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Occurence times of rare events for mixing dynamical systems

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ABSTRACT. – Let $(\Omega, \mathscr{A}, S, \mu)$ be a dynamical system where $(\Omega, \mathscr{A})$ is a Lebesgue space, S a measurable transformation on Ω leaving the measure of probability μ invariant. We suppose that $(\Omega, \mathscr{A}, S)$ admits a generator $\mathscr{P}$ for which μ is exponentially mixing. We prove that the hitting times T_n of $\mathscr{P}$-measurable rare events D_n, renormalized by a suitable sequence (β_n), converge in law to the exponential law of parameter $+1$. We prove how the asymptotic behavior of the β_n's is. We give finally examples in the case of expanding maps, illustrating the theory of large fluctuations.

RÉSUMÉ. – Soit $(\Omega, \mathscr{A}, S, \mu)$ un système dynamique où $(\Omega, \mathscr{A})$ est un espace de Lebesgue, S une transformation mesurable de Ω qui laisse la mesure de probabilité μ invariante. Nous supposons que $(\Omega, \mathscr{A}, S)$ admet un générateur $\mathscr{P}$ pour lequel μ est exponentiellement mélangeante. Nous démontrons que le temps d'atteinte T_n d'un événement rare D_n, $\mathscr{P}$ mesurable, normalisé par une suite adéquate (β_n), converge en loi vers la loi

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exponentielle de paramètre $+1$. Nous donnons le comportement asymptotique de la suite β_n. Nous illustrons ces résultats de grande déviation sur des exemples de transformations dilatantes.

Dedicated to Prof. T. E. Harris on his 70th birthday.

1. INTRODUCTION

How long does it take for the orbit of a dissipative system to perform a large fluctuation? In this paper we show that for some mixing finite dimensional dynamical systems, the first occurence time of a large class of rare events converge in distribution to exponential random times. Moreover we show that the right time scaling to catch the large fluctuation is related to a free energy-like function of the system.

In recent years it has been recognized that physical systems which behave in a chaotic way share many of the qualitative features of the time evolution of finite dimensional dynamical systems. The study of the ergodic theory of such systems is a topic of current interest in statistical physics and in probability theory (for a good review of the field we refer the reader to [1]).

However, as far as we know, from a strictly statistical point of view the situation is less satisfactory. Experimental measurements of ergodically relevant parameters are, in many cases, made without any serious statistical control. In particular no attention has been given to the exact distribution of relevant hitting times. It is hardly necessary to point out that making statistical inference is a much more difficult task when the probabilistic law of the variable being measured is unknown.

In this paper we study the asymptotical distribution of hitting times of rare events for Gibbsian dynamical systems. A typical example of the situation we consider is the following. Let I and J be two disjoint closed sub-intervals of $[0, 1]$ and $f: \mathrm{I} \cup \mathrm{J} \to [0, 1]$ be a $\mathrm{C}^{1+\varepsilon}$ function such that $|f'| \geqq \sigma > 1$ and $f(\mathrm{I}) = f(\mathrm{J}) = [0, 1]$. Let $\mathrm{K} = \bigcap_{n \geqq 1} f^{-n}([0, 1])$ be the Cantor-invariant set under f. There is a "natural" probability measure, the so-called Gibbs measure μ on K invariant under f. It can be defined as the limit, as $n \to \infty$, of the image by f^n of the uniform distribution on $[0, 1]$, renormalised by a constant C^n. This is the probability measure we see when we perform a computer simulation of the system. For instance, if we choose a point x at random, according to the uniform distribution in $\bigcap_{n=1}^{N} f^{-n}([0,1])$, then the proportion $\frac{1}{N}\#\{n=1,\ldots,N: f^n(x)\in I\}$ converges to $\rho=\mu(I)$, as $N\to\infty$.

We are interested in the distribution of the number of steps an orbit of the system makes, until, for the first time, it spends in I a proportion of time different from ρ. More precisely, let us take $q>\rho$ and for every $N\geqq 1$, define the rare event:

$$A_N^q=\left\{x\in K; \frac{1}{N}\sum_{i=0}^{N-1} 1_I(f^i(x))>q\right\}.$$

Since $q>\rho$, by Birkhoff's ergodic theorem, then $\lim_{N\to\infty}\mu(A_N^q)=0$. Nevertheless any typical orbit of the system will enter A_N^q. We can define the hitting time:

$$T_N^q(x)=\inf\{n\geqq 1; f^n(x)\in A_N^q\},$$

for any x belonging to K.

Our theorem states there is an increasing sequence of positive integers β_N, $N\geqq 1$, such that $\frac{T_N}{\beta_N}$ converges in law to an exponentiel random time, as $N\to\infty$.

The theorem says also that this scaling factor β_N is logarithmically equivalent to $\exp(N\varphi(q))$, where $\varphi(q)$-the free energy associated to the ratio q-is defined as:

$$\varphi(q)=-\lim_{N\to\infty}\frac{1}{N}\operatorname{Log}\mu(A_N^q).$$

This suggests a dynamical way to estimate the free energy of the system.

The fact that the limit law of $\frac{T_N}{\beta_N}$ is exponential means that the time needed to perform the large fluctuation is unpredictable: to know that the orbit evolved for a certain amount of time without performing the rare event does not give us further information about the future step in which it will occur.

This unpredictability is a consequence of the sensitive dependence on the initial condition of the system. Two orbits, even if they start very close one to the other, behave in distinct ways, and reach the rare event at completely different times.

This type of phenomenon was first pointed out by R. Bellman and T. E. Harris in the context of Markov chains ([2], [3]). It plays a crucial role in the so called "pathwise approach to metastability" ([4], [5], [6]).

The relationship between free-energy functions and occurence times of rare events for stochastic spin systems was stressed in [7], [8]. Related results for escape times for expansive dynamical systems were presented in [9], [10].

In the next section, we will state and prove our main theorem. In section 3 we will prove the conditions of the theorem are fulfilled in the case of the example presented in this introduction.

2. LIMIT LAW OF HITTING TIMES OF RARE EVENTS

Notations; statement and proof of the theorem

Let $(\Omega, \mathscr{A}, S, \mu)$ by a dynamical system where $(\Omega, \mathscr{A}, \mu)$ is a Lebesgue space and S a measurable transformation on Ω leaving the measure of probability μ invariant.

If P is a finite measurable partition of Ω, we note by $\bigvee_{i=0}^{n-1} S^{-i}P$ the measurable partition of Ω the atoms of which are $\bigcap_{i=0}^{n-1} S^{-i}C_{ji}$, C_{ji} being an atom of P. We will note by C_n an atom of $\bigvee_{i=0}^{n-1} S^{-i}P$, or $C_n(x)$ the atom of this partition containing x.

The dynamical systems we will study now are exponentially strong mixing, that is to say:

(i) There exists a finite measurable partition P such that: $\bigvee_{i=0}^{\infty} S^{-i}P = \mathscr{A}$.

(ii) $\exists\, K>0$, $\exists\, 0<\gamma<1$ such that:

$$\forall C_n \in \bigvee_{i=0}^{n-1} S^{-i}P, \quad \forall C_m \in \bigvee_{i=0}^{m-1} S^{-i}P \qquad \text{then,} \quad \forall k \geqq n,$$

$$|\mu(C_n \cap S^{-k}C_m) - \mu(C_n)\mu(C_m)| \leqq K\gamma^{k-n}\mu(C_n)\mu(C_m).$$

Remark. — This exponentially strong mixing property implies in particular that the dynamical system $(\Omega, \mathscr{A}, S, \mu)$ is weak-Bernoulli for the partition P (refer to [11]).

Let $(D_n)_{n \in N}$ be a sequence of measurable events verifying the following properties:

1. D_n is $\bigvee_{0}^{n-1} S^{-i}P$-measurable.

2. $\frac{1}{n}\operatorname{Log}\mu(D_n) \overset{n\to\infty}{\to} -\theta$, $\theta>0$.

We consider the sequence of random stopping times T_n associated with D_n and defined on Ω by:

$$T_n(\omega)=\operatorname{Inf}\{k\in\mathbf{N}; S^k\omega\in D_n\}.$$

We define:

$$\beta_n=\operatorname{Max}\{k\in\mathbf{N}/\mu(T_n\geqq k)\geqq e^{-1}\},$$

and

$$g_n(t)=\mu(T_n\geqq\beta_n t),\qquad t\in\mathbf{N}.$$

Finally, we note by $\operatorname{Exp}(+1)$ the random variable defined on $\mathbf{R}^+$, the law of probability of which is the exponentiel law of parameter $+1$.

THEOREM:

1. *The sequence* $\left(\dfrac{T_n}{\beta_n}\right)_{n\in\mathbf{N}}$ *converges in law to* $\operatorname{Exp}(+1)$ *when* $n\to\infty$.
2. $\lim_n \dfrac{1}{n}\operatorname{Log}\beta_n=\theta.$
3. $\lim_{n\to\infty}\dfrac{\mathbf{E}\,T_n}{\beta_n}=1.$

We are going to prove the first point of the theorem. It is sufficient to prove that:

$$\forall s,t\in\mathbf{N},\quad \lim_{n\to\infty}(g_n(t+s)-g_n(t)\,g_n(s))=0. \tag{0}$$

Indeed, let us suppose this property is true. By definition of β_n:

$$\mu(T_n>\beta_n)\geqq e^{-1}\geqq\mu(T_n>\beta_n+1).$$

On the other hand, it is clear that:

$$\mu(T_n>\beta_n)-\mu(T_n>\beta_n+1)\leqq\mu(D_n).$$

Since $\lim_n\mu(D_n)=0$ by hypothesis, we can conclude that:

$$\lim_n\mu(T_n>\beta_n)=e^{-1}.$$

This means that: $\lim_{n\to\infty} g_n(1)=e^{-1}$.

Following (0), we can infer that: $\lim_{n\to\infty} g_n(k)=e^{-k}$, $\forall k\in\mathbf{N}$.

This property extends in a classical way to the rationals, and then to real numbers, so that: $\lim_n\mu(T_n>\beta_n t)=e^{-t}$, $\forall t\in\mathbf{R}^+$, which is equivalent to the convergence in law of $\left(\dfrac{T_n}{\beta_n}\right)_{n\in\mathbf{N}}$ to $\operatorname{Exp}(+1)$.

Let us define:

$$\tilde{g}_n(t+s)=\mu\left\{\bigcap_{k=0}^{\beta_n t-d_n} S^{-k} D_n^c \cap \bigcap_{k=\beta_n t}^{\beta_n (t+s)} S^{-k} D_n^c\right\},$$

where d_n is a sequence of integers with properties we will later define.

We have:

$$|g_n(t+s)-g_n(t)g_n(s)| \leqq |g_n(t+s)-\tilde{g}_n(t+s)| + \left|\tilde{g}_n(t+s)-g_n\left(t-\frac{d_n}{\beta_n}\right).g_n(s)\right| + g_n(s)\left|g_n\left(t-\frac{d_n}{\beta_n}\right)-g_n(t)\right|.$$

It is clear that:

$$|g_n(t+s)-\tilde{g}_n(t+s)| \leqq \sum_{k=\beta_n t-d_n+1}^{\beta_n t-1} \mu(S^{-k} D_n).$$

Following the condition 2, one can find a constant $c>0$, such that, for n large enough:

$$|g_n(t+s)-\tilde{g}_n(t+s)| \leqq d_n e^{-nc}. \qquad (1)$$

The condition (ii) implies that if D_n (resp. D_m) is $\bigvee_0^{n-1} S^{-i} P$ $\left(\text{resp. } \bigvee_0^{m-1} S^{-i} P\right)$-measurable, then:

$$|\mu(D_n \cap S^{-k} D_m)-\mu(D_n)\mu(D_m)| \leqq K\gamma^{k-n}\mu(D_n)\mu(D_m).$$

Since $\bigcap_{k=0}^{\beta_n t-d_n} S^{-k} D_n^c \left(\text{resp. } \bigcap_{k=0}^{\beta_n s} S^{-k} D_n^c\right)$ is $\bigvee_{i=0}^{\beta_n t-d_n+n-1} S^{-i} P$ $\left(\text{resp. } \bigvee_{i=0}^{\beta_n s+n-1} S^{-i} P\right)$ measurable, we have:

$$\left|\tilde{g}_n(t+s)-g_n\left(t-\frac{dn}{\beta_n}\right)g_n(s)\right| \leqq K\gamma^{\beta_n t-[\beta_n t-d_n+n-1]} = K\gamma^{d_n-n+1}. \qquad (2)$$

Finally, using the same argument as for (1), we obtain:

$$\left|g_n\left(t-\frac{d_n}{\beta_n}\right)-g_n(t)\right| \leqq d_n e^{-nc}, \quad \text{for } n \text{ large enough}. \qquad (3)$$

Choosing a sequence (d_n) of integers so that $d_n-n+1 \to +\infty$, $d_n e^{-nc} \to 0$ when n tends to infinity, the result is proved.

We have to prove the point 2 of the theorem. One can first notice that:

$$\mu(T_n<t)=\sum_{s=0}^{t-1} \mu\left(\sum_{k=0}^{s-1} S^{-k} D_n^c \cap S^{-s} D_n\right) \leqq t\,\mu(D_n).$$

Then:

$$\mu(T_n \geqq t) \geqq 1 - t\,\mu(D_n)$$

This inequality means that β_n is larger than any integer t such that: $1 - t\mu(D_n) > e^{-1}$. In other words:

$$\forall \delta > 0, \quad \exists N(\delta); \qquad \forall n \geqq N(\delta) \text{ then } \beta_n > e^{n(\theta-\delta)}. \tag{4}$$

On the other hand, let α be a real number such that $\alpha > \theta$. It is easy to verify that:

$$\{T_n \geqq e^{\alpha n}\} \subset \bigcap_{J=0}^{[e^{\alpha \cdot n}/nn_0]-1} S^{-jnn_0} D_n^c, \tag{5}$$

for any integer n_0.

Following the mixing property (ii) of μ, it is clear that for every $\bigvee_0^{n-1} S^{-i} P \left(\text{resp. } \bigvee_0^{m-1} S^{-i} P\right)$-measurable sets E_n (resp. E_m) we have:

$$|\mu(E_n \cap S^{-k} E_m) - \mu(E_n)\mu(E_m)| \leqq K\gamma^{k-n}\mu(E_n)\mu(E_m). \tag{6}$$

And so, after (5) and (6), we have:

$$\mu(T_n \geqq e^{\alpha . n}) \leqq (\mu(D_n^c))^{[e^{\alpha . n}/nn_0]}(1 + K\gamma^{nn_0 - n})^{[e^{\alpha . n}/nn_0]}. \tag{7}$$

Now, let us choose an integer n_0 such that:

$$\alpha \leqq -(n_0 - 1)\operatorname{Log}\gamma; \tag{8}$$

we are going to study each of the righthand terms of (7).

$$(1 + K\gamma^{n(n_0-1)})^{[e^{\alpha . n}/nn_0]} \leqq \exp\left[\frac{e^{\alpha . n}}{nn_0} K\gamma^{(n_0-1)n}\right]$$

$$\leqq \exp\left(\frac{K}{nn_0}\right),$$

because of (8).

On the other hand, since $\frac{1}{n}\operatorname{Log}\mu(D_n) \xrightarrow{n \to +\infty} -\theta < 0$, we have $\mu(D_n) \geqq e^{-n(\theta+\delta)}$ for every $\delta > 0$, and n large enough. That is to say:

$$\forall \delta > 0, \quad \exists N(\delta), \qquad \forall n \geqq N(\delta) \quad \Rightarrow \quad \mu(D_n^c) \leqq 1 - e^{-n(\theta+\delta)}. \tag{9}$$

Then:

$$(\mu(D_n^c))^{[e^{\alpha . n}/nn_0]} \leqq (1 - e^{-n(\theta+\delta)})^{[e^{\alpha . n}/nn_0]}, \qquad \forall n \geqq N(\delta).$$

But:

$$\left[\frac{e^{\alpha . n}}{nn_0}\right]\operatorname{Log}(1 - e^{-n(\theta+\delta)}) \quad \text{is equivalent to } -\frac{e^{n(\alpha-\theta-\delta)}}{nn_0},$$

when n tends to infinity and so following (7) and (9), we have: $\mu(T_n \geqq e^{\alpha . n}) \xrightarrow{n \to \infty} 0$, for every $\delta > 0$ such that $\alpha - \delta - \theta > 0$.

We conclude that: $\mu(T_n \geqq e^{\alpha . n})$ tends to zero when n tends to infinity for any $\alpha > \theta$, and so $\beta_n \leqq e^{\alpha . n}$ for any such α.

This property and (4) imply the point 2 of the theorem.

It remains to prove 3.

We first notice that:

$$\frac{E(T_n)}{\beta_n} = \frac{1}{\beta_n} \int_0^\infty \mu(T_n \geqq t)\, dt = \int_0^\infty \mu(T_n \geqq \beta_n s)\, ds.$$

Let B be the following measurable subset of Ω, defined by: $B = \{ T_n > \beta_n p \}$, where p is an integer.

We have:

$$g_n(3\, qp) \leqq \mu\left(\bigcap_{j=0}^{3\, q-1} S^{-\beta_n jp} B \right) \leqq \mu\left(\bigcap_{l=0}^{q-1} S^{-3\, \beta_n lp} B \right).$$

Now, $B \in \bigvee_{i=0}^{\beta_n p} S^{-i} P$ and the gaps between the sets $S^{-3\, \beta_n lp} B$ is larger than $\beta_n\, p$. So applying (ii), we have:

$$g_n(3\, qp) \leqq (1 + K\, \gamma^{\beta_n p})^q (g_n(p))^q.$$

By part 1 and 2 of the theorem, $\lim\limits_{n \to \infty} g_n(p) = e^{-p}$ and $\lim\limits_{n} \beta_n = +\infty$.

Then let us choose an integer p such that: $e^{-p} \leqq \frac{1}{4}$; we can find an integer n^* such that, for every $n \geqq n^*$, we have:

$$(1 + K\, \gamma^{\beta\, np})\, g_n(p) \leqq \frac{1}{2}.$$

And so:

$$\forall\, n \geqq n^*, \quad g_n(3\, qp) \leqq \left(\frac{1}{2}\right)^q.$$

Therefore, $g_n(t)$ is bounded by a function of type $O(1) \left(\frac{1}{2}\right)^{[1/3\, p]}$ for n large enough, this function belonging to $L^1(\mathbf{R}_+, dt)$.

This enables us to use the dominated convergence theorem to get the limit:

$$\lim_{n\to\infty}\frac{E(T_n)}{\beta_n}=\lim_{n\to\infty}\int_0^\infty g_n(s)\,ds=\int_0^\infty \lim_{n\to\infty} g_n(s)\,ds=\int_0^\infty e^{-s}\,ds=1.$$

Q.E.D.

Two examples of rare events

a. The sets of non generic frequencies

To illustrate this paragraph, we consider a piecewise expanding $C^{1+\varepsilon}$ transformation defined on the interval [0, 1]. In fact, we will prove that the sets of non generic frequencies have the properties 1 and 2 of rare events in case of two branches, but the case of a finite number can be treated in the same way.

More precisely, let I_0 and I_1 be two disjoints or adjacents sub-intervals of [0, 1] and f a transformation defined on $I_0 \cup I_1$ such that:

A. $f(I_0)=f(I_1)=[0, 1]$.

B. f/I_i is $C^{1+\varepsilon}$, $i=1, 2$; moreover, there exists a constant $\sigma>1$ such that: $|f'/I_i|\geqq\sigma$.

It is well-known that among the f-invariant measures, there exists an unique one μ having the Gibbs properties; in particular μ verifies the properties (i), (ii), of paragraph 2, relatively to the partition $\mathscr{P}=\{I_0, I_1\}$. Moreover this measure μ has another metric property, that is to say the near bernoulli property:

(iii) $\exists L>0$, $\exists M>0$, $\forall C_n\in \bigvee_0^{n-1} f^{-i}\mathscr{P}$, $\forall C_m\in \bigvee_0^{m-1} f^{-i}\mathscr{P}$, then:

$$L\leqq\frac{\mu(C_n\cap f^{-n}C_m)}{\mu(C_n)\,\mu(C_m)}\leqq M.$$

(*see* [11]; [12] for more details).

Because of the ergodic properties of μ, the following generic property is true:

$$\frac{1}{n}\sum_{i=1}^{n-1} 1_{I_0}(f^i(x))\xrightarrow{\mu-\text{a. e.}}\mu(I_0)=\rho.$$

So, if we consider, for a fixed real number $\alpha>\rho$ (resp. $\alpha<\rho$), the $\bigvee_0^{n-1} f^{-i}\mathscr{P}$-measurable set:

$$A_n^\alpha=\left\{x\in[0, 1];\ \frac{1}{n}\sum_{i=0}^{n-1} 1_{I_0}(f^i(x))>\alpha\quad(\text{resp.} <\alpha)\right\},$$

it is clear that:

$$\lim_{n \to \infty} \mu(A_n^\alpha) = 0.$$

We will prove that in fact, the subsets A_n^α have the properties 1 and 2.

Remarks: 1. When the sub-intervals I_0 and I_1 are disjoint, we have to work on the Cantor-set:

$$K = \bigcap_{n \geq 0} f^{-i}([0, 1]),$$

where all the iterates of f are defined, and the definition of A_n^α is:

$$A_n^\alpha = \left\{ x \in K; \frac{1}{n} \sum_{i=0}^{n-1} 1_{I_0}(f^i(x)) > \alpha \right\}.$$

2. The property 2 for A_n^α could be proved using the theory of large deviations by Ellis [13]; our purpose here is to give a short and simple proof using probabilistic properties only.

LEMMA 3.1. — *The sequence* $(\mu(A_n^\alpha))_{n \in \mathbb{N}}$ *is sub-multiplicative modulo the constant* L.

Indeed, clearly we have:

$$A_{n+m}^\alpha \supset A_n^\alpha \cap f^{-n} A_m^\alpha.$$

Then:

$$\mu(A_{n+m}^\alpha) \geq \mu(A_n^\alpha \cap f^{-n} A_m^\alpha).$$

But the sets A_{n+m}^α, A_n^α, A_m^α being respectively measurable for $\bigvee_0^{n+m-1} f^{-i}\mathscr{P}$, $\bigvee_0^{n-1} f^{-i}\mathscr{P}$, it follows simply from (iii), that:

$$\mu(A_n^\alpha \cap f^{-n} A_m^\alpha) \geq L\, \mu(A_n^\alpha)\, \mu(A_m^\alpha).$$

The lemma is proved and the following limit exists:

$$c(\alpha) = \lim_{n \to \infty} \frac{1}{n} \operatorname{Log} \mu(A_n^\alpha).$$

LEMMA 3.2. — *The limit* $c(\alpha)$ *is strictly negative for* $\alpha > \rho$; *of course,* $c(\alpha) = 0$ *for* $\alpha < \rho$.

Let α and α' two real numbers such that $\rho < \alpha' < \alpha$, and $\delta = \dfrac{\alpha - \alpha'}{1 - \alpha'}$. For all integers n, $k \geq 1$, we have:

$$A_{nk}^\alpha \subset \bigcup_{\substack{J \subset \{0, \ldots, k-1\} \\ |J| \geq \delta k}} \left(\bigcap_{j \in J} f^{-jn}(A_n^{\alpha'}) \right).$$

Indeed, let x be an element of A^{α}_{nk}; it verifies the property:

$$\frac{1}{nk}\sum_{j=0}^{nk-1} 1_{I_0} f^j(x) > \alpha.$$

Let J_1 be subset of $J = \{0, 1, \ldots, k-1\}$ defined by:

$$j \in J_1 \quad \Leftrightarrow \quad \frac{1}{n}\sum_{r=0}^{n-1} f^{nj+r}(x) > \alpha'.$$

We have:

$$\begin{aligned}\alpha < \frac{1}{nk}\sum_{j=0}^{nk-1} 1_{I_0} f^j(x) &= \frac{1}{k}\sum_{t=0}^{k-1}\left(\frac{1}{n}\sum_{r=0}^{n-1} f^{nj+r}(x)\right)\\ &= \frac{1}{k}\sum_{l\in J_1}\left(\frac{1}{n}\sum_{r=0}^{n-1} f^{ln+r}(x)\right) + \frac{1}{k}\sum_{l\in J/J_1}\left(\frac{1}{n}\sum_{0}^{n-1} f^{nl+r}(x)\right)\\ &\leqq \frac{1}{k}(1-\alpha')\,|J_1| + \alpha'.\end{aligned}$$

Therefore:

$$|J_1| \geqq k\frac{\alpha-\alpha'}{1-\alpha'}.$$

We can conclude that x belongs to $\bigcup_{\substack{J \subset \{0, 1, \ldots, k-1\}\\ |J| \geqq \delta k}} \left(\bigcap_{j\in J} f^{-jn}(A^{\alpha'}_n)\right)$.

Following (10), we have the inequality:

$$\mu(A^{\alpha}_{nk}) \leqq \sum_{\substack{J_1 \subset \{0, \ldots, k-1\}\\ |J_1| \geqq \delta k}} \mu\left(\bigcap_{j\in J_1} f^{-jn} A^{\alpha'}_n\right).$$

Now, using the mixing property (ii), we have:

$$\begin{aligned}\mu(A^{\alpha}_{nk}) &\leqq \sum_{\substack{J_1 \subset \{0, 1, \ldots, k-1\}\\ |J_1| \geqq \delta k}} (1+K)^{|J_1|-1}(\mu(A^{\alpha'}_n))^{|J_1|}\\ &\leqq (\mu(A^{\alpha'}_n))^{\delta k}.(2+K)^k.\end{aligned}$$

Let us choose now n_0 large enough, so that:

$$(2+K)(\mu(A^{\alpha'}_{n_0}))^{\delta} = \theta < 1. \tag{12}$$

By (11) and (12), we have:

$$\frac{1}{n_0 k} \operatorname{Log} \mu (A^{\alpha}_{n_0 k}) \leqq \frac{1}{n_0} \operatorname{Log} \theta < 0; \qquad \forall k \in \mathbf{N}.$$

Q.E.D.

Remark. – In a general way, we can be interested in large deviations properties for functions of the same kind as 1_{I_0}: the functions $1_{\bigcap_{i=0}^{k-1} f^{-i}(I_{j_i})}$, where $j_i = 0, 1$.

Using once more the ergodic theorem:

$$\frac{1}{n} \sum_{k=0}^{n-1} 1_{\bigcap_{i=0}^{k-1} f^{-i}(I_{j_i})} (f^k x) \xrightarrow{\mu-\text{a. e.}} \mu \left(\bigcap_{i=0}^{k-1} f^{-i}(I_{j_i}) \right)$$

and so rare events can be defined which would have the same properties 1 and 2.

b. The sets of non generic expansitivity

We consider the previous example. It is well-known a generic Lyapunov exponant exists χ such that:

$$\frac{1}{n} \operatorname{Log} | (f^n)'(x) | \xrightarrow{\mu\text{.a. e.}} \chi,$$

the measure of probability μ being the Gibbs measure previously defined.

Now we note by $C_n(x)$ the atom of $\bigvee_{i=0}^{n-1} f^{-i} \mathscr{P}$ containing the point x of the Cantor-set K, and $l_n(x) = \lambda (C_n(x))$ where λ is the Lebesgue probability on [0, 1].

The expansivity of f implies the following distorsion property:

$$\exists K_1 > 0, \quad \exists K_2 > 0; \qquad \forall y \in C_n(x) \quad \Rightarrow \quad K_1 \leqq \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \leqq K_2.$$

An immediate corollary for the function l_n is that:

$$K_1 \leqq \frac{l_n(x)}{|(f^n)'(y)|} \leqq K_2, \qquad \forall y \in C_n(x) \tag{13}$$

So it is clear that following the mean value theorem we have:

$$-\frac{1}{n} \operatorname{Log} l_n(x) \xrightarrow{\mu-\text{a. e.}} \chi.$$

It is natural to consider subsets of type:

$$A^{\alpha}_n = \left\{ x \in K; -\frac{1}{n} \operatorname{Log} | (l_n)(x) | > \alpha, \text{for } \alpha > \chi \right\},$$

$\left(\text{resp.}\left\{x\in K;\ -\frac{1}{n}\operatorname{Log}|(l_n)(x)|<\alpha,\ \text{for}\ \alpha<\chi\right\}\right).$

These subsets are cylindrical events and rare events for the measure μ. In order to prove they are relevant of our theorem, we have to prove that:

$$\frac{1}{n}\log\mu(A_n^\alpha)\xrightarrow{n\to\infty} d(\alpha)<0,\qquad \text{for}\quad \alpha>\chi. \tag{14}$$

The method used for subsets of non generic frequencies is not relevant here and we will follow Ellis's argumentation [13].

We consider;

$$c_n(t)=\frac{1}{n}\log E_\mu[l_n(x)^t]=\frac{1}{n}\log\sum_{C_n\in\bigvee_{i=0}^{n-1}f^{-1}\mathscr{P}}[\lambda(C_n)]^t\mu(C_n),$$

where t is a real number.

With our notations we have:

$$c_{n+m}(t)=\frac{1}{n+m}\operatorname{Log}\sum_{\substack{c_n\in\bigvee_0^{n-1}f^{-i}\mathscr{P}\\ C_m\in\bigvee_0^{m-1}f^{-i}\mathscr{P}}}[\lambda(C_n\cap f^{-n}C_m)]^t\mu(C_n\cap f^{-n}C_m).$$

But, by the property (iii) of the measure μ and (1), two constants exist $K_3^{(t)}<K_4^{(t)}$ such that:

$$K_3^{(t)}\leqq\frac{[\lambda(C_n\cap f^{-n}C_m)]^t\mu(C_n\cap f^{-n}C_m)}{[\lambda(C_n)]^t[\lambda(C_m)]^t\mu(C_n)\mu(C_m)}\leqq K_4^{(t)}, \tag{15}$$

for all C_n and C_m.

It follows that the sequence $(c_n(t))$ verifies a property of sub-additivity modulo $K_3^{(t)}$ and $K_4^{(t)}$ and we can conclude that the following limit exists:

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{Log}c_n(t)=c(t).$$

D. Ruelle [14] proved in this expanding case that the function $c(t)$ is analytic in t; moreover, it is convex. Following Ellis, we can define the Legendre transform of $c(t)$:

$$I(s)=\operatorname*{Sup}_{t\in\mathbf{R}}[st-c(t)],$$

and:

$$\frac{1}{n}\operatorname{Log}\mu(\mathrm{A}_n^\alpha)\to\operatorname*{Inf}_{s>\alpha}-\mathrm{I}(s).$$

Because of the fact that $c(t)$ is analytic, it is obvious that:

$$\mathrm{I}(s)=st_0-c(t_0),$$

where t_0 is the unique real number verifying $c'(t_0)=s$. Moreover, the function I has the unique infimum 0 at the point χ. So we have the property (2), and these rare events A_n^α have the property of "exponential hitting times".

Question. – It is easy to see that $c(t)$ is the following limit too:

$$c(t)=\lim_{n\to\infty}\frac{1}{n}\operatorname{Log}\mathrm{E}_\mu\left(\frac{1}{|(f^n)'|}\right).$$

We can define the subsets:

$$\mathrm{B}_n^\alpha=\left\{x\in\mathrm{K};\frac{1}{n}\operatorname{Log}|(f^n)'|>\alpha\right\},$$

for a real $\alpha>\chi$. The property 2 remains valid for the B_n^α's. But the events B_n^α are not cylindrical, and so we loose the property (ii) of exponential mixing.

Is the property of "exponential hitting times" always true for the B_n^α's?

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