely, for a function $f(x)$ with a finite integral (1.11) on the real line, there exists an analytic extension $f(z)$ such that $f(z)\in A(\infty, \alpha)$ satisfying the identity (1.11).

Remark 4. — The identity (1.10) of Lemma 1 is similar to Lemma 2.2 of [4]. But, their result is different from ours, since they only proved that there exist positive constants C_1 and C_2 such that

$$C_1\sum_{j=0}^{\infty}\frac{(r'')^{2j}}{(j!)^2}|\partial_x^j f|_2^2 \leqq \int_{-r'}^{r'}\int_{\mathbb{R}}|f(z)|^2\,dx\,dy$$
$$\leqq C_2\sum_{j=0}^{\infty}\frac{r^{2j}}{(j!)^2}|\partial_x^j f|_2^2 \qquad \text{for} \quad r''<r'<r.$$

2. PROOF OF RESULTS

PROOF OF LEMMA 1

By the Taylor expansion

$$f(x+iy)=\sum_{j=0}^{\infty}\partial_x^j f(x)\,(iy)^j/j! \qquad \text{of} \quad f(z)\in A_1(r),$$

we have

$$|f(x+iy)|^2=\left|\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}y^{2j}\partial_x^{2j}f(x) + iy\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}y^{2j}\partial_x^{2j+1}f(x)\right|^2$$
$$=E(x, y)+O(x, y), \qquad (2.1)$$

where

$$E(x, y)=\left|\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}y^{2j}\partial_x^{2j}f(x)\right|^2+\left|\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}y^{(2j+1)}\partial_x^{2j+1}f(x)\right|^2$$

is an even function of y and

$$O(x, y) = -2\,\mathrm{Re}\left\{ iy \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} y^{2j} \partial_x^{2j} f(x) \right) \times \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} y^{2j} \partial_x^{2j+1} \overline{f(x)} \right) \right\}$$

is an odd function of y. By a simple calculation we obtain

$$\begin{aligned} \mathrm{E}(x, y) = {} & \sum_{j=0}^{\infty} \frac{y^{2 \cdot 2j}}{((2j)!)^2} |\partial_x^{2j} f(x)|^2 \\ & + 2\,\mathrm{Re} \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \frac{(-1)^{j+k}}{(2j)!\,(2k)!} y^{2(j+k)} . \partial_x^{2j} f(x) . \partial_x^{2k} \overline{f(x)} \\ & + 2\,\mathrm{Re} \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \frac{(-1)^{j+k}}{(2j+1)!\,(2k+1)!} y^{2(j+k+1)} . \partial_x^{2j+1} f(x) \\ & \times \partial_x^{2k+1} \overline{f(x)} + \sum_{j=0}^{\infty} \frac{y^{2(2j+1)}}{((2j+1)!)^2} |\partial_x^{2j+1} f(x)|^2. \end{aligned} \tag{2.2}$$

Since $f(z) \in \mathrm{A}_1(r)$ implies that $\partial_x^n f \in \mathrm{L}^2(\mathbb{R})$ for any n (*see* [4], section 4), by integration by parts we see that for all j and k

$$\begin{aligned} 2\,\mathrm{Re} \int_{\mathbb{R}} \partial_x^{2j} f(x) . \partial_x^{2k} \overline{f(x)}\, dx & = 2(-1)^{k-j} \int_{\mathbb{R}} |\partial_x^{k+j} f(x)|^2\, dx \\ & = 2(-1)^{k-j} |\partial_x^{k+j} f|_2^2. \end{aligned} \tag{2.3}$$

We have by (2.2) and (2.3)

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{E}(x, y)\, dx = {} & \sum_{j=0}^{\infty} \frac{y^{2j}}{(j!)^2} |\partial_x^j f|_2^2 \\ & + 2 \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \frac{y^{2(j+k)}}{(2j)!\,(2k)!} |\partial_x^{j+k} f|_2^2 \\ & + 2 \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \frac{y^{2(j+k+1)}}{(2j+1)!\,(2k+1)!} |\partial_x^{j+k+1} f|_2^2. \end{aligned} \tag{2.4}$$

We find easily that the coefficient of the term $|\partial_x^n f|_2^2$ in the R.H.S. of (2.4) is equal to

$$\left(\frac{1}{(n!)^2} + 2 \sum_{j=0}^{n-1} \frac{1}{j!\,(2n-j)!} \right) y^{2n}. \tag{2.5}$$

By a simple calculation we obtain

$$\frac{2n!}{(n!)^2}+2\sum_{j=0}^{n-1}\frac{2n!}{j!(2n-j)!}$$

$$=\binom{2n}{n}+2\sum_{j=0}^{n-1}\binom{2n}{j}$$

$$=\binom{2n}{n}+\sum_{j=0}^{n-1}\binom{2n}{j}+\sum_{j=n+1}^{2n}\binom{2n}{j}$$

$$=\sum_{j=0}^{2n}\binom{2n}{j}=2^{2n}. \quad (2.6)$$

Thus we have by (2.4), (2.5) and (2.6)

$$\int_{\mathbb{R}} \mathrm{E}(x, y)\,dx=\sum_{n=0}^{\infty}\frac{2^{2n}}{2n!}y^{2n}|\partial_x^n f|_2^2. \quad (2.7)$$

The identities (2.1) and (2.7) lead us to

$$\int_{\mathbb{R}}|f(x+iy)|^2\,dx=\sum_{n=0}^{\infty}\frac{2^{2n}}{2n!}y^{2n}|\partial_x^n f|_2^2+\int_{\mathbb{R}} O(x, y)\,dx. \quad (2.8)$$

Integration in y shows (1.10), since $\int_{\mathbb{R}} O(x, y)\,dx$ is an odd function.

Conversely, for a function $f(x)$ on $\mathbb{R}$ with a finite integral (1.9), we define the function

$$f(x+iy)=\sum_{j=0}^{\infty}\partial_x^j f(x)\,(iy)^j/j!$$

and we have the identity (1.10), conversely. From this fact, we have the desired assertion.

We next prove (1.11). Multiplying both sides of (2.8) by $\frac{1}{\sqrt{\alpha\pi}}e^{-(y^2/\alpha)}$ and integrating in y, we have

$$\frac{1}{\sqrt{\alpha\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(y^2/\alpha)}|f(z)|^2\,dx\,dy$$

$$=\sum_{n=0}^{\infty}\frac{2^{2n}}{2n!}\frac{1}{\sqrt{\alpha\pi}}|\partial_x^n f|_2^2\int_{\mathbb{R}} y^{2n}e^{-(y^2/\alpha)}\,dy$$

$$=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|\partial_x^n f|_2^2,$$

since

$$\frac{1}{\sqrt{\alpha\pi}}\int_{\mathbb{R}} y^{2n} e^{-(y^2/\alpha)}\,dy = \frac{2n!}{2^{2n}n!}\alpha^n.$$

From this, the identity (1.11) follows. We also have the converse statement as in the above. This completes the proof of Lemma 1.

The proofs stated below are rather formal, but they are justified since it is well-known that the equation (1.1)-(1.2) [or (1.3)-(1.4)] has a sufficiently regular solution for a sufficiently regular initial data.

Proof of Theorem 1

We define the operator D by

$$\mathrm{D} = e^{(ix^2/2t)}\, it\,\partial_x\, e^{-(ix^2/2t)} = (x + it\,\partial_x).$$

It is known that D commutes with the operator

$$\mathrm{L} = i\,\partial_t + \frac{1}{2}\partial_x^2,$$

namely we have $[\mathrm{L}, \mathrm{D}] = \mathrm{LD} - \mathrm{DL} = 0$. Applying D^j to the equation (1.1)-(1.2) we obtain

$$i\,\partial_t \mathrm{D}^j u + \frac{1}{2}\partial_x^2 \mathrm{D}^j u = 0, \tag{2.9}$$

$$\mathrm{D}^j u(0, x) = x^j \varphi(x). \tag{2.10}$$

Multiplying both sides of (2.9) by $\overline{\mathrm{D}^j u}$, integrating in x and taking the imaginary part, we have

$$\frac{d}{dt}|\mathrm{D}^j u(t, x)|_2^2 = 0, \tag{2.11}$$

from which we see that

$$|\mathrm{D}^j u(t, x)|_2^2 = t^{2j}\int_{\mathbb{R}} |\partial_x^j e^{-(ix^2/2t)} u(t, x)|^2\,dx$$
$$= \int_{\mathbb{R}} x^{2j}|\varphi(x)|^2\,dx, \qquad \text{for} \quad t\in\mathbb{R}\setminus\{0\}. \tag{2.12}$$

The identity (2.12) leads us to

$$\sum_{j=0}^{\infty} \frac{t^{2j}}{j!} \int_{\mathbb{R}} | \partial_x^j e^{-(ix^2/2t)} u(t, x) |^2 dx$$
$$= \sum_{j=0}^{\infty} \int_{\mathbb{R}} \frac{(x^2)^j}{j!} | \varphi(x) |^2 dx$$
$$= \int_{\mathbb{R}} e^{x^2} | \varphi(x) |^2 dx < \infty, \qquad \text{for} \quad t \in \mathbb{R} \setminus \{0\}. \quad (2.13)$$

From Lemma 1 (1.11) with $\alpha = t^2 > 0$, $f(x) = e^{-(ix^2/2t)} u(t, x)$, and (2.13), we have (1.5) and (1.7). The identity (1.6) follows from (2.12) with $j = 0$.

We next prove (II). We multiply both sides of (2.12) by $\frac{2^{2j}}{(2j+1)!}$. Then, we have

$$\sum_{j=0}^{\infty} \frac{(2t)^{2j}}{(2j+1)!} \int_{\mathbb{R}} | \partial_x^j e^{-(ix^2/2t)} u(t, x) |^2 dx$$
$$= \sum_{j=0}^{\infty} \int_{\mathbb{R}} \frac{(2x)^{2j}}{(2j+1)!} | \varphi(x) |^2 dx. \quad (2.14)$$

The equality (2.14) and Lemma 1 (1.10) with $r = |t|$ imply (1.8) and (1.9). This completes the proof of Theorem 1.

Remark 5. — The operator D has been used to investigate the property of solutions for the Schrödinger equations by many authors, *see*, e. g., [1], [2] and [3].

Proof of Theorem 2

We put

$$F(t) = \frac{1}{2|t|} \int_{-|t|}^{|t|} \int_{\mathbb{R}} | e^{-(iz^2/2t)} u(t, z) |^2 dx\, dy$$
$$= \sum_{j=0}^{\infty} \frac{(2t)^{2j}}{(2j+1)!} | \partial_x^j e^{-(ix^2/2t)} u(t, x) |_2^2.$$

In the same way as in the proof of (2.11), we have

$$\frac{1}{2} \frac{d}{dt} | D^j u(t, x) |_2^2 = \mathrm{Im}\,(D^j (V(x) u(t, x)), D^j u(t, x)). \quad (2.15)$$

We multiply both sides of (2.15) by $\frac{2^{2j}}{(2j+1)!}$ and use the Schwarz inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{\infty}\frac{2^{2j}}{(2j+1)!}|D^j u(t,x)|_2^2 \leqq \left(\sum_{j=0}^{\infty}\frac{2^{2j}}{(2j+1)!}|D^j(V(x)u(t,x))|_2^2\right)^{1/2} \times\left(\sum_{j=0}^{\infty}\frac{2^{2j}}{(2j+1)!}|D^j u(t,x)|_2^2\right)^{1/2}. \quad (2.16)$$

From Lemma 1 (1.9) we have

$$\sum_{j=0}^{\infty}\frac{2^{2j}}{(2j+1)!}|D^j(V(x)u(t,x))|_2^2 = \frac{1}{2|t|}\int_{-|t|}^{|t|}\int_{\mathbb{R}}|e^{-(iz^2/2t)}V(z)u(t,z)|^2\,dx\,dy \leqq M(t)F(t) \quad (2.17)$$

where

$$M(t)=\sup_{\substack{-\infty<x<\infty\\ -|t|<y<|t|}}|V(z)|.$$

From (2.16) and (2.17) it follows that

$$\frac{1}{2}\frac{d}{dt}F(t)\leqq M(t)F(t). \quad (2.18)$$

We have by (2.18), for all $|t|\leqq r$

$$F(t)\leqq F(0)\exp\left[2\int_0^t M(\tau)\,d\tau\right].$$

This implies that the solution of (1.3)-(1.4) satisfies (1.8) for all $|t|\leqq r$. The identity (1.6) follows from (2.15) with $j=0$. This completes the proof of Theorem 2.

ACKNOWLEDGEMENTS

The authors would like to thank the referee for helpful comments, and Professor H. Aikawa for useful discussions on this subject.

REFERENCES

[1] J. GINIBRE and G. VELO, On a Class of Nonlinear Schrödinger Equations I, II, *J. Funct. Anal.*, Vol. **32**, 1979, pp. 1-32 and 33-71.

[2] N. HAYASHI and T. OZAWA, Smoothing Effect for Some Schrödinger Equations, *J. Funct. Anal.* Vol. **85**, 1989, pp. 307-348.
[3] A. JENSEN, Commutator Methods and a Smoothing Property of the Schrödinger Evolution Group, *Math. Z.*, Vol. **191**, 1986, pp. 53-59.
[4] T. KATO and K. MASUDA, Nonlinear Evolution Equations and Analyticity I, *Ann. Inst. Henri-Poincaré, Analyse non linéaire*, Vol. **3**, 1986, pp. 455-467.
[5] S. SAITOH, *Isometrical Indentities and Inverse Formulas in the One Dimensional Schrödinger's Equation, Complex Variables* (to appear).

(*Manuscript received February* 24, 1989)
(*Revised version received May* 15, 1989).)