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Kaehler coherent state orbits for representations of semisimple Lie groups

by

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ABSTRACT. — We show that an irreducible unitary representation (with discrete kernel) of a (noncompact) semisimple Lie group admits a Kähler coherent state orbit (*i.e.* a complex orbit on the space of all lines in the representation space) if and only if it is a highest weight representation.

RÉSUMÉ. — Nous prouvons qu'une représentation unitaire irréductible (avec noyau discret) d'un groupe de Lie semi-simple (non compacte) admet une orbite de Kähler d'états cohérents (*i.e.* une orbite complexe sur l'espace de toutes les droites de l'espace de la représentation) si et seulement si c'est une représentation de poids maximal.

1. INTRODUCTION

The notion of coherent states goes back to the early days of Quantum Mechanics, but it was only in 1972 when it was formalized (using representation theoretic terms) by Perelomov [Pe1] (*see* also [Pe2]). According to his definition, a system of coherent states for an irreducible unitary representation of a Lie group G on a Hilbert space $\mathscr{H}$ (or for a quantum mechanical system with symmetry group G in physicists' terminology) is a G-orbit on the projective space $\mathbf{P}(\mathscr{H})$ (which is the quantum mechanical phase space).

Of particular interest are orbits of coherent states which are "closest to the classical ones". Geometrically, these are complex submanifolds of $\mathbf{P}(\mathscr{H})$. Now $\mathbf{P}(\mathscr{H})$ carries a natural Kähler structure (induced by the scalar product on $\mathscr{H}$) and this is inherited by any complex submanifold. In particular, $\mathbf{P}(\mathscr{H})$ is a symplectic manifold, which allows one to treat, formally, the quantum mechanics as a "classical mechanics" on $\mathbf{P}(\mathscr{H})$ (*cf*. [Tu]). In many cases a complex (or, more generally, symplectic) coherent state orbit may be interpreted as the corresponding classical phase space. Thus, in such a case, the classical phase space is embedded into the quantum one. If we start from a classical system, then to construct such an emedding amounts to the same thing as to quantize the system (*see* [Od]).

Thus it is natural to consider the following problem: given a Lie group G, classify all its (irreducible unitary) representations which admit Kähler coherent state orbits; in particular, classify all groups possesing such representations.

In case of compact G this problem was solved by Kostant and Sternberg [KoS] who showed that for any representation of G the orbit through the projectivized highest weight vector is the only Kähler coherent state orbit.

In the present paper we show that if G is semisimple (and noncomcompact), this result still holds for the highest weight representations (and only for them) [*see* Theorem (3.8)]. (Contrary to the compact case, these representations (whose definition is now somewhat more subtle) form only a small part of the unitary dual of G; moreover, they exist only for those G which are of Hermitian type). Since the highest weight representations are classified ([EHW], [Ja]), this gives a complete solution to our problem in this special case.

We also briefly discuss "holomorphic realizations" of the representations which admit a Kähler coherent state orbit and their connection with geometric quantization and vector coherent state theory [RRG]. Finally, we indicate that the above mentioned interpretation of Kähler coherent state orbits as classical phase spaces is, in general, possible only for square-integrable representations. The non-square-integrable case is illustrated by an example concerning the ladder representations of SU (2,2).

Concluding this introduction let us remark that the general classification problem stated above could be attacked along the lines of this paper. Our main technical tool here is a structure theory (due to Borel [Bo]) of homogeneous Kähler manifolds of semisimple Lie groups. Thanks to an affirmative solution (due to Dorfmeister and Nakajima [DN]) of the famous Vinberg-Gindikin conjecture [VG] such a theory is now available for arbitrary homogeneous Kähler manifolds. A solution of the general classification problem based on this theory would involve a classification of homogeneous bounded domains which is, as yet, not complete. It is possible, however, to obtain a complete solution for the, rather large, class of all unimodular Lie groups. This will be carried out in a future publication.

2. SYMPLECTIC AND KAEHLER ORBITS IN THE GENERAL CASE

(2.1) Let $\mathscr{H}$ be a complex Hilbert space and $\mathbf{P}(\mathscr{H})$ the corresponding projective space. For each $v \in \mathscr{H} \setminus \{0\}$, we shall denote by $[v]$ the complex line $\mathbb{C}\, v$. $\mathbf{P}(\mathscr{H})$ has a unique structure of a complex Hilbert manifold such that the map

$$p: \quad \mathscr{H} \setminus \{0\} \to \mathbf{P}(\mathscr{H}), \qquad v \mapsto [v],$$

is a holomorphic submersion. Moreover, it carries a natural Kähler structure (Fubini-Study metric). This can be defined as follows. Consider the unit sphere $S(\mathscr{H})$ in $\mathscr{H}$. This may be viewed as the total space of a principal circle bundle over $\mathbf{P}(\mathscr{H})$ with projection $v \mapsto [v]$. There is a natural connection on $S(\mathscr{H})$ whose horizontal space at v is the orthogonal complement of v in $\mathscr{H}$ [identified in a natural manner with a subspace of $T_v S(\mathscr{H})$]. The corresponding connection 1-form α is given by

$$\langle \alpha_v, w \rangle = -i(v \mid w),$$

where $(v \mid w)$ is the scalar product in $\mathscr{H}$ of v and w (which is assumed to be linear in the second variable). It is easy to see that the curvature form ω of α is a type (1,1) symplectic form on $\mathbf{P}(\mathscr{H})$ and that it gives rise to a Kähler structure. In the terminology of geometric quantization, $(S(\mathscr{H}), \alpha)$ is a prequantization (in the sense of Souriau) of the symplectic manifold $(\mathbf{P}(\mathscr{H}), \omega)$. Note that the Banach Lie group $\mathbf{U}(\mathscr{H})$ of unitary operators in $\mathscr{H}$ acts via prequantization automorphisms (*i.e.* automorphisms of a principal circle bundle with connection) on $(S(\mathscr{H}), \alpha)$ and via Kähler manifold automorphisms on $(\mathbf{P}(\mathscr{H}), \omega)$.

(2.2) Associated to the principal bundle $S(\mathscr{H}) \to \mathbf{P}(\mathscr{H})$, there is a holomorphic line bundle $\mathbb{E} \to \mathbf{P}(\mathscr{H})$ (Kostant's prequantization of $(\mathbf{P}(\mathscr{H}), \omega)$) whose fiber $\mathbb{E}_{[v]}$ at $[v]$ is the dual space $[v]^*$ of the line $[v]$. Each *linear* functional φ on $\mathscr{H}$ gives rise to a holomorphic section $[v] \mapsto \varphi|_{[v]}$ of $\mathbb{E}$. Thus we obtain a natural mapping

$$\mathscr{H}^* \to \Gamma(\mathbf{P}(\mathscr{H}), \mathbb{E})$$

from the dual of $\mathscr{H}$ to holomorphic sections of $\mathbb{E}$, which is easily seen to be a linear isomorphism (*cf*. [Tu]).

(2.3) Let G be a connected real Lie group with Lie algebra $\mathfrak{g}_0$ and let $(\pi, \mathscr{H})$ be an irreducible unitary representation of G on a Hilbert space $\mathscr{H}$. Thus $\pi: G \to \mathbf{U}(\mathscr{H})$ is a group homomorphism and the corresponding left action $G \times \mathscr{H} \to \mathscr{H}$ is continuous. This action induces a continuous action of G on $\mathbf{P}(\mathscr{H})$. Let $\mathscr{H}^\infty$ denote the space of smooth vectors of $(\pi, \mathscr{H})$. It is well known that $\mathscr{H}^\infty$ is a G-invariant dense subspace of $\mathscr{H}$ and carries a natural structure of a $\mathfrak{g}$-module, $\mathfrak{g}$ being the complexification of $\mathfrak{g}_0$. Denote by $\mathbf{P}(\mathscr{H}^\infty)$ the space of smooth lines. Then we have the following.

(2.4) PROPOSITION. – *The orbit* $G.[v]$ *is a smooth (immersed) submanifold of* $\mathbf{P}(\mathscr{H})$ *iff* $[v] \in \mathbf{P}(\mathscr{H}^\infty)$.

Proof. – Let $[v] \in \mathbf{P}(\mathscr{H}^\infty)$. Then $v \in \mathscr{H}^\infty$, *i.e.* the orbit map

$$\rho_v : G \to \mathscr{H}, \qquad g \mapsto g.v,$$

is smooth. Thus the orbit map at $[v]$, $\rho_{[v]} = p \circ \rho_v$, is also smooth. Moreover, it is a subimmersion (since G is finite-dimensional), which implies that $G.[v]$ is a smooth immersed submanifold of $\mathbf{P}(\mathscr{H})$.

Conversely, suppose that $G.[v]$ is an (immersed) submanifold. For each $g \in G$, the corresponding transformation of $\mathbf{P}(\mathscr{H})$ is a diffeomorphism and maps $G.[v]$ onto itself, hence it induces a diffeomorphism of $G.[v]$ onto itself. This implies that the action of G on $G.[v]$ is smooth (*see* [MZ], pp. 208 and 212). Thus the orbit map $\rho_{[v]}$ is smooth. Since $\rho_{[v]} = p \circ \rho_v$, ρ_v is also smooth, which means that $[v] \in \mathbf{P}(\mathscr{H}^\infty)$. ■

(2.5) We say $G.[v]$ is a *symplectic orbit* if it is smooth, *i.e.* $[v] \in \mathbf{P}(\mathscr{H}^\infty)$, and $(G.[v], \iota^* \omega)$ is a symplectic manifold ($\iota : G.[v] \to \mathbf{P}(\mathscr{H})$ being the canonical immersion). If this is the case, the pull-back $(\iota^* \mathbf{S}(\mathscr{H}), \iota^* \alpha)$ is a prequantization of $(G.[v], \iota^* \omega)$. Since π is unitary and $\mathbf{U}(\mathscr{H})$ is an automorphism group of $(\mathbf{S}(\mathscr{H}), \alpha)$, G acts on $(\iota^* \mathbf{S}(\mathscr{H}), \iota^* \alpha)$ by prequantization automorphisms. Thus, by a result of Kostant ([Ko], Th. 4.5.1), $(G.[v], \iota^* \omega)$ carries a natural structure of Hamiltonian G-space with momentum mapping $J : G[v] \to \mathfrak{g}_0^*$ given by

$$\langle {}^* J([v]), X \rangle = -\langle \alpha_v, X.v \rangle = i(v \mid X.v), \qquad \forall X \in \mathfrak{g}_0,$$

where dot denotes the action of $\mathfrak{g}_0$ on $\mathscr{H}^\infty$ and where we assume, as we may, that $v \in \mathbf{S}(\mathscr{H})$. Moreover, being prequantizable, $(G.[v], \iota^* \omega)$ is *integral* in the sense that there exists a unitary character χ of the stabilizer $G_{[v]}$ of $[v]$ such that $L(\chi) = i J([v])$, $L(\chi)$ being Lie algebra character corresponding to χ.

We shall now consider complex G-orbits on $\mathbf{P}(\mathscr{H})$.

(2.6) Because of homogeneity, $G.[v]$ is a complex submanifold of $\mathbf{P}(\mathscr{H})$ iff $T_{[v]}G.[v]$ is a complex subspace of $T_{[v]}\mathbf{P}(\mathscr{H})$. It is convenient to rephrase this as follows. Let

(2.6.1) $$F = T_{[v]}^{(0,1)}\mathbf{P}(\mathscr{H}) \cap T_{[v]}^{\mathbb{C}}G.[v],$$

where $T_{[v]}^{(0,1)}\mathbf{P}(\mathscr{H})$ is the subspace of antiholomorphic tangent vectors and $T_{[v]}^{\mathbb{C}}G.[v]$, the complexification of $T_{[v]}G.[v]$, is naturally identified with a subspace of $T_{[v]}^{\mathbb{C}}\mathbf{P}(\mathscr{H})$. Then $G.[v]$ *is a complex submanifold of* $\mathbf{P}(\mathscr{H})$ *if*

(2.6.2) $$F + \bar{F} = T_{[v]}^{\mathbb{C}}G.[v]$$

(here bar denotes the conjugation in $T_{[v]}^{\mathbb{C}}G.[v]$).

(2.7) We shall now characterize complex orbits in terms of polarizations. Recall that *a positive totally complex* polarization for $f \in \mathfrak{g}_0^*$ is a complex subalgebra $\mathfrak{q} \subset \mathfrak{g}$ with the following properties

(i) $\langle f, [\mathfrak{q}, \mathfrak{q}] \rangle = 0$;

(ii) $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{g}$;

(iii) $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{g}_f$;

(iv) $i\langle f, [X, \bar{X}] \rangle \geqq 0, \quad \forall X \in \mathfrak{q}$,

where bar denotes the complex conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, and $\mathfrak{g}_f$ is the stabilizer of (the complexification of) f in $\mathfrak{g}$. If M is a homogeneous Hamiltonian G-space with momentum mapping J, then, for a fixed $m \in M$, there is a 1-1 correspondence between G_f-invariant positive totally complex polarizations for $f = J(m)$ and G-invariant Kähler structures on M such that the symplectic form coincides with the imaginary part of the Kähler metric. It is obtained by letting $T_m^{\mathbb{C}}\rho_m(\mathfrak{q})$, where $T_m^{\mathbb{C}}\rho_m$ is the complexification of the tangent of the orbit map at m, be the subspace of antiholomorphic vectors at m.

(2.8) PROPOSITION. — *Let* $G.[v]$ *be a smooth orbit and let* $\mathfrak{g}_{[v]} = \{X \in \mathfrak{g} \mid X.v \in \mathbb{C}v\}$ *be the stabilizer of* $[v]$ *in* $\mathfrak{g}$. *Then the following conditions are equivalent.*

(i) $G.[v]$ *is a complex orbit;*

(ii) $G.[v]$ *is a symplectic orbit and* $\mathfrak{g}_{[v]}$ *is a positive totally complex polarization for* $f = J(m)$;

(iii) $\mathfrak{g}_{[v]} + \bar{\mathfrak{g}}_{[v]} = \mathfrak{g}$.

Proof. — Suppose $G.[v]$ is a complex orbit and consider the tangent

(2.8.1) $$T_{[v]}\rho_{[v]} : \mathfrak{g}_0 \to T_{[v]}G.[v]$$

of the orbit map at $[v]$. Viewing $T_{[v]}G.[v]$ as complex vector space, we may extend this map to a complex linear map

$$\rho'_{[v]} : \mathfrak{g} \to T_{[v]}G.[v],$$

which can be written as the composition

$$\mathfrak{g} \to T^{\mathbb{C}}_{[v]} G.[v] \to T^{(1,0)}_{[v]} G.[v] \to T_{[v]} G.[v]$$

of the complexification of (2.8.1), the projection onto the subspace of holomorphic vectors and the canonical isomorphism of this subspace onto $T_{[v]} G.[v]$. It follows that

$$\mathfrak{g}_{[v]} = \operatorname{Ker} \rho'_{[v]} = (T^{\mathbb{C}}_{[v]} \rho_{[v]})^{-1} (T^{(0,1)}_{[v]} G.[v])$$

and hence is a positive totally complex polarization for f. We have thus shown that (i) ⇒ (ii). The implication (ii) ⇒ (iii) is obvious. Finally, noting that $T^{\mathbb{C}}_{[v]} \rho_{[v]} (\mathfrak{g}_{[v]}) = F$, where F is defined by (2.6.1), we see that (iii) implies (2.6.2) which is equivalent to (i), ■

Remark. – Perelomov [Pe2] calls the algebras $\mathfrak{g}_{[v]}$ with property (iii) "maximal" and remarks that they are totally complex polarizations. This is made precise by the equivalence (ii) ⇔ (iii).

(2.9) Let $G.[v] \subset \mathbf{P}(\mathscr{H})$ be a complex orbit and $\iota : G.[v] \to \mathbf{P}(\mathscr{H})$ the natural (holomorphic) immersion. The pull-back $\mathbb{L} = \iota^* \mathbb{E}$ of $\mathbb{E}$ [*see* (2.2)] is a holomorphic line bundle (Kostant's prequantization of $(G.[v], \iota^* \omega)$). Write $\Gamma(G.[v], \mathbb{L})$ for holomorphic sections of $\mathbb{L}$ and consider the composition of the natural isomorphism $\mathscr{H}^* \to \Gamma(\mathbf{P}(\mathscr{H}), \mathbb{E})$ and the restriction map $\Gamma(\mathbf{P}(\mathscr{H}), \mathbb{E}) \to \Gamma(G.[v], \mathbb{L})$. Since the linear span of $G.v$ is dense in $\mathscr{H}$, the resulting G-equivariant mapping

$$\mathscr{H}^* \to \Gamma(G.[v], \mathbb{L}) \tag{2.9.1}$$

is injective. Thus we obtain a "holomorphic realization" of the representation $(\pi^\vee, \mathscr{H}^*)$ contragredient to $(\pi, \mathscr{H})$. $(\pi, \mathscr{H})$ itself has a holomorphic realization corresponding to the complex orbit on $\mathbf{P}(\mathscr{H}^*)$ which is the image of $G.[v]$ under the antiholomorphic isomorphism $\mathbf{P}(\mathscr{H}) \to \mathbf{P}(\mathscr{H}^*)$ induced by the Fréchet-Riesz isomorphism.

3. THE SEMISIMPLE CASE

(3.1) From now on we assume that G is a noncompact connected semisimple Lie group. As before, $\mathfrak{g}_0$ and $\mathfrak{g}$ denote, respectively, the Lie algebra of G and its complexification. Vector subspaces of $\mathfrak{g}_0$ are denoted by lower case German letters with subscript 0, dropping of which means complexification. We fix once and for all a Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

and we let K denote the Lie subgroup of G corresponding to $\mathfrak{k}_0$.

If $(\pi, \mathscr{H})$ is an irreducible unitary representation of G, we shall write $K(\pi)$ [resp. $PK(\pi)$] for its kernel (resp. projective kernel). Note that

$K(\pi) \subset PK(\pi)$ and the quotient group is discrete. From now on we assume that $K(\pi)$ is *discrete* (hence so is $PK(\pi)$) and π is *nontrivial* (*i.e.* $\dim \mathscr{H} \neq 1$).

(3.2) LEMMA. — *Given a smooth line* $[v] \in \mathbf{P}(\mathscr{H}^\infty)$, *let* $G_{[v]}$ *be its stabilizer in* G *and* $\mathfrak{g}_{[v]}$ *its stabilizer in* $\mathfrak{g}$. *Then*

(i) $G_{[v]}/PK(\pi)$ *is compact;*

(ii) $\mathfrak{g}_{[v]} \cap \bar{\mathfrak{g}}_{[v]}$ *is reductive in* $\mathfrak{g}$.

Proof. — π being irreducible, the linear span of $G.v$ is dense in $\mathscr{H}$. Thus $g \in G$ induces the identity transformation of $G.[v]$ iff $g \in PK(\pi)$, *i.e.* $G/PK(\pi)$ acts effectively on $G.[v]$. Since $G.[v]$ carries a G-invariant Riemannian metric [induced from $\mathbf{P}(\mathscr{H})$], we conclude that $G_{[v]}/PK(\pi)$ is compact. Hence (i) is proved. Now (ii) follows from (i) since, in virtue of the discreteness of $PK(\pi)$, $\mathfrak{g}_{[v]} \cap \bar{\mathfrak{g}}_{[v]}$ is the complexification of the Lie algebra of $G_{[v]}/PK(\pi)$. ■

Recall that an element $f \in \mathfrak{g}_0$ is called *elliptic* if the element $X \in \mathfrak{g}_0$ that corresponds to it under the isomorphism induced by the Killing form is so, *i.e.* $\mathrm{ad}(X)$ is a semisimple endomorphism with purely imaginary eigenvalues.

(3.3) PROPOSITION. — *Suppose* $G.[v]$ *is a symplectic orbit; let* J *be the corresponding momentum map* [see (2.5)] *and* $f = J([v])$. *Then*

(i) f *is elliptic;*

(ii) $\mathrm{rank}\, \mathfrak{g}_0 = \mathrm{rank}\, \mathfrak{k}_0$;

(iii) $G.[v]$ *is simply connected and* J *maps it diffeomorphically onto* $G.f$.

Proof. — Since $G.[v]$ is symplectic, $(\mathfrak{g}_0)_f = (\mathfrak{g}_0)_{[v]}$ and (3.2) (i) implies that $(\mathfrak{g}_0)_{[v]} \subset \mathfrak{k}_0$ (for an appropriate choice of $[v]$). Thus f may be identified with an element of $\mathfrak{k}_0$ and so is elliptic. In particular, it is semisimple hence $(\mathfrak{g}_0)_f$ contains a Cartan subalgebra of $\mathfrak{g}_0$. This proves (i) and (ii). Now (iii) follows from (i) and the well-known fact that elliptic orbits are simply connected. ■

Remark. — Condition (ii) above coincides with Harish-Chandra's criterion for the existence of the discrete series of G. Now it follows easily from the results of [MV] that for any square-integrable representation $(\pi, \mathscr{H})$ of G there exists a symplectic orbit on $\mathbf{P}(\mathscr{H})$. However, as we shall see below, symplectic orbits may exist also for non-square-integrable representations.

We now turn to complex orbits.

(3.4) PROPOSITION. — $G.[v]$ *is a complex orbit on* $\mathbf{P}(\mathscr{H})$ *iff* $\mathfrak{g}_{[v]}$ *is a parabolic subalgebra of* $\mathfrak{g}$.

Proof. — If $G.[v]$ is complex, $\mathfrak{g}_{[v]}$ is a (positive totally complex) polarization for $f = J([v])$ (2.8) hence it is a parabolic subalgebra by a result of [OW].

Conversely, suppose $\mathfrak{g}_{[v]}$ is parabolic and let $\mathfrak{n}$ be its nilradical. Then

$$2\dim \mathfrak{g}_{[v]} = \dim \mathfrak{g} + \dim \mathfrak{g}_{[v]}/\mathfrak{n}.$$

On the other hand, $\mathfrak{g}_{[v]} \cap \bar{\mathfrak{g}}_{[v]} \cap \mathfrak{n} = \{0\}$, because $\mathfrak{g}_{[v]} \cap \bar{\mathfrak{g}}_{[v]}$ is reductive in $\mathfrak{g}$ (3.2) (ii), hence

$$\dim \mathfrak{g}_{[v]} \cap \bar{\mathfrak{g}}_{[v]} \leqq \dim \mathfrak{g}_{[v]}/\mathfrak{n}.$$

Together with the previous equality this shows that

$$\mathfrak{g}_{[v]} + \bar{\mathfrak{g}}_{[v]} = \mathfrak{g},$$

which according to (2.8) implies that G.$[v]$ is complex. ■

(3.5) Recall that G is said to be of *Hermitian type* if G/K is a Hermitian symmetric space (*i.e.* carries a G-invariant complex structure). In virtue of our assumptions about G, this amounts to saying that, for each noncompact simple ideal of $\mathfrak{g}_0$, the corresponding maximal compact subalgebra has one-dimensional center.

Identify $\mathfrak{p}$ with the complexified tangent space at the origin of G/K and let $\mathfrak{m} \subset \mathfrak{p}$ be the subspace of antiholomorphic vectors for a given G-invariant complex structure on G/K. Then $\mathfrak{m}$ is an abelian subalgebra and $\mathfrak{k} + \mathfrak{m}$ is a parabolic subalgebra with nilradical $\mathfrak{m}$. Thus $\mathfrak{k} + \mathfrak{m}$ contains a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and any such $\mathfrak{b}$ must be of the form

$$\mathfrak{b} = \mathfrak{b} \cap \mathfrak{k} + \mathfrak{m},$$

where $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$. A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ which is contained in some $\mathfrak{k} + \mathfrak{m}$ will be called *admissible*. Since all Borel subalgebras of $\mathfrak{k}$ are Ad(K)-conjugate, the Ad(K)-conjugacy classes of admissible Borel subalgebras are in 1-1 correspondence with the G-invariant complex structures on G/K.

(3.6) Recall that, by a classical theorem of Borel [Bo], an effective homogeneous space M of G admits an invariant Kähler structure iff G is of Hermitian type and the stabilizer G_m of some point $m \in M$ is the centralizer of a torus in K. Moreover, the fibration $G/G_m \to G/K$ is holomorphic (with respect to an appropriate G-invariant complex structure on G/K uniquely determined by that on M) and the fiber K/G_m is a simply connected K-homogeneous Kähler manifold (*i.e.* a flag manifold).

Now if G.$[v]$ is a complex orbit on $\mathbf{P}(\mathscr{H})$, it is an effective homogeneous Kähler manifold of Ad(G) [*cf.* (3.3)] so the first part of Borel's theorem implies that G is of Hermitian type and $[v]$ can be chosen such that $G_{[v]} \subset K$, and the second part, together with (3.4), implies that $\mathfrak{g}_{[v]}$ is of the form

$$\mathfrak{g}_{[v]} = \mathfrak{g}_{[v]} \cap \mathfrak{k} + \mathfrak{m},$$

where $\mathfrak{g}_{[v]} \cap \mathfrak{k}$ is a positive totally complex polarization for $J([v])|_{\mathfrak{k}_0}$. It follows that $\mathfrak{g}_{[v]}$ *contains an admissible Borel subalgebra of* $\mathfrak{g}$.

(3.7) A vector $v \in \mathscr{H}$ is called K-*finite* if K.v spans a finite-dimensional linear subspace of $\mathscr{H}$. Any K-finite vector is smooth (in fact analytic) and the subspace $\mathscr{H}_K$ of all such vectors is an (algebraically) irreducible $\mathfrak{g}$-module, the Harish-Chandra module of $(\pi, \mathscr{H})$. (One usually requires in this definition that G has finite center, but for our purposes this is inessential. As a matter of fact, if G has a symplectic orbit on $\mathbf{P}(\mathscr{H})$, then G/K(π) must have finite center). We shall say that $(\pi, \mathscr{H})$ is a *highest weight representation* if G is of Hermitian type and $\mathscr{H}_K$ is a highest weight module relative to an admissible Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$., *i.e.* there exists a nonzero vector $v \in \mathscr{H}_K$ such that $\mathfrak{b}.v = \mathbb{C}v$. v is then called a highest weight vector and $[v]$ the highest weight line. For a fixed $\mathfrak{b}$, there is exactly one highest weight line. (*See* [Ha], [Va]; note that if we fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$, there is a 1-1 correspondence between Borel subalgebras containing $\mathfrak{h}$ and positive systems of roots of $(\mathfrak{g}, \mathfrak{h})$, and admissible Borel subalgebras correspond to admissible systems in [Va].)

Remark. — By a result of Harish-Chandra [Ha], the assumptions about $\mathfrak{g}_0$ and $\mathfrak{b}$ we have made above are necessary for the existence of highest weight representations. We shall give a simple geometric proof of this fact below [*see* (3.9)].

We can now state our main result.

(3.8) THEOREM. — *Suppose* G *is a (noncompact) connected semisimple Lie group and* $(\pi, \mathscr{H})$ *a nontrivial irreducible unitary representation of* G *with discrete kernel. Then* G *has a complex orbit on* $\mathbf{P}(\mathscr{H})$ *if and only if* $(\pi, \mathscr{H})$ *is a highest weight representation. If this is the case, there is exactly one such orbit, namely the orbit through the highest weight line.*

Proof. — Suppose $(\pi, \mathscr{H})$ is a highest weight representation (relative to an admissible Borel subalgebra $\mathfrak{b}$ and take the orbit G.$[v]$ through the highest weight line. Then $\mathfrak{b}$ is contained in $\mathfrak{g}_{[v]}$ so the latter is parabolic and G.$[v]$ is complex by (3.4). [We could have also proceeded more directly noting that $\mathfrak{b} + \bar{\mathfrak{b}} = \mathfrak{g}$ and then applying (2.8)]

Conversely, suppose there exists a complex orbit G.$[v] \subset \mathbf{P}(\mathscr{H})$. Then, as we have already noticed in (3.6), G is of Hermitian type and v can be chosen such that $\mathfrak{g}_{[v]}$ contains an admissible Borel subalgebra $\mathfrak{b}$, *i.e.* $\mathfrak{b}.v = \mathbb{C}v$ (recall that we have excluded the trivial representation). We shall show that v is K-finite. Let $\mathscr{F}$ be the closure of the linear span of K.v and $(\pi|_K, \mathscr{F})$ the natural unitary representation of K on $\mathscr{F}$. With our choice of v, K.$[v]$ is a fiber of the holomorphic fibration G.$[v] \to$ G/K of Borel's theorem (3.6) so it is a compact complex K-orbit on $\mathbf{P}(\mathscr{F})$. Let $\mathbb{L} \to$ K.$[v]$ be the holomorphic line bundle introduced in (2.9). K.$[v]$ being compact, the space Γ(K.$[v]$, $\mathbb{L}$) of holomorphic sections of this bundle is finite-dimensional. Considering the natural mapping $\mathscr{F}^* \to \Gamma$(K.$[v]$, $\mathbb{L}$) of (2.9), which is now injective by the very definition of $\mathscr{F}$, we conclude that $\mathscr{F}^*$ is finite-dimensional and hence so is $\mathscr{F}$. Thus v is K-finite and so $(\pi, \mathscr{H})$ is a highest weight representation.

Finally, to prove the uniqueness assertion, it suffices to show that if v and v' are highest weight vectors of $\mathscr{H}_K$ relative to admissible Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ respectively, then $\mathfrak{b}$ and $\mathfrak{b}'$ are Ad(K)-conjugate. Suppose, to the contrary, this is not the case. We may assume without loss of generality that $\mathfrak{g}_0$ is simple, so that G/K has two G-invariant complex structures corresponding to $\mathfrak{m}$ and $\bar{\mathfrak{m}}$, and take $\mathfrak{b}=\mathfrak{r}+\mathfrak{m}$, $\mathfrak{b}'=\mathfrak{r}+\bar{\mathfrak{m}}$, where $\mathfrak{r}$ is a Borel subalgebra of $\mathfrak{k}$. Since $\mathscr{H}_K$ is a highest weight module relative to $\mathfrak{b}$, any nonzero element Y of $\bar{\mathfrak{m}}$ acts freely on $\mathscr{H}_K$ (*i.e.*, for any $w \in \mathscr{H}_K$, $Y.w=0 \Rightarrow w=0$). On the other hand, v' is a nonzero vector annihilated by $\bar{\mathfrak{m}}$, which contradicts the preceding conclusion. The proof is complete. ■

Together with (3.4) this theorem implies the following.

(3.9) COROLLARY. – G *and* $(\pi, \mathscr{H})$ *being as above, suppose that there exist a smooth vector* v *and a Borel subalgebra* $\mathfrak{b} \subset \mathfrak{g}$ *such that* $\mathfrak{b}.v=\mathbb{C}v$. *Then* G *is of Hermitian type*, $\mathfrak{b}$ *is admissible and* v *is* K*-finite, so that* $(\pi, \mathscr{H})$ *is a highest weight representation.* ■

4. HOLOMORPHIC REALIZATIONS OF REPRESENTATIONS AND PHYSICAL INTERPRETATION OF KAEHLER COHERENT STATE ORBITS

(4.1) It is clear that Theorem (3.8) holds, with obvious modifications, also for compact G (we have excluded this case to simplify the statement). This is the result of Kostant and Sternberg [KoS] mentioned in the introduction. Note that in this case the natural map (2.9.1) is an isomorphism, which is just a restatement of the celebrated Borel-Weil theorem (*cf.* [GS].

(4.2) Returning to the situation of section 3, consider a complex G-orbit $G.[v]$ on $\mathbf{P}(\mathscr{H})$. Let $G.[v] \to G/K$ be the holomorphic fibration of (3.6) and $K.[v]$ the fiber over the origin of G/K. According to (2.9), there is a natural holomorphic line bundle $\mathbb{L}$ over $G.[v]$. Its restriction $\mathbb{L}|_{K.[v]}$ to $K.[v]$ is still holomorphic. The space $V=\Gamma(K.[v], \mathbb{L}|_{K.[v]})$ of holomorphic sections is finite-dimensional and naturally isomorphic to the dual of the space spanned by $K.v$ [*see* the proof of (3.8) and (4.1); note that K need not be compact but we may always pass to its compact quotient $K/K(\pi)$]. Let $\mathbb{V}$ be a holomorphic G-bundle over G/K having V as the fiber at the origin. Each holomorphic section ψ of $\mathbb{L}$ determines a holomorphic section of $\mathbb{V}$ whose value at the origin is $\psi|_{K.[v]}$. The resulting map

$\Gamma(G.[v], \mathbb{L}) \to \Gamma(G/K, \mathbb{V})$ is easily seen to be a G-equivariant isomorphism. We thus obtain an injection

(4.2.1) $$\mathscr{H}^* \to \Gamma(G/K, \mathbb{V}),$$

which realizes $(\pi^\vee, \mathscr{H}^*)$ as a vector coherent state representation in the sense of [RRG]. From our point of view there is no principal difference between "vector coherent states" ($\dim V > 1$ or, equivalently, $\dim K.[v] > 0$) of [RRG] and the "scalar" ones ($\dim V = 1$) of [Pe2]. The realization corresponding to $\mathbb{V}$ is just technically more convenient than that corresponding to $\mathbb{L}$, since $\mathbb{V}$ is holomorphically trivial while $\mathbb{L}$ is never so (in the "vector" case of course).

(4.3) Suppose for the moment that G is unimodular (not necessarily semisimple) and $(\pi, \mathscr{H})$ is a square-integrable (mod $K(\pi)$) representation such that G has a complex orbit $G.[v]$ on $\mathbf{P}(\mathscr{H})$. It is not hard to show that the image of (2.9.1) consists of those sections that are square-integrable with respect to the Liouville measure. Thus $(\pi^\vee, \mathscr{H}^*)$ can be recovered by the quantization of $J(G.[v])$ with respect to the polarization $\mathfrak{g}_{[v]}$. If G is semisimple and of Hermitian type, "most" of the highest weight representations are square-integrable; they constitute the so-called holomorphic discrete series of G [Ha]. For non-square-integrable ones (which exist for any such G [EHW], [Ja]) there seems to be no general description of the image of (2.9.1) which would fit the "quantization picture". Moreover, $J(G.[v])$ is then not the orbit one should associate with $(\pi^\vee, \mathscr{H}^*)$ according to the Orbit Method.

(4.4) Remaining in the above setting, let $G.f = G.J([v])$, where J is the momentum mapping (2.5). If $(\pi, \mathscr{H})$ is square-integrable, the orbit associated with it via the orbit method is the "dual" $G.(-f)$ of $G.f$. It is clear that the map $G.(-f) \to G.f$, $g.(-f) \mapsto g.f$, is antiholomorphic with respect to the complex structures corresponding to the positive totally complex polarizations $\bar{\mathfrak{g}}_{[v]}$ and $\mathfrak{g}_{[v]}$ for $-f$ and f respectively. Composing it with the inverse of J [*cf.* (3.3)] we get a G-equivariant antiholomorphic immersion

(4.4.1) $$G.(-f) \to \mathbf{P}(\mathscr{H}).$$

If $(\pi, \mathscr{H})$ describes a quantum mechanical system with symmetry group G, $G(-f)$ may be interpreted as the corresponding classical phase space. (4.4.1) is then the "quantization of states" studied in [Od]. For non-square-integrable $(\pi, \mathscr{H})$, $G.(-f)$ may not, in general, be given such an interpretation, as is shown by the following example.

(4.5) Example. — Let $G = SU(2, 2)$, the group of linear automorphisms of $\mathbb{C}^4$ of determinant 1 preserving a Hermitian form Φ of signature

$(++--)$. We shall need the following homogeneous complex manifolds of G.

$\mathbb{P}^+ = \{\text{positive lines in } \mathbb{C}^4\}$;

$\mathbb{M}^+ = \{\text{positive planes in } \mathbb{C}^4\}$;

$\mathbb{F}^+ = \{\text{positive flags in } \mathbb{C}^4\}$

(here flag means a pair consisting of a plane and a line contained in it, and positivity of a subspace $W \subset \mathbb{C}^4$ means that $\Phi|_{W \times W}$ is positive definite). Note that $\mathbb{M}^+$ is G-isomorphic to G/K [where $K = S(U(2) \times U(2))$] is a maximal compact subgroup of G), which gives rise to one of the two G-invariant complex structures on G/K.

G has a remarkable family $(\pi_n, \mathscr{H}_n)$, $n \in \mathbb{Z}$, of non-square-integrable unitary highest weight representations, the so-called ladder representations ([JaV], [SW]). We shall consider those with $n > 0$. The contragredient representation $(\pi_n^\vee, \mathscr{H}_n^*)$ is also a highest weight representation, so according to (3.8) G has a unique complex orbit $G.[v_n]$ on $\mathbf{P}(\mathscr{H}_n^*)$. Its image $J_n(G.[v_n])$ under the momentum mapping J_n (2.5) is a 10-dimensional elliptic orbit on $\mathfrak{g}_0^*$. As a complex manifold, $G.[v_n]$ is naturally isomorphic to $\mathbb{F}^+$ (the holomorphic fibration $G.[v_n] \to G/K$ (3.6) corresponds to the natural map $\mathbb{F}^+ \to \mathbb{M}^+$). It follows from (4.2) that $(\pi_n, \mathscr{H}_n)$ has a holomorphic realization

$$\mathscr{H}_n \to \Gamma(\mathbb{F}^+, \mathbb{L}_n) \cong \Gamma(\mathbb{M}^+, \mathbb{V}_n) \tag{4.5.1}$$

($\mathbb{V}_n$ is now a spinor bundle).

As is well known ([JaV], [Ma]), $(\pi_n, \mathscr{H}_n)$ describes a massless particle with helicity $s = n/2$. The classical phase space of such a particle is a 6-dimensional elliptic orbit $\mathcal{O}_n \subset \mathfrak{g}_0^*$ [*as a homogeneous space*, $\mathcal{O}_n = SU(2, 2)/S(U(1) \times U(1, 2))$]. Clearly $J_n(G.[v_n]) \neq \mathcal{O}_n$. (Note that $\mathcal{O}_n$ cannot be embedded into $\mathbf{P}(\mathscr{H}_n^*)$, even as a sympletic manifold, because its points have noncompact stabilizers [*cf*. (3.2) and (3.3)].)

We see that $G.[v_n]$ cannot be interpreted as the classical phase space of the quantum system described by $(\pi_n, \mathscr{H}_n)$. Nevertheless there is a certain connection between $G.[v_n]$ and the "true" phase space $\mathcal{O}_n$, which is provided by the twistor theory. First note that $\mathcal{O}_n$ has a (nonpositive) totally complex polarization (it satisfies the conditions (i)-(iii), but not (iv), of (2.7)) which gives it the structure of a complex manifold isomorphic to $\mathbb{P}^+$. Next consider the "quantum bundle" $\mathbb{H}^{-n-2}$, which is the restriction to $\mathbb{P}^+$ of the $(-n-2)$nd tensor power of the hyperplane bundle $\mathbb{H} \to \mathbf{P}(\mathbb{C}^4)$ (whose dual is the bundle $\mathbb{E}$ of (2.2) for $\mathscr{H} = \mathbb{C}^4$). Let $H^1(\mathbb{P}^+, \mathbb{H}^{-n-2})$ be the 1st Dolbeault cohomology group of $\mathbb{H}^{-n-2}$. Then the celebrated Penrose transform

$$\mathscr{P}_n:\ H^1(\mathbb{P}^+, \mathscr{H}^{-n-2}) \to \Gamma(\mathbb{M}^+, \mathbb{V}_n)$$

is an isomorphism onto the subspace $\mathscr{Z}_n(\mathbb{M}^+)$ of holomorphic solutions of helicity $s = n/2$ of the zero-rest-mass field equations on $\mathbb{M}^+$ [EPW]. The natural representation of G on $H^1(\mathbb{P}^+, \mathbb{H}^{-n-2})$ is not unitary. It can be unitarized using a representation theoretic version of the Gupta-Bleuler procedure in QED [RSW], and the resulting unitary representation is equivalent to $(\pi_n, \mathcal{H}_n)$. This means that we may view $\mathcal{H}_n$ as a subspace of $H^1(\mathbb{P}^+, \mathbb{H}^{-n-2})$. Due to the irreducibility of $(\pi_n, \mathcal{H}_n)$, the restriction $\mathcal{P}_n|_{\mathcal{H}_n}$ coincides (up to a constant factor) with (4.5.1). $\mathcal{P}_n$ itself may also be interpreted within the framework of coherent states. To this end consider the subspace $(\mathcal{H}_n^*)^\omega$ of analytic vectors in $\mathcal{H}_n^*$ and let $\mathcal{H}_n^{-\omega}$ be its dual (the space of hyperfunction vectors in the terminology of [Sch]). Since $G.[v_n]$ passes through a K-finite line, it is contained in the subspace $\mathbf{P}((\mathcal{H}_n^*)^\omega)$ of analytic lines, and so (4.5.1) can be extended to a G-equivariant injective map

$$\mathcal{H}_n^{-\omega} \to \Gamma(\mathbb{M}^+, \mathbb{V}_n) \tag{4.5.2}$$

[*cf.* (2.9) and (2.2)]. Now there is a natural isomorphism $\mathcal{H}^1(\mathbb{P}^+, \mathbb{H}^{-n-2}) \to \mathcal{H}_n^{-\omega}$ due to the fact that both of these spaces are "maximal globalizations" of the Harish-Chandra module $(\mathcal{H}_n)_K$ [Sch]. Composing this isomorphism with (4.5.2) we get the Penrose transform $\mathcal{P}_n$.

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