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Matched pairs of topological Lie algebras corresponding to Lie bialgebra structures on diff(S^1) and diff$(\mathbf{R})$

by

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ABSTRACT. — We give a rigorous construction of Lie bialgebra-type structures on diff$(S^1)_c$ and diff$(\mathbf{R})$ in the form of matched pairs of topological Lie algebras. These correspond to an heuristic class of Lie bialgebra structures on diff(S^1) proposed by Witten, obtained from a class of solutions of the Classical Yang-Baxter Equations. We identify a novel topological obstruction to constructing a matched pair based on diff(S^1) in the non-complexified case. Because of this obstruction, the problem of exponentiation to a matched pair of groups based on Diff(S^1) remains open.

RÉSUMÉ. — Nous construisons rigoureusement des structures de bigèbre de Lie sur diff$(S^1)_c$ et sur diff$(\mathbf{R})$ dans la forme de paires assorties (qui amènent à des algèbres de Lie bicroissées). Ces paires assorties correspondent à une classe heuristique de structures de bigèbre de Lie sur diff(S^1) proposée par Witten, obtenue à partir d'une classe de solutions des équations de Yang-Baxter classiques. Nous identifions une obstruction nouvelle dans le cas de diff(S^1) ne pas complexifié. A cause de cette obstruction,

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le problème d'exponentiation à une paire assortie de groupes basée sur $\mathrm{Diff}(S^1)$ reste.

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1. INTRODUCTION AND PRELIMINARIES

The notion of a *matched pair* of groups was studied in [Mack] [Tak] and more recently (and independently) in [M1] [M2] and in [Lu W] and (at the Lie algebra level) in [KM] [AK], among others. Two groups G_1, G_2 are a matched pair if they act on each other and these actions α, β obey the condition

$$\forall u, v \in G_1, \quad s, t \in G_2, \qquad \alpha_u(e)=e, \quad \beta_s(e)=e$$
$$\alpha_u(st)=\alpha_{\beta_t(u)}(s)\,\alpha_u(t), \qquad \beta_s(uv)=\beta_{\alpha_v(s)}(u)\,\beta_s(v),$$

where e denotes the relevant group identity. A pair of topological groups is matched if such α and β exist and are continuous. When (G_1, G_2) are a matched pair of groups, one may form a *bicrossproduct group* $G_1 \bowtie G_2$ defined on $G_1 \times G_2$ by

$$(u, s)(v, t)=(\beta_{t^{-1}}(u)\,v,\ s\,\alpha_u(t^{-1})^{-1}).$$

In the case when G_1 and G_2 are finite, one may also form a *bicrossproduct Hopf algebra* $k[G_1] \bowtie k(G_2)$. Here k is a field, $k[G_1]$ the convolution algebra on G_1 and $k(G_2)$ the function algebra on G_2. In a similar context of algebraic groups, [Tak] was able to show that a certain Hopf algebra of Taft and Wilson was of bicrossproduct form. New examples in this Hopf algebra context were given in [M1] in connection with a new algebraic approach to systems combining quantum mechanics and gravity [M3]. Purely differential-geometric aspects needed for this were given in [M2].

Matched pairs of Lie groups and their bicrossproducts are connected at the Lie algebra level to the Classical Yang-Baxter Equations (CYBE) and to the notion of *Lie bialgebras* introduced in [Dri]. This connection in the finite dimensional case was the basis of [M2] and of [Lu W] [KM] [AK]. In the present paper we supply the necessary functional analysis needed to extend this connection to the context of the infinite dimensional Lie algebras $\mathrm{diff}(S^1)_c$ and $\mathrm{diff}(\mathbf{R})$. This enables us to give a rigorous version of a class of Lie bialgebra structures, proposed in [Wit] in the context of string theory.

We now recall the relevant algebraic theory from [Dri] and [M1, Section 4]. See also [Lu W] [KM] [AK]. The notion of a matched pair of Lie groups linearizes to the notion of a matched pair of Lie algebras as follows. Two Lie algebras (g_1, g_2) are a matched pair if g_1 is represented on the vector space of g_2 by α and g_2 is represented on the vector space of g_1 by β, and for all $\xi, \xi_1, \xi_2 \in g_1$ and $\eta, \eta_1, \eta_2 \in g_2$

$$\alpha_\xi([\eta_1, \eta_2]) = [\alpha_\xi(\eta_1), \eta_2] + [\eta_1, \alpha_\xi(\eta_2)] + \alpha_{\beta_{\eta_2}(\xi)}(\eta_1) - \alpha_{\beta_{\eta_1}(\xi)}(\eta_2),$$
$$\beta_\eta([\xi_1, \xi_2]) = [\beta_\eta(\xi_1), \xi_2] + [\xi_1, \beta_\eta(\xi_2)] + \beta_{\alpha_{\xi_2}(\eta)}(\xi_1) - \beta_{\alpha_{\xi_1}(\eta)}(\xi_2).$$

For concreteness we shall work over **R** or **C**. If the Lie algebras are topological, we require the actions α, β to be continuous. A *coadjoint matched pair* of Lie algebras is a matched pair of the form (g, g^*) where g^* is the dual space of g and where the actions α, β are the mutual coadjoint actions. Explicitly, this means a matched pair (g_1, g_2) in which g_1 and g_2 are paired by a bilinear pairing $\langle\ ,\ \rangle$ and the (left) actions α, β are defined via

$$\langle \xi_1, \alpha_{\xi_2}(\eta) \rangle = \langle \eta, [\xi_1, \xi_2] \rangle, \qquad \langle \eta_1, \beta_{\eta_2}(\xi) \rangle = \langle \xi, [\eta_1, \eta_2] \rangle,$$
$$\forall \xi, \xi_1, \xi_2 \in g_1, \qquad \eta, \eta_1, \eta_2 \in g_2.$$

The pairing should be non-degenerate, *i. e.* such that $\langle\ , g_2 \rangle = g_1^*$, *i. e* the map $g_2 \to g_1^*$ defined by the pairing is an isomorphism. (And similarly $g_1 \to g_2^*$ is an isomorphism.) In the topological case $g_2 = g_1'$ the continuous dual. If g_1 and g_2 are paired topological Lie algebras and α and β obey the above but the pairing is not non-degenerate or only partially defined, we say that the matched pair is of *coadjoint type*.

The notion of a Lie bialgebra as introduced by Drinfeld [Dri] in connection with Poisson structures on Lie groups, is the following. A Lie bialgebra is a pair (g, δ) where g is a Lie algebra and $\delta: g \to g \otimes g$ is a Lie coalgebra [Mic] and in addition $\delta \in Z^1_{\mathrm{ad}}(g, g \otimes g)$ *i. e.* a one-cocycle in the Lie algebra cohomology [Hil, Chapter VII, Section 4] of g with values in $g \otimes g$. Here g acts on $g \otimes g$ in the adjoint representation of g on g, extended as derivations to higher tensor products. Explicity, the requirement that δ define a Lie coalgebra is

$$(\mathrm{id} \otimes \delta)\delta\xi + \text{cyclic permutations in } g \otimes g \otimes g = 0, \qquad \forall \xi \in g$$

(the coJacobi identity). Explicitly, the requirement that δ is a one-cocycle as required is

$$\delta(\mathrm{ad}_{\xi_1}(\xi_2)) = \mathrm{ad}_{\xi_1}\delta(\xi_2) - \mathrm{ad}_{\xi_2}\delta(\xi_1)$$

where $\mathrm{ad}_{\xi_1}(\xi_2) = [\xi_1, \xi_2]$, the adjoint action on g, extends to $g \otimes g$ as a derivation.

It turns out, *e. g.* [M1, Section 4], that in the finite dimensional case the notions of a coadjoint matched pair and of a Lie bialgebra coincide.

Thus (g, g^*) are both Lie algebras and matched by the mutual coadjoint actions iff g is a Lie bialgebra. The map δ is equivalent to a Lie algebra structure on g^* via $\langle \delta\xi, \eta_1 \otimes \eta_2 \rangle = \langle \xi, [\eta_1, \eta_2] \rangle$. However, in the infinite dimensional case these two algebraic notions diverge. The fundamental reason for this is that a Lie algebra structure on g^*, *i. e.* a suitable map $g^* \otimes g^* \to g^*$ defines a dual map $\delta : g^{**} \to (g^* \otimes g^*)^*$. Since $(g^* \otimes g^*)^* \supset g^{**} \otimes g^{**} \supset g \otimes g$, this may restrict to a map $g \to g \otimes g$, but does not necessarily do so. With suitable topologies the two notions can again coincide. It will be convenient to formulate our constructions in terms of matched pairs, rather than bialgebras.

In particular, we shall see below that the analogues of the heuristic construction of Lie bialgebra structures on $\mathrm{diff}(S^1)$ indicated in [Wit, Appendix], lead in fact to natural examples of matched pairs of topological Lie algebras based on $\mathrm{diff}(\mathbf{R})$ and $\mathrm{diff}(S^1)_c$. The motivation for [Wit] came from string theory, in connection with representations of the Virasoro algebra. It is possible that some ideas in the present paper could be generalized in this context to the Virasoro algebra, or further, to Gelfand-Dickey algebras [Bak].

Given a matched pair of Lie algebras over $\mathbf{R}$ or $\mathbf{C}$, it is natural to attempt to exponentiate these to a matched pair of groups or semigroups. This was done in the general finite dimensional case in [M2] and a physical application given in [M3]. The exponentiation was also given independently in [Lu W] with some of the steps also obtained in [KM, Section 4]. In the final section of the paper, we consider how this exponentiation could be carried out for matched pairs based on $\mathrm{diff}(S^1)$ [rather than on $\mathrm{diff}(S^1)_c$, which is not suitable for exponentiation to a Lie group]. We identify a novel topological obstruction. From the physical point of view, it may be possible to interpret such an exponentiation along the lines of [M3] as the motion of a particle in configuration space given by an orbit in $\mathrm{Diff}(S^1)$. One may also consider here $\mathrm{Diff}(S^1)_{c+}$, Segal's partial complexification of $\mathrm{Diff}(S^1)$ [Seg]. In addition, bicrossproduct groups in general are connected with the Riemann-Hilbert problem and dressing transformations in the context of classical integrable systems [Sem]. This connection was the motivation of [Lu W] [KM] [AK] via the theory of Manin triples. A final physical setting which may be relevant is [Yas]. Note that this exponentiation problem is very different, both mathematically and physically, from the problem of "exponentiation" of a solution of the CYBE equations to a solution of the Quantum Yang-Baxter Equations (QYBE).

Construction of Lie bialgebras: review of algebraic case

The strategy to obtain Lie bialgebras that will be used rests on work of Drinfeld. For clarity we describe it here in its finite dimensional setting.

It is not used directly in the paper but motivates the definitions that will be made in the infinite-dimensional examples of matched pairs. Let g be a finite dimensional Lie algebra over **R** or **C**. Let ω be a non-degenerate 2-cocycle on g. In the real case, this extends to a left-invariant simplectic form on G the simply connected Lie group with Lie algebra g. Non-degenerate means that when the two-cocycle $\omega \in g^* \otimes g^*$ is viewed as a map $\omega : g \to g^* : \xi \mapsto \omega(\xi,\ )$, it is invertible (corresponding to the symplectic form on G being non-degenerate). Let $r=\omega^{-1}$. Then Drinfeld [Dri] showed that r viewed as $r \in g \otimes g$ obeys the CYBE,

$$\sum [r^{(1)\,i}, r^{(1)\,j}] \otimes r^{(2)\,i} \otimes r^{(2)\,j} + r^{(1)\,i} \otimes [r^{(2)\,i}, r^{(1)\,j}] \otimes r^{(2)\,j} + r^{(1)\,i} \otimes r^{(1)\,j} \otimes [r^{(2)\,i}, r^{(2)\,j}] = 0.$$

(Here $r \equiv \sum r^{(1)} \otimes r^{(2)}$ is the formal sum notation for elements of $g \otimes g$ and i, j distinguish the two formal sums.) Indeed, non-degenerate (skew symmetric) solutions of the CYBE are precisely equivalent to non-degenerate two-cocycles in this way. In the real case they correspond to Poisson bracket structures on $C^\infty(G)$. Next, Drinfeld showed [Dri] how to obtain from this a Lie bialgebra structure on g. Namely, let $\delta = dr$ where d is the coboundary operator in the Lie algebra complex with values in $g \otimes g$ under the adjoint action. Explicitly,

$$\delta(\xi) = \mathrm{ad}_\xi(r) = \sum [\xi, r^{(1)}] \otimes r^{(2)} + r^{(1)} \otimes [\xi, r^{(2)}].$$

The corresponding Lie algebra structure on g^* is

$$[\eta_1, \eta_2] = \alpha_{r(\eta_1)}(\eta_2) + \alpha_{r^*(\eta_2)}(\eta_1)$$

where $r(\eta) = r(\eta,\ )$ and α is the coadjoint action of g on g^* as above. The coadjoint action of g^* on g is then given by

$$\beta_\eta(\xi) = r \circ \alpha_\xi(\eta) + [r(\eta), \xi].$$

The bracket on g can also be recovered in the form

$$[\xi_1, \xi_2] = \beta_{\omega(\xi_1)}(\xi_2) - \beta_{\omega(\xi_2)}(\xi_1)$$

and α can be recovered in the form $\alpha_\xi(\eta) = \omega \circ \beta_\eta(\xi) + [\omega(\xi), \eta]$ so that the construction is completely symmetric between g, g^* and r, ω.

Note that in forming the Lie bialgebra (g, r) it is not necessary that r be non-degenerate or the inverse of a 2-cocycle ω. For the above bracket on g^* to be skew symmetric it is only required that the self-adjoint part of r be ad-invariant. (In our examples, however, r will arranged to be skew adjoint.) In forming the coadjoint matched pair (g, g^*, α, β), r is needed only in the form of this map $r : g^* \to g$. Also, for the coadjoint matched pair, it is not necessary to construct the map δ, which need not exist in the infinite dimensional case. In the construction, the CYBE for r is used in the form $r([\eta_1, \eta_2]) = [r(\eta_1), r(\eta_2)]$ to show that the above bracket on g^* manifestly obeys the Jacobi identity. It provides a useful homomorphism $g^* \to g$.

The heuristic construction of Lie bialgebra structures on $\mathrm{diff}(S^1)$ described in [Wit] proceeds by analogy with the above finite dimensional construction in the case when the 2-cocycle ω is exact. Thus let ω_b be the two-cocycle on a suitable Lie algebra g defined by an element $b \in g^*$ as

$$\omega_b(\xi_1, \xi_2) = -\langle b, [\xi_1, \xi_2]\rangle.$$

In the present example, for the r_b defined via ω_b by $b \in g^*$, we obtain

$$[\eta_1, \eta_2] = \alpha_{[r_b(\eta_1), r_b(\eta_2)]}(b)$$

and $r_b : g^* \to g$ an isomorphism.

The above constructions of Lie algebra matched pairs can be summarized in terms of Lie algebra cohomology. Thus $(g_1, g_2, \alpha, \beta)$ is a matched pair of Lie groups if the actions α and β obey

$$\alpha \in Z^1_{\beta^* \otimes \mathrm{ad}}(g_2, g_1^* \otimes g_2), \qquad \beta \in Z^1_{\alpha^* \otimes \mathrm{ad}}(g_1, g_2^* \otimes g_1)$$

where α^* and β^* are the actions on g_2^* and g_1^* respectively, obtained as adjoints of α and β. The coadjoint matched pairs that we shall be considering are all cohomologically trivial: $\alpha = -d\omega$ and $\beta = -dr$ in the relevant complexes and $\omega \in g^* \otimes g^*$, $r \in g \otimes g$ (the minus signs here can be avoided by working with right coadjoint actions, rather than the left coadjoint actions). Moreover, the specific ω_b and $r_b = \omega_b^{-1}$ that we shall be considering are also cohomologically trivial, $\omega_b = db$ in the usual Lie algebra cohomology complex, and $b \in g^*$. This finite dimensional construction of certain coadjoint matched pairs motivates the construction of the infinite dimensional matched pairs of the present paper.

When $(g_1, g_2, \alpha, \beta)$ are a matched pair of Lie algebras, the *bicrossproduct Lie algebra* on $g_1 \oplus g_2$ is defined by

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] \\ = ([\xi_1, \xi_2] + \beta_{\eta_1}(\xi_2) - \beta_{\eta_2}(\xi_1), [\eta_1, \eta_2] + \alpha_{\xi_1}(\eta_2) - \alpha_{\xi_2}(\eta_1)).$$

As with Lie bialgebras, these are interesting even though the structure maps are cohomologically trivial in the relevant complex. For example, the bicrossproducts of such cohomologically trivial real coadjoint matched pairs have a natural complex structure,

PROPOSITION 1.1. – *Let $(g_1, g_2, \alpha, \beta)$ be the real coadjoint matched pair defined by $\alpha = -d\omega$ and $\beta = -dr$ in the relevant complexes. Here ω is a 2-cocycle and as maps, $r = \omega^{-1}$. The linear map* $\mathrm{J} = \begin{pmatrix} 0 & -r \\ \omega & 0 \end{pmatrix}$: *$g_1 \bowtie g_2 \to g_1 \bowtie g_2$ makes $g_1 \bowtie g_2$ naturally into a complex vector space (but not a complex Lie algebra as the Lie bracket on $g_1 \bowtie g_2$ need not respect* J). *Moreover, there is a natural 2-cocycle on $g_1 \bowtie g_2$ given as a map*

$\begin{pmatrix} \omega & 1 \\ -1 & -r \end{pmatrix}$: $g_1 \bowtie g_2 \to (g_1 \bowtie g_2)^*$, *and a natural solution of the CYBE on* $g_1 \bowtie g_2$ *given as a map* $\begin{pmatrix} r & -1 \\ 1 & -\omega \end{pmatrix}$: $(g_1 \bowtie g_2)^* \to g_1 \bowtie g_2$. *These are both degenerate (and so not inverse to each other.)*

Proof. — The proof follows by direct computation using the expressions above. Note that $g_1 \bowtie g_2$ is built explicitly on $g \oplus g^*$ where $g_1 = g$, $g_2 = g^*$. The complex structure J defines an action of $(x + \iota y) \in \mathbf{C}$ as $x + \mathrm{J}\, y\, (\mathrm{J}^2 = -1)$. g is finite dimensional and we identify $(g_1 \bowtie g_2)^*$ with $g^* \oplus g$. The stated 2-cocycle and solution of the CYBE can then be directly verified in this form. It should be mentioned that Drinfeld has also observed a (different) solution of the CYBE on $g_1 \bowtie g_2$. □

2. MATCHED PAIRS WITH diff(R)

This section extends the construction of a coadjoint matched pair g, g^* defined by an element $b \in g^*$, to the infinite dimensional case $g = \mathrm{diff}(\mathbf{R})$. More precisely, instead of working with $g_1 = \mathrm{diff}(\mathbf{R})$, we shall work with smooth bounded vector fields on $\mathbf{R}$. [Note that this is not all of the Lie algebra of the usual $\mathrm{Diff}(\mathbf{R})$.] Instead of working with g_2 the dual linear space, we shall take the space of smooth quadratic differentials. The example will be of coadjoint type.

DEFINITION 2.1. — *Let* $\mathbf{R}$ *be endowed with standard co-ordinates* $x \in \mathbf{R}$ *and co-ordinate basis* $\dfrac{\partial}{\partial x}$ *of the tangent bundle. Let* g_1 *be the topological Lie algebra of smooth bounded vector fields on* $\mathbf{R}$, *which we identify with* $C^\infty_{bd}(\mathbf{R})$ *by the basis. This consists of real-valued smooth functions, bounded in the seminorms*

$$\|\xi\|_n = \sup_{\mathbf{R}} |\xi^{(n)}|, \qquad n = 0, 1, 2, \ldots, \qquad \xi \in g_1.$$

$\|\xi\|_0$ *is abbreviated to* $\|\xi\|$. *The Lie bracket on* $g_1 = C^\infty_{bd}(\mathbf{R})$ *is explicitly given by*

$$[\xi_1, \xi_2] = \xi_1 \xi_2' - \xi_2 \xi_1', \qquad \forall\, \xi_1, \xi_2 \in g_1$$

and is jointly continuous.

That the Lie bracket shown on g_1 is indeed defined and jointly continuous follows immediately from the following elementary lemma and formulae of the form

$$\sup|(\xi_1\xi_2')^{(n)}| \leqq \sum_{r=0}^{r=n}\binom{n}{r}\sup|\xi_1^{(n-r)}|\sup|\xi_2^{(r+1)}|.$$

LEMMA 2.2. — *Let* U, V, W *be topological vector spaces with topology determined by families of seminorms* $\|\ \|_n$. *A linear map* $\varphi: \mathrm{V}\to\mathrm{V}$ *is continuous iff for each* n, $\|\varphi(v)\|_n$ *can be bounded by finite linear combinations of the* $\|v\|_m$ *(uniformly in* V*). A bilinear map* $\varphi: \mathrm{U}\times\mathrm{V}\to\mathrm{W}$ *is jointly continuous iff for each* n, $\|\varphi(u, v)\|_n$ *can be bounded by finite linear combinations of* $\|u\|_{m_1}\|v\|_{m_2}$ *(uniformly in* $\mathrm{U}\times\mathrm{V}$*).*

DEFINITION 2.3. — *Let* g_2 *be the topological linear space of certain smooth bounded quadratic differentials on* **R**, *identified via basis* $dx\otimes dx$ *with a subset of* $\mathrm{C}_{bd}^{\infty}(\mathbf{R})$. *More precisely, let* $g_2=\mathrm{C}_{bd}^{\infty}(\mathbf{R})\cap\mathrm{L}_{\infty}^{1}(\mathbf{R})$. *This consists of smooth functions bounded in the seminorms*

$$\|\eta\|_n=\sup_{\mathbf{R}}|\eta^{(n)}|,\qquad \|\eta\|_m^1=\int_{\mathbf{R}} dx\,|\eta^{(m)}(x)|,$$

$$n, m=0, 1, 2, \ldots,\quad \eta\in g_2.$$

$\|\eta\|_0$ *and* $\|\eta\|_0^1$ *are abbreviated to* $\|\eta\|$ *and* $\|\eta\|^1$ *respectively.*

There is a pairing between g_1 and g_2 defined by

$$\langle\xi, \eta\rangle=\int_{\mathbf{R}} dx\,\xi(x)\,\eta(x),\qquad \xi\in g_1,\quad \eta\in g_2.$$

This pairing is continuous since $|\langle\xi, \eta\rangle|\leqq\|\xi\|\,\|\eta\|^1$. Using this, we let α denote the left coadjoint action of g_1 on g_2. Explicitly, this is defined by

$$\alpha_\xi(\eta)=2\eta\xi'+\eta'\xi,\qquad \forall\xi\in g_1,\quad \eta\in g_2.$$

This is because

$$\langle\xi_1, \alpha_{\xi_2}(\eta)\rangle\equiv\langle\eta, [\xi_1, \xi_2]\rangle$$
$$=\int_{\mathbf{R}} dx\,\eta(\xi_1\xi_2'-\xi_2\xi_1')=\int_{\mathbf{R}} dx\,\xi_1((\eta\xi_2)'+\eta\xi_2').$$

To see the last equality, use integration by parts between the finite limits $-\mathrm{L}$ and R. The boundary term is

$$[\eta\xi_1\xi_2]_{-\mathrm{L}}^{\mathrm{R}}=\int_{-\mathrm{L}}^{\mathrm{R}}\frac{d}{dx}(\eta\xi_1\xi_2).dx=\int_{-\mathrm{L}}^{\mathrm{R}}(\eta\xi_1)'\xi_2\,dx+\int_{-\mathrm{L}}^{\mathrm{R}}(\eta\xi_1)\xi_2'\,dx.$$

Since the various integrands are in L^1, the expression on the right tends to a limit as R and L tend to infinity. Hence $[\eta\xi_1\xi_2]$ tends to a limit as

$R \to \infty$, and as $\eta\xi_1\xi_2$ is in L^1, that limit can only be zero. A similar argument applies as $L \to \infty$, so it is possible to use integration by parts over all of $\mathbf{R}$.

PROPOSITION 2.4. – *Let g_1 be the topological Lie algebra $g_1 = C^\infty_{bd}(\mathbf{R})$ defined in Definition 2.1 and g_2 the topological linear space $g_2 = C^\infty_{bd}(\mathbf{R}) \cap L^1_\infty(\mathbf{R})$ defined in Definition 2.3. Then the map*

$$\alpha : g_1 \times g_2 \to g_2, \qquad \alpha_\xi(\eta) = 2\eta\xi' + \eta'\xi, \quad \forall \xi \in g_1, \quad \eta \in g_2$$

is defined and is a jointly continuous action of g_1 on g_2.

Proof. – The seminorms $\|\alpha_\xi(\eta)\|_n$ are bounded bilinearly in the $\|\eta\|_{m_1}$ and the $\|\xi\|_{m_2}$ as in checking Definition 2.1. For the additional seminorms we have

$$\|\alpha_\xi(\eta)\|^1 = \int_{\mathbf{R}} |\alpha_\xi(\eta)|\,dx \leqq 2\|\xi'\| \int_{\mathbf{R}} |\eta|\,dx + \|\xi\| \int_{\mathbf{R}} |\eta'|\,dx$$
$$= 2\|\xi\|_1 \|\eta\|^1 + \|\xi\|\,\|\eta\|^1_1$$

and similarly for the higher $\|\alpha_\xi(\eta)\|^1_n$. Hence by Lemma 2.2, the map α is well defined and continuous. □

As explained in the finite dimensional case, the construction of a Lie algebra structure on g_2 and the action β of g_2 on g_1, begins with a two-cocycle ω_b on g_1 viewed in the finite dimensional case as a map $\omega_b : g_1 \to g_2 = g_1^*$. It was defined by $\langle \xi_1, \omega_b(\xi_2) \rangle = \langle b, [\xi_1, \xi_2] \rangle$. We use an analogous definition.

DEFINITION 2.5. – *Let b be a fixed every-where positive element of $C^\infty_{bd}(\mathbf{R})$ bounded away from zero, and let ω_b be the linear map*

$$\omega_b : g_1 \to C^\infty_{bd}(\mathbf{R}), \qquad \omega_b(\xi) = 2b\xi' + b'\xi.$$

PROPOSITION 2.6. – *Let g_1 be the topological Lie algebra, g_2 the topological linear space and α, ω_b the maps defined above. Then the map ω_b has image containing g_2 and a unique skew adjoint inverse $r_b : g_2 \to g_1$. It is given by*

$$r_b(\eta)(x) = \int_{-\infty}^{\infty} dy \frac{\operatorname{sgn}(x-y)}{4\sqrt{b(x)}\sqrt{b(y)}} \eta(y)$$

and is continuous. Here $\operatorname{sgn}(x)$ *denotes* ± 1 *according to the sign of x. Skew adjoint means here that $\langle r_b(\eta_1), \eta_2 \rangle = -\langle r_b(\eta_2), \eta_1 \rangle$ for all $\eta_1, \eta_2 \in g_2$.*

Proof. – The inverse of ω_b is the map $\eta \mapsto r_b(\eta)$ where $r_b(\eta)$ is the solution r to the differential equation $2br' + b'r = \eta$. Since b is bounded away from zero and smooth, this has a unique solution of the form $r(x) = \frac{1}{\sqrt{b}}\left(\int_{-\infty}^{x} \sqrt{b}\frac{\eta}{2b}\,dy + A(\eta)\right)$ where $A(\eta)$ is a constant. This is fixed by the requirement of skew adjointness in the form stated. Note that the skew adjointness can be checked on η_1, η_2 of compact support, which are dense in g_2; this extends to all of g_2 by continuity of the pairing. The result is

$$r_b(\eta)(x) = \frac{1}{2\sqrt{b}}\left(\int_{-\infty}^{x} \frac{\eta}{\sqrt{b}}\,dx - \int_{-\infty}^{\infty} \frac{\eta}{2\sqrt{b}}\,dx\right).$$

The integral kernel stated for r can also be found directly using the method of Green's functions. The integral kernel $r_y(x) = r(x, y)$ is the solution of the distributional differential equation $r'_y + \frac{b'}{2b} r_y = \frac{\delta_y}{2b(y)}$. Here δ_y is the delta function distribution at y. This evidently has solutions $r_\pm = A_\pm/\sqrt{b}$ for the two domains $x \langle y$ and $x \rangle y$. Here A_- is arbitrary and $A_+ = A_- + \frac{1}{2\sqrt{b(y)}}$. Antisymmetry of the kernel then fixes

$$A_- = -1/4\sqrt{b(y)}.$$

Note that although $\ker \omega_b$ is nontrivial $\left(\text{it is spanned by } \xi = \frac{1}{\sqrt{b}}\right)$, this degeneracy is fixed by the skew adjointness requirement. Also note that if b, η are smooth then $r = r_b(\eta)$ is also smooth. (To see this, if r is continuous then r' is continuous since $r' = \frac{\eta - b' r}{2b}$. Similarly, all derivatives to order k continuous implies $r^{(k+1)}$ continuous. We require for this that b is smooth and bounded away from zero.) We now check that the map $r_b : \eta \mapsto r_b(\eta)$ is well defined and continuous. Firstly note,

$$|r_b(\eta)(x)| \leqq \frac{1}{4\sqrt{b(x)}} \left\| \frac{\eta}{\sqrt{b}} \right\|^1 \leqq \frac{1}{4 \inf_{\mathbf{R}} b} \|\eta\|^1.$$

Next, by $2br' + b'r = \eta$,

$$2\|r_b(\eta)\|_1 \leqq \left\|\frac{1}{b}\right\| \|\eta\| + \left\|\frac{b'}{b}\right\| \|r_b(\eta)\|,$$

and similarly for the higher derivatives. Hence by Lemma 2.2, r_b is defined and continuous. This completes the derivation of $r_b(\eta)$. □

PROPOSITION 2.7. – *Let g_1, g_2, α and r_b be as above. The resulting bracket on g_2,*

$$[\eta_1, \eta_2] = \alpha_{r_b(\eta_1)}(\eta_2) - \alpha_{r_b(\eta_2)}(\eta_1), \qquad \eta_1, \eta_2 \in g_2$$

makes g_2 into a topological Lie algebra and the map

$$\beta_\eta(\xi) = r_b \circ \alpha_\xi(\eta) + [r_b(\eta), \xi]$$

defines an action of g_2 on g_1. With these actions, (g_1, g_2) is a matched pair of coadjoint type.

Proof. — It is straightforward to check that the integral kernel of r_b obeys the CYBE as it must by construction. Equivalently [M1], $r_b([\eta_1, \eta_2])=[r_b(\eta_1), r_b(\eta_2)]$ where $[\eta_1, \eta_2]$ is the Lie bracket on g_2 in the form stated. This is immediate given that r_b and α are well defined and continuous for g_1, g_2 as stated. Also, the stated form of β is well defined and an action. As in the algebraic case *cf.* [M1, Lemma 4.3], these facts are enough to ensure that α and β are matched. □

Example 2.8. — *Let $b=\frac{1}{2}$. Then $\omega_b(\xi)=\xi'$ and*

$$r_b(\eta)(x)=\int_{-\infty}^{x}\eta-\frac{1}{2}\int_{-\infty}^{\infty}\eta.$$

The Lie bracket on g_2 and its action β then take the form

$$[\eta_1, \eta_2]=r_b(\eta_1)\eta_2'-r_b(\eta_2)\eta_1', \qquad \beta_\eta(\xi)=r_b(\eta\xi')+r_b(\eta)\xi'$$

on $\xi\in g_1$, η, η_1, $\eta_2\in g_2$.

3. MATCHED PAIRS WITH diff $(S^1)_C$

In this and the next section we shall consider matched pairs based on diff(S^1) rather than vector fields on **R** as in the last section. The construction will follow the same strategy. Given a function $b:S^1\to\mathbf{R}$ we shall construct a map r_b obeying the CYBE, as the solution of a differential equation on S^1 (*cf.* proof of Proposition 2.6). If b has no zeros on S^1, the differential equation has non-singular coefficients and it is easy to show *cf.* [Wit, Section 3] that there are no solutions. In order to make sense of differential equations with singular coefficients we embed S^1 in **C** and use the theory of complex analytic continuation.

DEFINITION 3.1. — *Let $S^1\subset\mathbf{C}$ be the unit circle in the complex plane with angular co-ordinate θ and $\frac{\partial}{\partial\theta}$ a co-ordinate basis of the tangent bundle. Let g denote the space of* real-analytic *vector fields on S^1, which we identify with real-analytic functions: $S^1\to\mathbf{R}$ by the basis. By definition, $f: S^1\to\mathbf{R}$ is* real-analytic (*or* analytically extendible) *if there exists $a>1$ such that f extends to an analytic function $f: A_a\to\mathbf{C}$. Here A_a is the annulus $\left\{z\in\mathbf{C}:|z|\in\left(\frac{1}{a}, a\right)\right\}$.*

Let g_a denote the space of analytic functions $A_a \to \mathbf{C}$ that are real on S^1, endowed with the compact open topology. We give g the direct limit topology, $g = \lim_{a \to 1} g_a$.

By the Stone-Weierstrass theorem, such real-analytic functions are dense in the continuous real functions on the circle with the compact open topology. Thus it is reasonable to work with this topological vector space g in place of $\operatorname{diff}(S^1)$, with a corresponding Lie algebra structure. Eventually, b will also be taken an element of this space. We shall return in the next section to the problem of constructing a matched pair on the topological vector spaces $g_1 = g_2 = g$, the $\operatorname{diff}(S^1)$ Lie algebra structure on g_1 and a Lie algebra structure on g_2 determined by r_b. (We will find an obstruction.)

In the present section we consider the easier case g_a^+ of functions analytic on an annulus of radii $(1, a)$, and without any reality condition. For suitable b we shall obtain a matched pair on the topological vector spaces $g_1 = g_2 = g^+$ as the direct limit of matched pairs on the topological vector spaces $g_1 = g_2 = g_a^+$. This direct limit g^+ plays the role in our framework of $\operatorname{diff}(S^1)_{\mathbf{c}}$. Thus,

DEFINITION 3.2. — *Let g_a^+ denote the space of complex valued analytic vector fields in the annulus A_a^+, endowed with the compact open topology. Here A_a^+ denotes the open subset $\{z \in \mathbf{C} : |z| \in (1, a)\}$ and $a > 1$. Let $g^+ = \lim_{a \to 1} g_a^+$ with the direct limit topology.*

We define a Lie algebra structure in the g_a and the g_a^+ by extending the usual $\operatorname{diff}(S^1)$ Lie bracket, $[\xi_1, \xi_2] = \xi_1 \frac{d\xi_2}{d\theta} - \xi_2 \frac{d\xi_1}{d\theta}$, to the complex case as

$$[\xi_1, \xi_2] = \iota z \left(\xi_1 \frac{d\xi_2}{dz} - \xi_2 \frac{d\xi_1}{dz} \right).$$

Note that $\frac{d}{d\theta} = \iota z \frac{d}{dz}$ at $z = e^{\iota\theta}$.

To proceed along the lines of Section 2, we want to solve the differential equation $2b \frac{dr}{d\theta} + r \frac{db}{d\theta} = \eta$ extended to an annulus A_a^+. The extension to an equation in $z \in \mathbf{C}$ is

$$2b \frac{dr}{dz} + r \frac{db}{dz} = -\iota \frac{\eta}{z}. \tag{1}$$

We assume that b is analytic in the annulus A_a^+ and has no zeros there.

PROPOSITION 3.3. — *Let b be analytic on A_a^+ and without zeros there. If b has odd winding number then the map $\eta \mapsto r_b(\eta): g_a^+ \to g_a^+$ giving the solution to the equation* $2br_b' + r_b b' = -\iota\frac{\eta}{z}$, exists and is continuous. It is given by

$$r_b(\eta)(z) = -\frac{\iota}{4\sqrt{b(z)}}\int_\gamma \frac{\eta}{w\sqrt{b}}\,dw$$

where γ is a curve in A_a^+ from z to itself with winding number 1.

Proof. — Using an integrating factor of $1/\sqrt{b}$, equation (1) can be written as

$$\frac{d}{dz}\left(\sqrt{b}r\right) = \frac{-\iota\eta}{2z\sqrt{b}}.$$

However, $\sqrt{b}$ may not be defined on the annulus, so for this to make sense we consider the equation in the double cover of the annulus, $\tilde{A}_a^+$ $(\theta \in [0, 4\pi))$. Given a starting point w_0 in the double cover, we can define a solution to the differential equation by

$$r(w) = -\frac{\iota}{2\sqrt{b(w)}}\int_\gamma \frac{\eta}{z\sqrt{b}}\,dz + \frac{\sqrt{b(w_0)}}{\sqrt{b(w)}}\,r(w_0),$$

where γ is a curve in the double cover from w_0 to w. But r is required to be a function on the original annulus, so it must have period 2π. That is, if $w_1 = e^{2\pi\iota}w_0$ and γ is a curve with winding number 1 in the original annulus from w_0 to w_1, then

$$r(w_1) = r(w_0) = -\frac{\iota}{2\sqrt{b(w_1)}}\int_\gamma \frac{\eta}{z\sqrt{b}}\,dz + \frac{\sqrt{b(w_0)}}{\sqrt{b(w_1)}}\,r(w_0).$$

There are two cases to deal with. Firstly, if $\sqrt{b}(w_1) = \sqrt{b}(w_0)$, the only possibility of a solution is if the integral vanishes, which will only be true for certain η, and if this condition is satisfied there will be an infinite number of solutions. Secondly, if $\sqrt{b}(w_1) = -\sqrt{b}(w_0)$, there is a unique solution for all η, namely

$$r(w_1) = -\frac{\iota}{4\sqrt{b(w_1)}}\int_\gamma \frac{\eta}{z\sqrt{b}}\,dz.$$

If b has no zeros in the open annulus A_a^+, then this formula gives a unique single valued solution r to the differential equation in A_a^+, and we can consider γ to be a curve in A_a^+ from w_1 to itself with winding number 1. The condition that $\sqrt{b(w_1)} = -\sqrt{b(w_0)}$ merely says that b has odd winding number about the annulus. Also note that if we consider the points

w_1 lying in a compact subset K of A_a^+, then we can choose the corresponding γ to lie in the compact set $S^1 K$. Thus uniform convergence of the function η on the compact set $S^1 K$ implies uniform convergence of r on the compact set K. Therefore the map $r_b : \eta \mapsto r$ is continuous. □

This result can now be combined along the lines of Section 2 to prove the following:

PROPOSITION 3.4. — *Suppose that b is analytic on the annulus* A_a^+ *and without zeros there, and that b has odd winding number. Let* $g_1 = g_2 = g_a^+$ *as topological vector spaces. Then there are jointly continuous Lie algebra brackets in* g_1 *and* g_2 *defined respectively by*

$$[\xi_1, \xi_2] = \imath z \left(\xi_1 \frac{d\xi_2}{dz} - \xi_2 \frac{d\xi_1}{dz} \right), \qquad \xi_1, \xi_2 \in g_1,$$

$$[\eta_1, \eta_2] = \alpha_{r_b(\eta_1)}(\eta_2) - \alpha_{r_b(\eta_2)}(\eta_1), \qquad \eta_1, \eta_2 \in g_2,$$

forming a matched pair of topological Lie algebras $(g_1, g_2, \alpha, \beta)$. *Here* r_b *is given in Proposition* 3.3 *and the actions are given by*

$$\alpha_\xi(\eta) = 2\eta\xi' + \eta'\xi, \qquad \beta_\eta(\xi) = r_b \circ \alpha_\xi(\eta) + [r_b(\eta), \xi].$$

If b extends to a real valued function on S^1, the winding number condition in the last result can be reformulated using the following well-known reflection principle:

LEMMA 3.5 (The Reflection Principle). — *If* $f : A_a^+ \to \mathbf{C}$ *is analytic, and if* f *extends continuously to a function on* $\{z \in \mathbf{C} : |z| \in [1, a)\}$ (*except perhaps for some finite set*) *which is real on* S^1, *then* f *can be extended to be analytic on the open annulus* A_a (*with perhaps finitely many isolated singularities on* S^1).

Outline of Proof. — Define $f(z)$ for $|z| \in \left(\frac{1}{a}, 1\right)$ by $f(z) = \overline{f(1/\bar{z})}$. Then the two definitions of f agree on S^1 by the reality condition (except at the finitely many points). If the set of such f were given the compact open topology on the semi-closed annulus $\{z \in \mathbf{C} : |z| \in [1, a)\}$ (except at the finitely many points), then the resulting extension would have the compact open topology on the open annulus $\left\{z \in \mathbf{C} : |z| \in \left(\frac{1}{a}, a\right)\right\}$ (except at the finitely many points). □

LEMMA 3.6. — *If* $f : S^1 \to \mathbf{R}$ *extends to an analytic function in some complex neighbourhood of* S^1, *then the winding number of* f *is one half of the number of zeros of* f *on the circle* (*counted multiply*).

Proof. – Suppose that f extends analytically to the open annulus $\left\{z\in\mathbf{C} : |z|\in\left(\frac{1}{a}, a\right)\right\}$, and that f has no zeros off the unit circle in the annulus (use the principle of isolated zeros). Choose a radius c between 1 and a, and count the number of times that the curve $\theta\mapsto f(ce^{\iota\theta})$ winds about the origin. By the reflection principle, $f\left(\frac{1}{c}e^{\iota\theta}\right)=\overline{f(ce^{\iota\theta})}$, so the curve $\theta\mapsto f\left(\frac{1}{c}e^{\iota\theta}\right)$ winds the opposite way about the origin. Thus we have the equation

$$2\times \text{winding no.}\,(f\,|_c)=\text{winding no.}\,(f\,|_c)-\text{winding no.}\left(f\,|\frac{1}{c}\right)=\int_\gamma \frac{f'}{f}dz$$

where γ is a path anticlockwise at radius c and clockwise at radius $1/c$. The result follows by Cauchy's theorem. □

The principle of isolated zeros now leads to the following proposition.

PROPOSITION 3.7. – *Suppose that b is a real-analytic function $S^1\to\mathbf{R}$. Then there exists an $a>1$ such that b is analytic on the annulus A_a and real on the circle, with no zeros on A_a^+. The winding number of b is one half the number of zeros of b on the circle (counted multiply). If this is odd, then Proposition 3.4 applies and $(g_1, g_2, \alpha, \beta)$ is a matched pair built on the topological spaces $g_1=g_2=g_a^+$.*

We now turn to the refinement of the above matched pairs, corresponding to $\text{diff}(S^1)_c$. As explained above, we consider not a fixed g_a^+, but the direct limit g^+. This can be defined formally as the disjoint union of the g_a^+ for all $a>1$, quotiented by the equivalence relation defined by agreement on the common domain of definition. Thus a set U is open in g^+ if and only if $U\cap g_a^+$ is open in g_a^+ for all $a>1$. One interesting fact about g^+ is that it is self dual, as we shall now show:

PROPOSITION 3.8. – *The topological vector space g^+ is self dual under the pairing*

$$<f, h> = -\iota\int_{cS^1}\frac{fh}{z}dz,\qquad \forall f, h\in g^+$$

where $c=c(f, h)>1$ is a number such that both f and h are defined on A_a^+ for some $a>c$. That is, the map $g^+\to g^{+\prime} : f\mapsto\langle f, \ \rangle$ given by the pairing is a 1-1 correspondence.

Proof. – By definition of the topology on g^+, to show that the pairing $\langle\ ,\ \rangle$ is continuous we only have to show that it is continuous on $g_a^+\times g_a^+$ for all $a>1$. This is automatic, since for any c in the interval $a>c>1$, cS^1 is a compact subset of A_a^+. To show that the map $g^+\to g^{+\prime}$

corresponding to the paring is 1-1, let $\{e_n(z)=z^{-n} \mid n\in\mathbf{Z}\}$ be a basis of g^+. Then

$$\langle f, e_n\rangle = 2\pi f_n$$

where f_n is the nth Laurent coefficient of f.

To show that this map $g^+ \to g^{+\prime}$ is onto, take a continuous map $\mathrm{T} : g^+ \to \mathbf{C}$, and define a function h as the function whose Laurent expansion is

$$h_n = \frac{1}{2\pi}\mathrm{T}(e_n).$$

We wish to show that the series for h converges in some annulus A_c^+, and thus show that $\mathrm{T}=\langle h, \ \rangle$. Now $\mathrm{T} : g_2^+ \to \mathbf{C}$ is continuous, so there is some compact set $\mathrm{K}\subset \mathrm{A}_2^+$ such that

$$|\mathrm{T}(e_n)| \leqq \mathrm{Const.}\sup\{z^{-n} : z\in\mathrm{K}\}.$$

Choose $c>1$ sufficiently small such that $\mathrm{A}_c^+\cap\mathrm{K}$ is empty. Then for $n\geqq 0$, $|h_n|\leqq \mathrm{Const.}\ c^{-n}$. Thus the sum of positive coefficients of h converges uniformly on compact subsets of A_c^+. Also for any $a>1$, $\mathrm{T} : g_a^+ \to \mathbf{C}$ is continuous, so there is some compact set $\mathrm{K}_a\subset\mathrm{A}_a^+$ such that

$$|\mathrm{T}(e_n)| \leqq \mathrm{Const.}\sup\{z^{-n} : z\in\mathrm{K}_a\}.$$

But if $n<0$, then $|h_n|\leqq \mathrm{Const.}\, a^n$, so the sum of negative coefficients of h converges uniformly on compact subsets of $|z|>a$ for all $a>1$, and thus uniformly on compact subsets of $|z|>1$. □

PROPOSITION 3.9. – *Let $g_1=g_2=g^+$ as topological vector spaces. Let b be a real-analytic function $\mathrm{S}^1 \to \mathbf{R}$ such that one half the number of zeros on S^1 (counted multiply) is odd. Then the topological vector spaces $g_1=g_2=g^+$ with the Lie algebra structures and actions defined as the direct limit of those of Propositions* 3.4 *and* 3.7, *form a coadjoint matched pair.*

Proof. – Continuity of the Lie bracket structures g_1 and g_2 on g_a^+, and of the maps ω_b and r_b were checked in obtaining Proposition 3.4. The direct limit topology is such that this extends to continuity on g^+. Thus $(g_1, g_2, \alpha, \beta)$ is a matched pair. That this matched pair is coadjoint (*i.e.* $g_2=g_1'$ as a topological vector space and α, β are the coadjoint actions) follows from Proposition 3.8. Note that in Proposition 3.8, if f, h in fact extend to functions on S^1, we can take $c=1$ and the pairing shown reduces to the usual bilinear pairing of functions on S^1 analogous to the L^2 pairing used in Section 2. □

4. TOPOLOGICAL OBSTRUCTION TO CASE OF REAL diff(S^1)

In this section we shall attempt to find matched pairs built on the topological vector spaces $g_1 = g_2 = g$ where g is the space of real-analytic functions on the circle. This space plays the role in our formulation of diff(S^1). It was defined in Definition 3.1 as the direct limit for $a > 1$ of g_a, the analytic functions on the annulus A_a that are real on the circle. The topology is given via the compact open topology on each g_a. We would like to give a version in this real case of the results for the complexified case treated in the last section. That is, we would like to show that if $g_1 = g_2 = g$ as topological vector spaces, then the definitions of Lie algebra structures on g_1 and g_2 and of the actions α and β along the lines of Proposition 3.4, give a matched pair. To do this we must show that r_b maps g_2 into g_1. It is not very difficult to see that the formula for r_b in Proposition 3.3 extends to continuous functions on the annulus $\{z \in \mathbf{C} : |z| \in [1, a)\}$ (excepting the zeros of b on S^1). However it is not obvious whether the reality condition on S^1 is preserved. In fact we shall see that the reality condition cannot be preserved because of the condition on b that it has odd winding number.

Recall that the function r_b was invented to be an inverse to the differential operator ω_b: $g \to g$ defined by $\omega_b(\xi) = 2b\xi' + b'\xi$. We found above that r_b really only makes sense if b has an odd winding number, and so in particular only if b has zeros by Proposition 3.7. Now ω_b is 1-1 since $r_b(\omega_b(\xi)) = \xi$, but is it onto? It is obviously not onto if b has a zero of order $\geqq 2$ at some point $z_0 \in S^1$, since then $\omega_b(\xi)(z_0) = 0$ for all ξ. If b has only order 1 zeros then ω_b is also not onto, but the proof will be rather more difficult.

PROPOSITION 4.1. — *Let* $\omega_b : g \to g : \xi \mapsto 2b\xi' + b'\xi$ *and let* r_b *be defined as in Proposition* 3.3. *Then* $\mathrm{image}(\omega_b) = \{\eta \in g : r_b(\eta) \in g\}$.

Proof. — If $\eta \in g$ is such that $r_b(\eta) \in g$, then $r_b \omega_b r_b(\eta) = r_b(\eta)$. But r_b is $1-1$, so $\eta = \omega_b r_b(\eta)$. Conversely, if $\eta = \omega_b(\xi)$, then $r_b(\eta) = r_b \omega_b(\xi) = \xi \in g$. □

PROPOSITION 4.2. — *If* b *has odd winding number, then* $\mathrm{image}(\omega_b)$ *is a finite codimension subspace of* g.

Proof. — If $r_b(\eta) \in g$, then two conditions must be satisfied:

(1) $r_b(\eta)$ is real on S^1, except at the zeros of b. This means that $r_b(\eta)$ can be extended to some annulus A_a except for the zeros of b.

(2) The singularities of $r_b(\eta)$ at the zeros of b are removable.

First we show that condition (1) is satisfied on a finite codimension subspace. Pick points $w_1, \ldots, w_n$ between the n zeros of b on S^1, and define $C : g \to \mathbf{R}^n$ by $C_k(\eta) = \mathscr{I}(r_b(\eta)(w_k))$. Here $\mathscr{I}$ denotes imaginary part.

Now if $r_b(\eta)(w_k)$ is real, then equation (1) implies that $r_b(\eta)(w)$ is real for all w which can be connected to w_k by a path on S^1 not including the zeros of b. Thus $C(\eta)=0$ if and only if $r_b(\eta)$ is real on the complement of the zeros of b in S^1.

Under the assumption that $C(\eta)=0$, we can use the reflection principle to extend $r_b(\eta)$ to an analytic function on A_a, except for the zeros of b. The condition that these singularities are removable is precisely that the negative index Laurent coefficients in the expansion about the zeros $z_1, \ldots, z_n$ of b vanish, *i.e.* that

$$\int_{\gamma_k} r_b(\eta)(z)(z-z_k)^m\,dz=0, \qquad \forall\, m\geqq 0, \quad k=1, \ldots, n$$

where γ_k is a small contour in A_a enclosing z_k. These integrands are continuous functions of η, and the finite codimension result will follow if we can show that there is a maximum m (depending only on b) for which the above integrands can be non-zero. Now $r_b(\eta)(w)$ was found in the previous section. It is the solution stated in Proposition 3.3 where γ is a contour from w to itself with winding number 1. Choose a zero z_k of b on S^1, and a radius $c\in(1, a)$. For w near z_k we know that

$$\left|\frac{\eta(w)}{w\sqrt{b(w)}}\right|\leqq\frac{\text{Const.}}{|w-z_k|^s}$$

for some s depending on the order of the zero of b at z_k. Let γ be the following curve: Move radially from w to the radius c circle, then around the circle anticlockwise, and then radially back to w. Thus we find a bound for $r_b(\eta)(w)$, for w near z_k:

$$|r_b(\eta)(w)|\leqq\frac{\text{Const.}}{|w-z_k|^s}\left(1+\frac{1}{|w-z_k|^s}\right).$$

This shows that the order of the singularity of $r_b(\eta)$ at z_k is bounded in terms of the order of the zero of b there, and completes the proof. □

PROPOSITION 4.3. — *If b has odd winding number, then there is $\eta\in g$ such that $r_b(\eta)\notin g$.*

Proof. — If b has a zero of order $\geqq 2$, then the result follows since we already know that ω_b is not onto. Now suppose that all the orders of the zeros of b are 1. We know that there must be at least two such zeros. In this case the function

$$z\mapsto\frac{\eta(z)}{z\sqrt{b(z)}}$$

is integrable on S^1, and the dominated convergence theorem shows that

$$r_b(\eta)(w) = -\frac{\imath}{4\sqrt{b(w)}} \int_{\gamma_w} \frac{\eta(z)}{z\sqrt{b(z)}} dz$$

for $w \in S^1$ and γ_w a path on S^1 from w to itself with winding number 1. Now the function

$$w \mapsto \int_{\gamma_w} \frac{\eta(z)}{z\sqrt{b(z)}} dz$$

varies continuously with $w \in S^1$. But $1/\sqrt{b(z)}$ blows up at a zero of b, so if $r_b(\eta)(w)$ is in g, then the integral

$$\int_{\gamma} \frac{\eta(z)}{z\sqrt{b(z)}} dz = 0$$

for γ the path from a zero of b to itself. Since g is dense in the continuous functions, this cannot be true for all η. $\square$

There results show that the constructions of this paper do not go through to obtain a matched pair built on the topological vector spaces $g_1 = g_2 = g$. Because $\omega_b : g \to g$ is not onto, and r_b does not map g into g, g_2 must be made smaller and/or g_1 must be made larger to obtain a map $r_b : g_2 \to g_1$ inverting ω_b. Unfortunately, there do not appear to be suitable choices for either of these that are closed under the relevant Lie brackets and for which α and β are defined.

In this section we have identified an obstruction to constructing a matched pair with real diff(S^1) along the lines of the complexified case given in Section 3. If this can be overcome, one would hope to exponentiate to a matched pair based on Diff(S^1). Note that the complex case of Section 3, diff$(S^1)_c$, is not the Lie algebra of any Lie group, so could not form the basis of an exponentiation to a matched pair of Lie groups. It may be however, that a matched pair could be obtained along the lines above with Diff$(S^1)_{c+}$, Segal's partial complexification of Diff(S^1). This would be interesting in the context described in the introduction.

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