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Non-hyperbolicity and invariant measures for unimodal maps

by

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There are several results dealing with the existence of an invariant probability measure, which is absolutely continuous with respect to the Lebesgue measure (acim in short), *see* for example [C-E], [K], [N-S]. In this note we want to describe a weak condition which guarantees the existence of such a measure. We believe this condition is even equivalent to the existence of acim's. We say that the critical point c of a smooth map f has order l if there are constants O_1, O_2 so that

$$O_1 |x - c|^{l-1} \leq |Df(x)| \leq O_2 |x - c|^{l-1} \quad (\text{NF}).$$

As usual let f^n be the n -th iterate of f and let $c_1 = f(c)$. Furthermore denote the Lebesgue measure of a measurable set I by $|I|$.

MAIN THEOREM. — *Suppose that f is unimodal, C^3 , has negative Schwarzian derivative and that the critical point of f is of order $l \geq 1$. Moreover assume that the growth-rate of $|Df^n(c_1)|$ is so fast that*

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/l} < \infty$$

holds, Then f has a unique absolutely continuous invariant probability measure μ which is ergodic and of positive entropy. Furthermore there exists a positive constant K such that

$$\mu(A) \leq K |A|^{1/l},$$

for any measurable set $A \subset (0, 1)$.

M. Benedicks and L. S. Young announced the existence of acim's for maps for which $|Df^n(c_1)|$ grows at least polynomially.

Of course the estimate $\mu(A) \leq K|A|^{1/l}$ shows that the poles of the invariant measure μ are at most of the form $|x - x_0|^{1/l-1}$. It is not hard to show that any absolutely continuous invariant probability measure has a pole of this order at the critical values $f^n(c)$, $n \geq 1$, and therefore this estimate is optimal. Even for maps for which $|Df^n(c_1)|$ grows exponentially this result is new (the results in [C-E] and [N-S] only give some bounds for the order of the poles).

A REFORMULATION OF THE MAIN THEOREM AND AN OUTLINE OF ITS PROOF

In [BL] it is shown that any unimodal map with negative Schwarzian derivative is ergodic (w.r.t. to the Lebesgue measure) and that any absolutely continuous invariant probability measure μ has positive metric entropy. Therefore, in order to prove the Main Theorem it is enough to establish the existence of an absolutely continuous invariant probability measure μ .

In order to prove the existence of this invariant measure we will use the strategy of [N-S]. Using general arguments one can show that f has an absolutely continuous invariant probability measure provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set A with $|A| < \delta$ one has that $|f^{-n}(A)| < \varepsilon$ for all $n > 0$. In fact in this paper we will prove the following more precise statement: there exists a constant K such that for every n and every measurable set A ,

$$|f^{-n}(A)| < K|A|^{1/l}. \quad (1)$$

One of the main results in [N-S] was to show that (1) can be deduced from the following: there exists a constant K' such that for any n and every $\varepsilon > 0$,

$$|f^{-n}(c_1 - \varepsilon, c_1)| < K' \varepsilon^{1/l} \quad (2)$$

where l is the order of the critical point of f . Because of the non-flatness condition at the critical point this is equivalent to: there exists a constant K'' such that for every $n > 0$ and every $\varepsilon > 0$

$$|f^{-n}(c - \varepsilon, c + \varepsilon)| < K'' \varepsilon.$$

From all this it follows that the Main Theorem can be deduced from

THEOREM. — *Suppose that f is unimodal, C^3 , has negative Schwarzian derivative and that the critical point of f is of order $l \geq 1$. Moreover assume*

that

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/l} < \infty$$

holds. Then there exists a constant $K < \infty$ such that for each $\varepsilon > 0$,

$$|f^{-n}(c - \varepsilon, c + \varepsilon)| < K\varepsilon. \quad (3)$$

Let us say a few words about the proof of inequality (3). The main idea is to show that each component of $f^{-n}(c - \varepsilon, c + \varepsilon)$ is either contained in or at least can be compared in size (this process we will call 'sliding') with a set of the form

$$f^{-(n-k)}\left(c - \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}, c + \frac{\varepsilon}{|Df^k(c_1)|^{1/l}}\right).$$

Using this and the summability condition, inequality (3) will then be proved by induction.

Details of the proof can be found in a paper which will appear in *Inventiones Mathematicae*.

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