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## On the asymptotics of scattering phases for the Schrödinger equation

by

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**ABSTRACT.** — Scattering phases are defined in terms of the asymptotics of solutions of the Schrödinger equation behaving as standing waves at infinity. The scattering phases are connected in a simple way with the eigenvalues of the unitary operator  $\Sigma = S\mathcal{R}$  where  $S$  is the scattering matrix and  $\mathcal{R}$  is the reflection operator. The eigenvalues of  $\Sigma$  can accumulate only at the points 1 and  $-1$ . It is shown that the leading terms of their asymptotics are determined only by the asymptotics of the even part of the potential at infinity. Explicit expressions for these terms are obtained.

**RÉSUMÉ.** — Des phases de diffusion sont définies en termes de l'asymptotique des solutions de l'équation de Schrödinger qui se comportent comme des ondes stationnaires à l'infini. Les phases de diffusion sont liées d'une manière simple aux valeurs propres de l'opérateur unitaire  $\Sigma = S\mathcal{R}$  où  $S$  est la matrice de diffusion et  $\mathcal{R}$  est l'opérateur de réflexion. Les valeurs propres de  $\Sigma$  peuvent s'accumuler seulement aux points 1 et  $-1$ . On montre que les termes principaux de leur comportement asymptotique sont déterminés seulement par l'asymptotique de la partie paire du potentiel à l'infini. Des expressions explicites pour ces termes sont obtenues.

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## 1. INTRODUCTION

Under the assumption

$$|q(x)| \leq C(1+|x|)^{-\beta}, \quad \beta > 1, \quad q = \bar{q}, \quad x \in \mathbf{R}^d, \quad d \geq 2, \quad (1.1)$$

one can construct solutions of the Schrödinger equation

$$-\Delta u + q(x)u = \lambda u, \quad \lambda = p^2 > 0, \quad (1.2)$$

whose asymptotic behaviour is that of a standing wave

$$\left. \begin{aligned} u(x) &\sim v(\omega) r^{-(d-1)/2} \sin(pr - (d-3)\pi/4 - \theta), \\ v &\in L_2(\mathbf{S}^{d-1}), \quad r = |x|, \quad x = r\omega, \quad \omega \in \mathbf{S}^{d-1}, \end{aligned} \right\} \quad (1.3)$$

as  $|x| \rightarrow \infty$ . Such solutions are usually studied for spherically symmetric potentials  $q(x) = q(r)$ , when the variables  $r$  and  $\omega$  can be separated in (1.2). In this case a solution of the equation (1.2) with the asymptotics (1.3) exists if  $\theta - l\pi/2 = \theta_l$  is a scattering phase (or a phase shift) corresponding to an orbital quantum number  $l = 0, 1, 2, \dots$ , and  $v = v_l$  is a spherical function. By analogy with the spherically symmetric case we shall call scattering phases those numbers  $\theta$  for which solutions of the equation (1.2) with the asymptotics (1.3) exist. Note that phases are defined up to a summand  $n\pi$  where  $n$  is an integer.

The scattering phases  $\theta$  and the corresponding functions  $v$  can be described in terms of the scattering matrix  $S(\lambda)$  (for a precise definition see section 3) of the Schrödinger equation (1.2). We recall that  $S(\lambda)$  is a unitary operator in the space  $\mathbf{H} = L_2(\mathbf{S}^{d-1})$  and that  $S(\lambda) - I$  is compact, where  $I$  is the identity operator. Let  $\mathcal{J}$ , defined by  $(\mathcal{J}f)(\omega) = f(-\omega)$ , be the reflection operator in  $\mathbf{H}$ . The product  $\Sigma(\lambda) = S(\lambda)\mathcal{J}$  is also a unitary operator so that its spectrum consists of eigenvalues lying on the unit circle  $\mathbf{T}$  and accumulating only at the points 1 and  $-1$ . It can be shown that a solution of the equation (1.2) with the asymptotics (1.3) exists if and only if  $\mu = \exp(-2i\theta)$  is an eigenvalue of the operator  $\Sigma(\lambda)$  and  $v$  is its eigenvector, *i.e.*  $\Sigma(\lambda)v = \mu v$ . Without going into details, we note that the asymptotics (1.3) should be understood in a natural averaged sense. On the other hand, we require that the relation (1.3) holds also for the first derivative with respect to  $r$ . This distinguishes a unique solution of the equation (1.2). Moreover, an expansion theorem in the space  $\mathcal{H} = L_2(\mathbf{R}^d)$  in terms of functions satisfying (1.2), (1.3) can be established. This theorem is similar in nature to the usual expansion (see e.g. [1], [2]) in a generalized Fourier integral. Such an expansion relies on those solutions of the equation (1.2) which behave asymptotically as plane waves at infinity. We emphasize that the expansion in standing waves is valid for arbitrary  $\beta > 1$  whereas the expansion in plane waves requires that  $2\beta > d + 1$ . Proofs of these results will be published elsewhere.

In the present paper we study eigenvalues of the operator  $\Sigma(\lambda) = S(\lambda)\mathcal{I}$ . More precisely, the convergence of eigenvalues to the points 1 and  $-1$  is investigated. Our main result (Theorem 4.2) is the evaluation of the leading term of their asymptotics. This problem is similar to the corresponding one for the scattering matrix  $S(\lambda)$  itself [3], [4]. We recall that in [3], [4] potentials decaying at infinity as homogeneous functions of a negative order  $-\beta < -1$  were considered. The explicit expression for the asymptotics of eigenvalues of  $S(\lambda)$  was derived in terms of the asymptotics of the potential at infinity.

The evaluation of the asymptotics of the scattering phases follows the scheme of [3], [4] but requires some new technical tools. First of all, we verify that it is sufficient to consider only the first Born approximation (the first term of the perturbation theory) to the scattering matrix. At this step the usual limiting absorption principle plays a crucial role. Further, due to the operator  $\mathcal{I}$  the problem is restricted to the subspaces of even and odd functions. Here we make use of the abstract Theorem 2.4 where a perturbation of an isolated eigenvalue of infinite multiplicity is investigated. We apply this Theorem to perturbations of the operator  $\mathcal{I}$  which has eigenvalues 1 and  $-1$  with the corresponding eigenfunctions being even and odd. Finally, we reduce the problem to the study of an operator constructed explicitly in terms of the even part  $q_e(x)$  of the potential  $q(x)$ . To this operator we apply results of [5] about the asymptotics of eigenvalues of pseudodifferential operator of negative order acting in the space  $\mathbf{H}$ . Thus the asymptotics of eigenvalues of  $\Sigma(\lambda)$  is evaluated in terms of the asymptotics of the function  $q_e(x)$  at infinity. As to the potential itself it is sufficient to assume the bound (1.1) with some  $\beta > (\alpha + 1)/2$ . This shows that the odd part of the potential can decay slower than  $q_e(x)$  and still not contribute to the asymptotics of the scattering phases.

The paper is organized as follows. Necessary information from abstract operator theory is collected in section 2. Scattering theory and pseudodifferential operators are discussed in section 3. Bounds and asymptotics of scattering phases are obtained in section 4.

## 2. PERTURBATION OF AN ISOLATED EIGENVALUE OF INFINITE MULTIPLICITY

1. We start with some well-known facts about compact operators. Their detailed presentation can be found e.g. in [6]. Let  $\mathbf{H}_1, \mathbf{H}_2$  be separable Hilbert spaces and  $\mathcal{K}_\infty = \mathcal{K}_\infty(\mathbf{H}_1, \mathbf{H}_2)$  be the class of compact operators  $K: \mathbf{H}_1 \rightarrow \mathbf{H}_2$ . For a self-adjoint operator  $K = K^* \in \mathcal{K}_\infty$  we denote by

$\lambda_n^+(K)$  [by  $-\lambda_n^-(K)$ ] its subsequent positive (negative) eigenvalues enumerated with their multiplicities. For an arbitrary compact operator  $K$  its singular (or simply  $s$ -) numbers are defined by the relation  $s_n(K) = \lambda_n^+((K^*K)^{1/2})$ .

Clearly, for any bounded operator  $A$

$$s_n(AK) \leq \|A\| s_n(K), \quad s_n(KA) \leq \|A\| s_n(K). \tag{2.1}$$

The singular numbers of a sum and of a product of compact operators satisfy the estimates

$$s_{n_1+n_2-1}(K_1+K_2) \leq s_{n_1}(K_1) + s_{n_2}(K_2), \tag{2.2}$$

$$s_{n_1+n_2-1}(K_1K_2) \leq s_{n_1}(K_1)s_{n_2}(K_2), \quad n_j \in \mathbb{N}. \tag{2.3}$$

It is convenient to introduce classes  $\mathcal{L}_\rho$  of compact operators  $K$  with  $s$ -numbers obeying a bound  $s_n(K) \leq Cn^{-\rho}$ ,  $\rho > 0$ . In other words,  $K \in \mathcal{L}_\rho$  if the functional

$$\langle K \rangle_\rho = [\sup_n (n^\rho s_n(K))]^\delta, \quad \delta = (\rho + 1)^{-1}, \tag{2.4}$$

is finite. It follows from (2.2) that the classes  $\mathcal{L}_\rho$  are linear. In fact, the functional (2.4) satisfies the triangle inequality

$$\langle K_1 + K_2 \rangle_\rho \leq \langle K_1 \rangle_\rho + \langle K_2 \rangle_\rho. \tag{2.5}$$

Moreover, it is quasi-homogeneous

$$\langle cK \rangle_\rho = |c|^\delta \langle K \rangle_\rho. \tag{2.6}$$

The closure of the finite-dimensional operators in “quasi-norm” (2.4) is denoted by  $\mathcal{L}_\rho^0$ . It consists of all compact operators  $K$  whose  $s$ -numbers obey a bound  $s_n(K) = o(n^{-\rho})$ . When  $\rho < 1$  the quasi-norm (2.4) is equivalent to the norm

$$\|K\|_\rho = \sup_n \left( n^{\rho-1} \sum_{m=1}^n s_m(K) \right). \tag{2.7}$$

The following assertion is frequently used when the asymptotics of a sum of compact operators is investigated. Note that whenever a relation contains the signs “ $\pm$ ”, it is understood as two separate relations.

**PROPOSITION 2.1.** — *Let  $K_j = K_j^* \in \mathcal{H}_\infty$ ,  $j = 1, 2$ . If*

$$\lambda_n^\pm(K_1) \sim k_\pm n^{-\rho}, \quad s_n(K_2) = o(n^{-\rho}), \quad n \rightarrow \infty,$$

then

$$\lambda_n^\pm(K_1 + K_2) \sim k_\pm n^{-\rho}, \quad n \rightarrow \infty.$$

2. We were not able to find the proof of the following elementary assertion in the literature\*.

LEMMA 2.2. — Let  $f \in C_0^\infty(\mathbf{R})$ . Then for any self-adjoint operators  $A, B$ , such that  $B - A \in \mathcal{L}_\rho$  (or  $\mathcal{L}_\rho^0$ ), the inclusion  $f(B) - f(A) \in \mathcal{L}_\rho$  (or  $\mathcal{L}_\rho^0$ ) holds. Moreover,

$$\langle f(B) - f(A) \rangle_\rho \leq C \langle B - A \rangle_\rho,$$

where  $C$  depends only on  $f$  and  $\rho$ .

*Proof.* — First we reduce the problem to the unitary case. In fact, let

$$U = (A - i)(A + i)^{-1}, \quad W = (B - i)(B + i)^{-1}$$

be the Cayley transforms of the operators  $A, B$ . Since

$$W - U = -2i[(B + i)^{-1} - (A + i)^{-1}] = 2i(B + i)^{-1}(B - A)(A + i)^{-1},$$

the operator  $W - U \in \mathcal{L}_\rho$  (or  $\mathcal{L}_\rho^0$ ) if  $B - A \in \mathcal{L}_\rho$  (or  $\mathcal{L}_\rho^0$ ). Define now a function  $\varphi$  on the unit circle  $\mathbf{T}$  by the equality  $\varphi(\mu) = f(\lambda)$  where  $\mu = (\lambda - i)(\lambda + i)^{-1}$ . Then  $\varphi(U) = f(A)$  and  $\varphi(W) = f(B)$ . Thus it is sufficient to prove that

$$\varphi(W) - \varphi(U) \in \mathcal{L}_\rho \quad (\text{or } \mathcal{L}_\rho^0) \quad (2.8)$$

and

$$\langle \varphi(W) - \varphi(U) \rangle_\rho \leq C \langle W - U \rangle_\rho. \quad (2.9)$$

Clearly,  $\varphi \in C^\infty(\mathbf{T})$  if  $f \in C_0^\infty(\mathbf{R})$ . In particular, the coefficients of its expansion in the Fourier series

$$\varphi(\mu) = \sum_{n=-\infty}^{\infty} a_n \mu^n$$

are rapidly decreasing. In terms of this expansion

$$\varphi(W) - \varphi(U) = \sum_{n=-\infty}^{\infty} a_n (W^n - U^n). \quad (2.10)$$

Let us show that  $W^n - U^n \in \mathcal{L}_\rho$  and

$$\langle W^n - U^n \rangle_\rho \leq |n| \langle W - U \rangle_\rho. \quad (2.11)$$

Since

$$W^{n+1} - U^{n+1} = W^n(W - U) + (W^n - U^n)U, \quad (2.12)$$

the triangle inequality (2.5) and the unitarity of  $U, W$  ensure that

$$\langle W^{n+1} - U^{n+1} \rangle_\rho \leq \langle W - U \rangle_\rho + \langle W^n - U^n \rangle_\rho.$$

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\* It can be deduced from general results of [9] about double operator integrals.

Now for  $n \geq 0$  we obtain (2.11) by induction. Going over to adjoint operators we extend (2.11) to  $n < 0$ .

Similar arguments show that

$$W^n - U^n \in \mathcal{L}_\rho^0 \tag{2.13}$$

if  $W - U \in \mathcal{L}_\rho^0$ . Actually, let  $W^j - U^j = T_m^{(j)} + R_m^{(j)}$ ,  $j = 1, \dots, n$ , where  $T_m^{(j)}$  are finite-dimensional and  $\langle R_m^{(j)} \rangle_\rho = o(1)$  as  $m \rightarrow \infty$ . Then according to (2.12)

$$W^{n+1} - U^{n+1} = (W^n T_m^{(1)} + T_m^{(n)} U) + (W^n R_m^{(1)} + R_m^{(n)} U).$$

The first term in the RHS is again finite-dimensional and the second one tends to zero in  $\mathcal{L}_\rho$  as  $m \rightarrow \infty$ . Therefore  $W^{n+1} - U^{n+1} \in \mathcal{L}_\rho^0$  if  $W - U \in \mathcal{L}_\rho^0$  and  $W^n - U^n \in \mathcal{L}_\rho^0$ . By induction this ensures (2.13).

Now we are able to estimate the sum (2.10). By (2.5) and (2.6)

$$\langle \varphi(W) - \varphi(U) \rangle_\rho \leq \sum_{n=-\infty}^{\infty} \langle a_n (W^n - U^n) \rangle_\rho = \sum_{n=-\infty}^{\infty} |a_n|^\delta \langle W^n - U^n \rangle_\rho, \\ \delta = (\rho + 1)^{-1}.$$

In virtue of (2.11) it gives the bound

$$\langle \varphi(W) - \varphi(U) \rangle_\rho \leq \sum_{n=-\infty}^{\infty} |a_n|^\delta |n| \langle W - U \rangle_\rho. \tag{2.14}$$

Since for  $\varphi \in C^\infty(\mathbf{T})$  the series in the RHS of (2.14) is convergent, we arrive at (2.8) and (2.9). This concludes the proof.

*Remark 2.3.* — The convergence of the series in (2.14) is sufficient for the validity of (2.8), (2.9). By the Hölder inequality one can rewrite this condition in a simpler form

$$\sum_{n=-\infty}^{\infty} |a_n| \cdot |n|^{1+2\rho+\varepsilon} < \infty, \quad \varepsilon > 0.$$

Thus for smaller  $\rho$  the relations (2.8), (2.9) hold true under less stringent assumptions on  $\varphi$ . When  $\rho < 1$ , using the norm (2.7) instead of the functional (2.4), one can obtain the estimate

$$\| \varphi(W) - \varphi(U) \|_\rho \leq C \| W - U \|_\rho, \\ C = \sum_{n=-\infty}^{\infty} |a_n| \cdot |n|,$$

where  $C$  is independent of  $\rho$ .

3. Now we prove a simple but general result about the asymptotics of eigenvalues arising by perturbation of an isolated eigenvalue of infinite multiplicity. We shall consider this problem in an abstract framework. Let  $A$  be some bounded self-adjoint operator in a Hilbert space  $\mathbf{H}$ . Assume that  $\lambda$  is an isolated eigenvalue of  $A$  of infinite multiplicity so that  $(\lambda, \lambda + \varepsilon]$

and  $[\lambda - \varepsilon, \lambda)$  are gaps in the spectrum of  $A$  if  $\varepsilon$  is small enough. By Weyl's theorem, for any compact operator  $K = K^*$  the essential spectra of the operators  $A$  and  $B = A + K$  coincide. This implies that the spectrum of  $B$  in  $(\lambda, \lambda + \varepsilon]$  and  $[\lambda - \varepsilon, \lambda)$  consists of eigenvalues. They have finite multiplicities and may accumulate only at the point  $\lambda$ . Denote by  $\mu_n^+$  and  $\mu_n^-$  the eigenvalues of the operator  $B$  lying in  $(\lambda, \lambda + \varepsilon]$  and  $[\lambda - \varepsilon, \lambda)$  and enumerated with their multiplicities so that  $\lambda < \mu_{n+1}^+ \leq \mu_n^+$  and  $\mu_n^- \leq \mu_{n+1}^- < \lambda$ .

Let  $P$  be an orthogonal projection onto the space of eigenvectors of  $A$  corresponding to the eigenvalue  $\lambda$ . It turns out that under rather weak assumptions on the operator  $K$  itself the asymptotics of the eigenvalues  $\mu_n^\pm$  are determined by the operator  $PKP$  alone.

THEOREM 2.4. — *Let for some  $\rho > 0$*

$$\lambda_n^\pm (PKP) \sim k_\pm n^{-\rho}, \quad n \rightarrow \infty, \quad (2.15)$$

and  $s_n(K) = o(n^{-\rho/2})$ . Then

$$\mu_n^\pm = \lambda \pm k_\pm n^{-\rho} + o(n^{-\rho}), \quad n \rightarrow \infty. \quad (2.16)$$

*Proof.* — By translation one can always assume that  $\lambda = 0$ . Let  $\theta = \bar{\theta} \in C_0^\infty(\mathbf{R})$ ,  $\theta(\lambda) = 1$  for sufficiently small  $|\lambda|$  and  $\theta(\lambda) = 0$  for  $|\lambda| \geq \varepsilon$ . Here  $\varepsilon$  is chosen in such a way that  $(0, \varepsilon]$  and  $[-\varepsilon, 0)$  are gaps in the spectrum of  $A$ . The function  $f(\lambda) = \lambda \theta^2(\lambda)$  equals  $\lambda$  in some neighbourhood of the point  $\lambda = 0$  and  $f(\lambda) = 0$  for  $|\lambda| \geq \varepsilon$ . Note that in the support of  $f$  the point  $\lambda = 0$  is the only point of accumulation of eigenvalues of the operator  $B$ . Therefore  $f(B) \in \mathcal{K}_\infty$  and up to a change of numeration by some finite number

$$\mu_n^\pm = \pm \lambda_n^\pm (f(B)). \quad (2.17)$$

Now we take into account that  $P = \theta(A)$  and  $A \theta(A) = 0$  by the construction of  $\theta$  and use the obvious identity

$$\begin{aligned} f(B) - PKP &= \theta(B) B \theta(B) - \theta(A) K \theta(A) \\ &= (\theta(B) - \theta(A)) K \theta(B) + (\theta(B) - \theta(A)) A (\theta(B) - \theta(A)) \\ &\quad + \theta(A) K (\theta(B) - \theta(A)). \end{aligned} \quad (2.18)$$

Since  $s_n(B - A) = o(n^{-\rho/2})$ , Lemma 2.2 ensures that

$$s_n(\theta(B) - \theta(A)) = o(n^{-\rho/2}).$$

So by estimates (2.1)–(2.3) the RHS of (2.18) is  $o(n^{-\rho})$  as  $n \rightarrow \infty$ . Thus

$$f(B) = PKP + \tilde{K}, \quad s_n(\tilde{K}) = o(n^{-\rho}).$$

In virtue of (2.15), Proposition 2.1 shows that

$$\lambda_n^\pm (f(B)) = k_\pm n^{-\rho} + o(n^{-\rho}).$$

To conclude the proof of (2.16) it suffices to take the equality (2.17) into account.



The bound for  $|\mu_n^\pm - \lambda|$  can be obtained more simply.

THEOREM 2.5. — *If for some  $\rho > 0$*

$$s_n(\text{PKP}) = O(n^{-\rho}), \quad s_n(\mathbf{K}) = O(n^{-\rho/2}),$$

then  $|\mu_n^\pm - \lambda_0| = O(n^{-\rho})$ .

*Proof.* — We assume again that  $\lambda = 0$  and use identity (2.18). Since  $\mathbf{K} \in \mathcal{L}_{\rho/2}$ , Lemma 2.2 ensures that  $\theta(\mathbf{B}) - \theta(\mathbf{A}) \in \mathcal{L}_{\rho/2}$ . So by estimates (2.1)–(2.3) the RHS of (2.18) belongs to  $\mathcal{L}_\rho$ . Recall now that  $\text{PKP} \in \mathcal{L}_\rho$ . It follows that  $f(\mathbf{B}) \in \mathcal{L}_\rho$  and therefore  $|\mu_n^\pm| = O(n^{-\rho})$  by (2.17).

### 3. BASIC AUXILIARY FACTS

1. We give now a precise definition of the scattering matrix  $\mathbf{S}(\lambda)$ . Denote by  $\mathbf{V}$  multiplication by a function  $q$  satisfying (1.1). Let  $\mathbf{H}_0 = -\Delta$  and  $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$  be self-adjoint operators in the Hilbert space  $\mathcal{H} = L_2(\mathbf{R}^d)$ . It is well-known (see e. g. [1], [2]) that the wave operators

$$\mathbf{W}_\pm = s\text{-}\lim_{t \rightarrow \pm\infty} \exp(i\mathbf{H}t)\exp(-i\mathbf{H}_0t)$$

exist and are complete, that is, their ranges coincide with the absolutely continuous subspace  $\mathcal{H}^{(a)}$  of the operator  $\mathbf{H}$ . Moreover,  $\mathcal{H}^{(a)} = E((0, \infty))\mathcal{H}$ , where  $E(\mathcal{X})$  is the spectral projection of  $\mathbf{H}$  corresponding to a set  $\mathcal{X} \subset \mathbf{R}$ . Since  $\mathbf{H}\mathbf{W}_\pm = \mathbf{W}_\pm\mathbf{H}_0$ , the scattering operator  $\mathbf{s} = \mathbf{W}_+^* \mathbf{W}_-$  commutes with  $\mathbf{H}_0$  and is unitary.

Let  $\hat{f}$  be the Fourier transform of a function  $f \in L_2(\mathbf{R}^d)$ , i. e.

$$\hat{f}(p) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} \exp(-i\langle x, p \rangle) f(x) dx, \quad p \in \Xi^d,$$

and let

$$(\Gamma_0(\lambda)f)(\omega) = 2^{-1/2} \lambda^{(d-2)/4} \hat{f}(\lambda^{1/2}\omega), \quad \lambda > 0, \quad (3.1)$$

be (up to the numerical factor) the restriction of  $\hat{f}$  onto the sphere of radius  $\lambda^{1/2}$ . Clearly, an operator  $\mathbf{F}$  defined by the relation  $(\mathbf{F}f)(\lambda) = \Gamma_0(\lambda)f$  maps  $\mathcal{H}$  unitarily onto the space  $\mathcal{H} = L_2(\mathbf{R}_+; \mathbf{H})$  of vector-functions on  $\mathbf{R}_+$  with values in the space  $\mathbf{H} = L_2(\mathbf{S}^{d-1})$ . Since  $\mathbf{F}\mathbf{H}_0\mathbf{F}^*$  acts as multiplication by the independent variable  $\lambda$ , the operator  $\mathbf{F}\mathbf{s}\mathbf{F}^*$  acts as multiplication by the unitary operator-function  $\mathbf{S}(\lambda): \mathbf{H} \rightarrow \mathbf{H}$  called the scattering matrix.

To describe a stationary representation for  $\mathbf{S}(\lambda)$  we need some analytical facts. Let  $\mathbf{X}_\gamma$  be multiplication by the function  $(1+x^2)^{-\gamma/2}$ . Then for  $\gamma > 1/2$  the operator

$$\mathbf{Z}_0(\lambda) = \mathbf{Z}_0^{(\gamma)}(\lambda) = \Gamma_0(\lambda)\mathbf{X}_\gamma: \mathcal{H} \rightarrow \mathbf{H}$$

is compact and depends continuously on the parameter  $\lambda > 0$ . This is a direct consequence of Sobolev's trace theorem. Sharp estimates on  $s$ -numbers of  $Z_0^{(\gamma)}$  are established in Corollary 3.3 below. By  $R_0(z) = (H_0 - z)^{-1}$  and  $R(z) = (H - z)^{-1}$ ,  $\Im z \neq 0$ , we denote the resolvents of operators  $H_0$  and  $H$ . The following assertion (see e.g. the original papers [7], [8] or monographs [1], [2]) is called the limiting absorption principle.

PROPOSITION 3.1. — *Let (1.1) be fulfilled and  $\gamma > 1/2$ . Then the operator*

$$G(z) = G^{(\gamma)}(z) = X_\gamma R(z) X_\gamma$$

*is continuous in norm with respect to the spectral parameter  $z$  in the complex plane cut along  $[0, \infty)$ , possibly with exception of the point  $z = 0$ .*

The stationary representation for the scattering matrix has the form

$$S(\lambda) = I - 2\pi i \Gamma_0(\lambda) (V - VR(\lambda + i0)V) \Gamma_0^*(\lambda). \quad (3.2)$$

The justification of this formula can be found, for example, in [7]. To be quite precise one should rewrite (3.2) in terms of the operators  $Z_0$  and  $G$ . Let  $\gamma_1 + \gamma_2 = \beta$ ,  $\gamma_j > 1/2$ , and

$$V = X_{\gamma_1} \mathcal{V} X_{\gamma_2} = X_{\gamma_2} \mathcal{V} X_{\gamma_1},$$

where  $\mathcal{V}$  is multiplication by the bounded function  $q(x)(1+x^2)^{\beta/2}$ . Then (3.2) takes the form

$$S(\lambda) = I - 2\pi i Z_0^{(\gamma_1)}(\lambda) \mathcal{V} (I - G^{(\gamma_2)}(\lambda + i0) \mathcal{V}) (Z_0^{(\gamma_1)}(\lambda))^*. \quad (3.3)$$

The RHS of (3.3) is a combination of bounded operators and thus it is well defined. It follows from (3.3) that  $S(\lambda)$  depends continuously on  $\lambda$  and that  $S(\lambda) - I$  is compact.

One can take formulas (3.2) or (3.3) as the definition of the scattering matrix. The unitarity of  $S(\lambda)$  can be easily deduced from this representation. To this end one should use the resolvent identity connecting  $R(z)$  with  $R_0(z)$  and the relation

$$2\pi i Z_0^*(\lambda) Z_0(\lambda) = G_0(\lambda + i0) - G_0(\lambda - i0), \\ G_0(z) = X_\gamma R_0(z) X_\gamma.$$

In its turn, the last equality is a consequence of the relation between the boundary values of a Cauchy integral and its density.

2. We need some information about the spectrum of the operator  $T_0 = T_0(\lambda) = \Gamma_0(\lambda) V \Gamma_0^*(\lambda)$ , which is called the first Born approximation to the scattering matrix. Actually, we shall consider a slightly more general operator  $Y_1 T_0 Y_2$  where  $Y_j$  is multiplication by the characteristic function of some set  $\mathcal{A}_j \subset \mathbb{S}^{d-1}$ . Denote by  $\Lambda_\omega$  the plane orthogonal to  $\omega$  and passing through the origin in  $\mathbb{E}^d$ . We assume that in case  $d > 2$  the sphere  $\mathbb{S}_\omega^{d-2} = \mathbb{S}^{d-1} \cap \Lambda_\omega$  is endowed with the usual  $(d-2)$ -surface measure. In

case  $d=2$  the set  $S_{\omega}^{d-2}$  consists of two points (corresponding to unit vectors orthogonal to  $\omega$ ) both of which have measure 1.

PROPOSITION 3.2. — *Let a function  $q(x)$  have the asymptotic form*

$$\left. \begin{aligned} q(x) &= |x|^{-\alpha} g(\omega) + o(|x|^{-\alpha}), \\ |x| &\rightarrow \infty, \quad \alpha > 1, \quad g \in C^{\infty}(S^{d-1}), \end{aligned} \right\} \quad (3.4)$$

at infinity. Set

$$\rho = (\alpha - 1)(d - 1)^{-1}, \quad (3.5)$$

$$\begin{aligned} \Omega(\omega, \psi) &= \int_0^{\pi} g(\omega \cos \theta + \psi \sin \theta) \sin^{\alpha-2} \theta \, d\theta, \quad \psi \in S_{\omega}^{d-2}, \\ \Omega_+ &= \max\{\Omega, 0\}, \quad \Omega_- = \Omega_+ - \Omega, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} \mathbf{a}_{\pm}(\mathcal{Y}) &= 2^{-1} (d-1)^{-\rho} (2\pi)^{-\alpha} \lambda^{-1+\alpha/2} \\ &\times \left( \int_{\mathcal{Y}} d\omega \int_{S_{\omega}^{d-2}} d\psi (\Omega_{\pm}(\omega, \psi))^{1/\rho} \right)^{\rho}. \end{aligned} \quad (3.7)$$

Then

$$\lambda_n^{\pm}(\mathbf{Y}T_0\mathbf{Y}) = \mathbf{a}_{\pm}(\mathcal{Y}) n^{-\rho} + o(n^{-\rho}). \quad (3.8)$$

If the  $(d-1)$ -surface measure of  $\mathcal{Y}_1 \cap \mathcal{Y}_2$  equals zero, then

$$s_n(\mathbf{Y}_1 T_0 \mathbf{Y}_2) = o(n^{-\rho}). \quad (3.9)$$

The detailed proof of this assertion can be found in [3]. Its basic steps are the following. First, we consider the case when  $q$  is a smooth function which equals  $|x|^{-\alpha} g(\omega)$  outside of some neighbourhood of the origin. Then, according to (3.1),  $T_0$  is an integral operator in  $\mathbf{H}$  with kernel

$$t(\omega, \omega') = 2^{-1} \lambda^{(d-2)/2} (2\pi)^{-d} \int_{\mathbf{R}^d} \exp(-i\lambda^{1/2} \langle \omega - \omega', x \rangle) q(x) \, dx.$$

This function has a singularity on the diagonal  $\omega = \omega'$  which determines the asymptotics of the spectrum of the operator  $\mathbf{Y}T_0\mathbf{Y}$ . However, it is more convenient to treat  $T_0$  as a pseudodifferential operator of negative order  $1 - \alpha$  on the sphere  $S^{d-1}$ . Essentially, its symbol has the form

$$\mathbf{q}(\omega, \xi) = (4\pi)^{-1} \lambda^{-1/2} \int_{-\infty}^{\infty} q(\omega t + \lambda^{-1/2} \xi) \, dt, \quad \xi \in \Lambda_{\omega}.$$

More precisely,  $S^{d-1}$  can be split into a finite number of pieces in such a way that on each piece the problem is reduced to the consideration of a pseudodifferential operator in Euclidean space of dimension  $d-1$ . We can apply Weyl's formula (proved in this situation in [5]) for asymptotics of eigenvalues of operators  $\mathbf{Y}T_0\mathbf{Y}$ . This ensures the asymptotics (3.8) for our special choice of  $q$ . In the particular case  $g(\omega) = 1$  this result shows

that the term  $o(|x|^{-\alpha})$  in (3.4) does not contribute to the asymptotics of the eigenvalues. Thus we can extend (3.8) to arbitrary bounded  $q$  satisfying (3.4). The expression (3.7) for  $\mathbf{a}_{\pm}(\mathcal{Y})$  can be obtained if the homogeneity of the function  $|x|^{-\alpha}g(\omega)$  is taken into account. The estimate (3.9) is established as a by-product of these considerations.

**COROLLARY 3.3.** — *Let  $\gamma > 1/2$ . Then the operator  $Z_0^{(\gamma)}(\lambda) = \Gamma_0(\lambda) X_{\gamma} : \mathcal{H} \rightarrow \mathbf{H}$  is compact and*

$$s_n(Z_0^{(\gamma)}(\lambda)) = O(n^{-(\gamma-1/2)/(d-1)}).$$

*Proof.* — Let us apply Proposition 3.2 in case  $q(x) = (1+x^2)^{-\gamma}$ ,  $Y = \mathbf{I}$ . Then it follows from (3.8) that

$$s_n^2(Z_0^{(\gamma)}) = (\lambda_n^+ (\Gamma_0 X_{2\gamma} \Gamma_0^*)) \leq C n^{-(2\gamma-1)/(d-1)}.$$

#### 4. MAIN THEOREM

1. Here we shall consider the asymptotics of the spectrum of the unitary operator  $\Sigma(\lambda) = S(\lambda)\mathcal{J}$  in the space  $\mathbf{H}$ . We recall that the precise definition of the scattering matrix  $S(\lambda)$  is given by the relation (3.3) and  $(\mathcal{J}f)(\omega) = f(-\omega)$ . Proposition 3.1 and Corollary 3.3 ensure that  $S(\lambda) - \mathbf{I}$  is compact so that

$$\Sigma(\lambda) - \mathcal{J} \in \mathcal{K}_{\infty}. \quad (4.1)$$

In the following the dependence of different objects on  $\lambda$  is often omitted. The spectrum of  $\mathcal{J}$  consists of the eigenvalues 1 and  $-1$  with corresponding eigenfunctions being even and odd. We denote by

$$P_{\pm} = 2^{-1}(\mathbf{I} \pm \mathcal{J}) \quad (4.2)$$

the orthogonal projections in  $\mathbf{H}$  onto the subspaces  $\mathbf{H}_+$  and  $\mathbf{H}_-$  of even and odd functions.

By Weyl's theorem two unitary operators with compact difference have the same essential spectra. Thus the following assertion is an immediate consequence of (4.1).

**LEMMA 4.1.** — *The spectrum of  $\Sigma$  consists of eigenvalues accumulating only at the points 1 and  $-1$ . Moreover, all eigenvalues except possibly the limit points 1 and  $-1$  have finite multiplicities.*

We shall denote the eigenvalues of the operator  $\Sigma$  accumulating at 1 by

$$\exp(\mp 2i\delta_n^{\pm}), \quad 0 < \delta_n^{\pm} \leq \pi/4, \quad \delta_{n+1}^{\pm} \leq \delta_n^{\pm},$$

and the eigenvalues accumulating at  $-1$  by

$$-\exp(\mp 2i\eta_n^{\pm}), \quad 0 < \eta_n^{\pm} < \pi/4, \quad \eta_{n+1}^{\pm} \leq \eta_n^{\pm}.$$

Here the numbers  $\delta_n^\pm$  and  $\eta_n^\pm$  are called scattering phases. Note that compared to section 1 the definition of scattering phases has been slightly changed. Namely, the phases  $\delta$  coincide with  $\theta$  and the phases  $\eta$  differ from  $\theta$  by  $\pi/2$ . This is consistent with a constant phase shift by  $l\pi/2$  which is taken into account in the spherically symmetric case. On the other hand, we accept enumeration of phases  $\delta_n^\pm$  and  $\eta_n^\pm$  according to their values whereas in the spherically symmetric case phases are usually enumerated by orbital quantum numbers.

Now we formulate our main result about asymptotics of the scattering phases as  $n \rightarrow \infty$ .

THEOREM 4.2. — *Let the even part*

$$q_e(x) = 2^{-1}(q(x) + q(-x))$$

*of a potential  $q(x)$  have the asymptotic form*

$$q_e(x) = |x|^{-\alpha} g(\omega) + o(|x|^{-\alpha}), \quad \left. \begin{array}{l} \alpha > 1, \\ g \in C^\infty(\mathbf{S}^{d-1}), \end{array} \right\} \quad (4.3)$$

*as  $|x| \rightarrow \infty$ . Assume also that  $q$  satisfies the bound (1.1) with  $\beta > (\alpha + 1)/2$ . Define the number  $\rho$  and the functions  $\Omega_\pm(\omega, \psi)$  by the relations (3.5) and (3.6). Then the following limits exist*

$$\lim_{n \rightarrow \infty} n^\rho \delta_n^\pm = \lim_{n \rightarrow \infty} n^\rho \eta_n^\pm = a_\pm, \quad (4.4)$$

*where*

$$a_\pm = 2^{-\rho-2} (d-1)^{-\rho} (2\pi)^{1-\alpha} \lambda^{-1+\alpha/2} \times \left( \int_{\mathbf{S}^{d-1}} d\omega \int_{\mathbf{S}_0^{d-2}} d\psi (\Omega_\pm(\omega, \psi))^{1/\rho} \right)^\rho. \quad (4.5)$$

We emphasize that according to (4.4) the leading terms of the asymptotics of  $\delta_n^\pm$  and  $\eta_n^\pm$  coincide with each other.

The result about the estimate of scattering phases is formulated in essentially the same (but simpler) way.

THEOREM 4.3. — *Let  $q_e(x) = O(|x|^{-\alpha})$  and  $q$  satisfy (1.1) with  $\beta > (\alpha + 1)/2$ . Then*

$$\delta_n^\pm = O(n^{-\rho}), \quad \eta_n^\pm = O(n^{-\rho}). \quad (4.6)$$

2. Let us start with the proof of Theorem 4.2. First we reduce the study of eigenvalues of the unitary operator  $\Sigma$  to that of some self-adjoint operator. It is convenient to choose

$$\mathbf{B} = 2^{1/2} \Im(\tau \Sigma) = 2^{-1/2} i(\bar{\tau} \mathcal{I} \mathbf{S}^* - \tau \mathbf{S} \mathcal{I}), \quad \begin{matrix} 3 \\ \tau = \exp(\pi i/4), \end{matrix} \quad (4.7)$$

as such an operator. The spectrum of  $\mathbf{B}$  consists of eigenvalues, accumulating possibly from both sides at the points 1 and  $-1$ . We denote them by

$\mu_n^\pm, \mu_n^\pm \rightarrow 1 \pm 0$ , and  $\nu_n^\pm, \nu_n^\pm \rightarrow -1 \pm 0$ , respectively. Clearly, up to a change of numeration by some finite number,

$$\mu_n^\pm = 2^{1/2} \Im(\tau \exp(\pm 2i\delta_n^\mp)), \quad \nu_n^\pm = -2^{1/2} \Im(\tau \exp(\mp 2i\eta_n^\pm)).$$

It follows that  $\pm 2\delta_n^\mp \sim \mu_n^\pm - 1$  as  $\mu_n^\pm \rightarrow 1$  and  $\pm 2\eta_n^\pm \sim \nu_n^\pm + 1$  as  $\nu_n^\pm \rightarrow -1$ . We state this assertion in a precise form.

**LEMMA 4.4.** — *Assume that  $\mu_n^\pm \sim 1 \pm b_\pm n^{-\rho}$  or  $\nu_n^\pm \sim -1 \pm c_\pm n^{-\rho}$  for some  $\rho > 0$  as  $n \rightarrow \infty$ . Then  $\delta_n^\mp \sim 2^{-1} b_\pm n^{-\rho}$  or  $\eta_n^\pm \sim 2^{-1} c_\pm n^{-\rho}$  as  $n \rightarrow \infty$ .*

Comparing (3.2) and (4.7) we obtain an explicit expression for the operator B. Let

$$T_0 = \Gamma_0 V \Gamma_0^*, \quad T_1 = -\Gamma_0 V R V \Gamma_0^*, \quad T = T_0 + T_1 \quad (4.8)$$

with  $R = R(\lambda + i0)$ . Denote

$$K_0 = \tau T_0 \mathcal{J} + \bar{\tau} \mathcal{J} T_0, \quad K_1 = \tau T_1 \mathcal{J} + \bar{\tau} \mathcal{J} T_1^* \quad (4.9)$$

and

$$K = -2^{1/2} \pi (K_0 + K_1). \quad (4.10)$$

Then  $B = \mathcal{J} + K$ .

The plan for proving Theorem 4.2 is the following. By using Proposition 3.1 and Corollary 3.3 the  $s$ -numbers of the operator K are estimated by  $o(n^{-\rho/2})$ . The asymptotics of the spectrum of the operator  $P_\pm K P_\pm$  are found with the help of Proposition 3.2. According to Theorem 2.4 this gives the asymptotics of the spectrum of the operator B.

The bound on  $s_n(K)$  is quite simple.

**LEMMA 4.5.** — *Let the assumption (1.1) hold. Then  $s_n(T_0) = O(n^{-\rho_0})$  and  $s_n(T_1) = O(n^{-\rho_1})$ , where  $\rho_0 = (\beta - 1)(d - 1)^{-1}$  and  $\rho_1$  is any number smaller than  $2(\beta - 1)(d - 1)^{-1}$ .*

*Proof.* — According to (1.1),  $V = X_{\beta/2} \mathcal{V} X_{\beta/2}$  with bounded  $\mathcal{V}$ . Therefore by (2.1) and (2.3)

$$s_{2n-1}(T_0) \leq \| \mathcal{V} \| s_n^2(\Gamma_0 X_{\beta/2}).$$

Thus Corollary 3.3 ensures the bound for  $s_n(T_0)$ .

Similarly,  $V = X_{\beta-\gamma} \mathcal{V} X_\gamma = X_\gamma \mathcal{V} X_{\beta-\gamma}$  with bounded  $\mathcal{V}$  and any  $\gamma \in (1/2, \beta - 1/2)$ . By Proposition 3.1 it follows that

$$s_{2n-1}(T_1) \leq s_n^2(\Gamma_0 X_{\beta-\gamma}) \| X_\gamma R X_\gamma \| \cdot \| \mathcal{V} \|^2 = O(n^{-(2\beta - 2\gamma - 1)/(d-1)}).$$

Since  $\gamma$  can be chosen arbitrary close to  $1/2$ , this concludes the proof.

Taking into account the inequalities (2.1) and (2.2) we obtain the estimate for the  $s$ -numbers of the operator K defined by (4.9), (4.10).

**COROLLARY 4.6.** — *Let the assumption (1.1) hold. Then  $s_n(K) = O(n^{-\rho_0})$ .*

3. Now we shall study the operator  $P_{\pm} K P_{\pm}$ . By Lemma 4.5

$$s_n(P_{\pm} K_1 P_{\pm}) = O(n^{-\rho_1}), \quad \rho_1 < 2(\beta - 1)(d - 1)^{-1}. \tag{4.11}$$

So we need only consider the asymptotics of the operator

$$P_{\pm} K_0 P_{\pm} =: 2^{1/2} L_{\pm}. \tag{4.12}$$

By (4.2) and (4.9) and because  $\mathcal{J}^2 = I$  we have

$$L_{\pm} = 2^{-1} (\pm T_e + T_e \mathcal{J}), \tag{4.13}$$

where

$$T_e = 2^{-1} (T_0 + \mathcal{J} T_0 \mathcal{J}). \tag{4.14}$$

Let  $V_e$  be multiplication by  $q_e(x)$ . Then according to (3.1), (4.8)

$$T_e = \Gamma_0 V_e \Gamma_0^*. \tag{4.15}$$

Since  $\mathcal{J}^2 = I$ , the operator (4.14) commutes with  $\mathcal{J}$  and

$$L_{\pm} \mathcal{J} = \mathcal{J} L_{\pm} = \pm L_{\pm}. \tag{4.16}$$

We can directly apply Proposition 3.2 to find the asymptotics of the spectrum of the operator (4.15). However, the operator (4.13) also contains the term  $T_e \mathcal{J}$ . Considered as an integral operator it has a singularity on the antidiagonal  $\omega = -\omega'$ . We shall show that  $T_e \mathcal{J}$  does not contribute to the asymptotics of the spectrum of  $L_{\pm}$ . To this end it is convenient to reduce the problem to the study of the operator  $2Y L_{\pm} Y$  where  $Y$  is multiplication by the characteristic function of any fixed hemisphere  $S_0 \subset S^{d-1}$ .

**LEMMA 4.7.** — *Let  $L_{\pm}$  be any compact operator obeying the relation (4.16). Then the operators  $L_{\pm}$  and  $M_{\pm} = 2Y L_{\pm} Y$  have common non-zero eigenvalues with the same multiplicities.*

*Proof.* — Set  $S'_0 = S^{d-1} \setminus S_0$ ,  $Y' = I - Y$ . In the representation  $L_2(S^{d-1}) = L_2(S_0) \oplus L_2(S'_0)$  the operator  $L_{\pm}$  can be written as the matrix

$$\begin{pmatrix} Y L_{\pm} Y & Y L_{\pm} Y' \\ Y' L_{\pm} Y & Y' L_{\pm} Y' \end{pmatrix}.$$

Taking into account the relations  $Y' = \mathcal{J} Y \mathcal{J}$  and (4.16) we find that this operator equals

$$2^{-1} \begin{pmatrix} M_{\pm} & \pm M_{\pm} \mathcal{J} \\ \pm \mathcal{J} M_{\pm} & \mathcal{J} M_{\pm} \mathcal{J} \end{pmatrix}.$$

The latter operator is obviously unitarily equivalent to the operator

$$2^{-1} \begin{pmatrix} M_{\pm} & \pm M_{\pm} \\ \pm M_{\pm} & M_{\pm} \end{pmatrix} \tag{4.17}$$

acting in the two-component space  $L_2(\mathbf{S}_0) \otimes \mathbb{C}^2$ . The non-zero spectrum of the operator (4.17) is the same as that of the operator  $M_{\pm}$ . This concludes the proof.

Now we apply this Lemma to the operator (4.13) when

$$M_{\pm} = \pm Y T_e Y + Y T_e Y' \mathcal{J}, \quad Y' = I - Y. \quad (4.18)$$

Proposition 3.2 allows us to find the asymptotics of its spectrum.

LEMMA 4.8. — *Let the condition (4.3) be satisfied and let the coefficient  $a_{\pm}$  be given by (4.5). Define the operator  $M_{\pm}$  by the relations (4.15), (4.18). Then for both signs  $\sigma = "+"$  and  $\sigma = "-"$*

$$\lambda_n^{\pm}(M_{\sigma}) = \pi^{-1} a_{\pm\sigma} n^{-\rho} + o(n^{-\rho}), \quad n \rightarrow \infty. \quad (4.19)$$

*Proof.* — Let us apply Proposition 3.2 to operators  $Y T_e Y$  and  $Y T_e Y'$ . According to (3.8)

$$\lambda_n^{\pm}(Y T_e Y) = \mathbf{a}_{\pm}(\mathbf{S}_0) n^{-\rho} + o(n^{-\rho}),$$

where  $\mathbf{a}_{\pm}(\mathbf{S}_0)$  is given by (3.7). Since for an even potential  $g(-\omega) = g(\omega)$ , we have that  $\Omega(-\omega, \psi) = \Omega(\omega, \psi)$ . Therefore the integral in (4.5) over  $\mathbf{S}^{d-1}$  equals twice the integral over  $\mathbf{S}_0$ . This shows that  $\mathbf{a}_{\pm}(\mathbf{S}_0) = \pi^{-1} a_{\pm}$ . Moreover, according to (3.9)

$$s_n(Y T_e Y') = o(n^{-\rho}).$$

Taking into account Proposition 2.1 we obtain the asymptotics (4.19) for the sum (4.18).

4. Now for the proof of Theorem 4.2 it suffices to combine the results obtained. Lemmas 4.7 and 4.8 ensure that for the operator (4.13) and both signs  $\sigma = "+"$  and  $\sigma = "-"$

$$\lambda_n^{\pm}(L_{\sigma}) = \pi^{-1} a_{\pm\sigma} n^{-\rho} + o(n^{-\rho}).$$

Since  $\beta > (\alpha + 1)/2$  the number  $\rho_1$  in (4.11) can be chosen larger than  $\rho$ . Thus, applying Proposition 2.1 to the operator

$$P_{\sigma} K P_{\sigma} = -2\pi L_{\sigma} - 2^{1/2} \pi P_{\sigma} K_1 P_{\sigma},$$

we find that

$$\lambda_n^{\pm}(P_{\sigma} K P_{\sigma}) = 2a_{\mp\sigma} n^{-\rho} + o(n^{-\rho}).$$

Moreover, since  $2\rho_0 > \rho$ , Corollary 4.6 shows that  $s_n(K) = o(n^{-\rho/2})$ . Therefore we can apply Theorem 2.4 to the operator  $B = \mathcal{J} + K$ . In our case  $\mathcal{J}$  plays role of  $A$  and  $\lambda$  equals 1 or  $-1$ . It follows that the eigenvalues of  $B$  are split into two series with asymptotics

$$\begin{aligned} \mu_n^{\pm} &= 1 \pm 2a_{\mp} n^{-\rho} + o(n^{-\rho}), \\ \nu_n^{\pm} &= -1 \pm 2a_{\pm} n^{-\rho} + o(n^{-\rho}). \end{aligned}$$

Finally, taking into account Lemma 4.4 we establish the asymptotics (4.4). This concludes the proof of Theorem 4.2.



The proof of Theorem 4.3 can be obtained in a similar (but considerably simpler) way to that of Theorem 4.2. In fact, under the assumption  $q_e(x) = O(|x|^{-\alpha})$ , Corollary 3.3 gives the bound  $s_n(L_\sigma) = O(n^{-\rho})$ . Taking into account (4.11) we obtain, according to Lemma 4.7 and inequality (2.2), that  $s_n(P_\sigma KP_\sigma) = O(n^{-\rho})$ . Moreover,  $s_n(K) = O(n^{-\rho/2})$  by Corollary 4.6. Thus Theorem 2.5 gives that

$$|\mu_n^\pm - 1| = O(n^{-\rho}), \quad |v_n^\pm + 1| = O(n^{-\rho}), \quad n \rightarrow \infty.$$

The bounds (4.6) are immediate consequences of these relations.

*Remark 4.9.* — As it is clear from the proofs, the limits (4.4) and the bounds (4.6) for  $\delta_n^\pm(\lambda)$ ,  $\eta_n^\pm(\lambda)$  are uniform with respect to  $\lambda \in [\lambda_0, \lambda_1]$  if  $0 < \lambda_0 < \lambda_1 < \infty$ .

Finally we note that for an even potential  $q(x) = q(-x)$  the proofs of Theorems 4.2 and 4.3 become essentially simpler. In this case the subspaces  $\mathbf{H}_+$  and  $\mathbf{H}_-$  of even and odd functions are invariant under the actions of  $S$  and  $\Sigma$ . Clearly, the spectra of these operators are the same on  $\mathbf{H}_+$  and differ only by the sign on  $\mathbf{H}_-$ . Thus the evaluation of the asymptotics of eigenvalues of  $\Sigma$  is reduced to that of  $S$  on the subspaces  $\mathbf{H}_+$  and  $\mathbf{H}_-$ .

On the other hand, for an odd potential  $q(x) = -q(-x)$  Theorems 4.2 or 4.3 give only a bound for the scattering phases. Namely, if an odd  $q$  obeys (1.1) then according to any of these Theorems,  $\delta_n^\pm = O(n^{-\rho_1})$  and  $\eta_n^\pm = O(n^{-\rho_1})$ , where  $\rho_1$  is an arbitrary number smaller than  $2(\beta - 1)(d - 1)^{-1}$ . Indeed, we do not know the asymptotics of the scattering phases even in the case of the dipole potential

$$q(x) = |x + a|^{-1} - |x - a|^{-1}, \quad a \in \mathbf{R}^d.$$

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