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The Cauchy problem for Hartree-Fock time-dependent equations

by

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ABSTRACT. – We study the Cauchy problem for time dependent Hartree-Fock equations for an infinite system of fermions with a single particle potential V and a two-body interaction potential v. We prove existence, uniqueness and globality of solutions in L^2 and in H^1 under suitable condition on V and on v.

RÉSUMÉ. – Nous étudions le problème de Cauchy pour les équations de Hartree-Fock dependant du temps pour un système de fermions infini avec un potentiel d'une seule particule V et un potentiel d'interaction à deux corps v. Nous démontrons existence, unicité et globalité des solutions in L^2 et in H^1 avec conditions convenables sur V et sur v.

0. INTRODUCTION

We consider the time-dependent Hartree-Fock equations for an infinite system of fermions:

$$i\frac{\partial \varphi_j}{\partial t} = -\Delta \varphi_j + V\varphi_j + \sum_{k\in\mathbf{N}} (\varphi_j v * |\varphi_k|^2 - \varphi_k v * (\varphi_j \bar{\varphi}_k)) \qquad j\in\mathrm{N} \tag{0.1}$$

where each φ_j is a single particle wave function defined on $I \times \mathbf{R}^n$, ($I=[0, T]$, $T \in \mathbf{R}^+$ or $I=[0, +\infty[$); V is a real single particle potential and v is a two-body interaction potential which is real and even, Δ is the Laplace operator on $\mathbf{R}^n$ (we assume $n \geqq 2$).

For a physical derivation of (0.1) one can see in [1].

In order to treat this infinite system of equations we introduce a sequence $\Phi = \{\varphi_j\}_{j \in \mathbf{N}}$ which collects all the states and rewrite (0.1) in a more compact form:

$$i\frac{\partial \Phi}{\partial t} = -\Delta \Phi + V\Phi + F_v(\Phi) \tag{0.2}$$

where $F_v(\Phi)$ accounts suitably for the nonlinearity, and we assume that if A a generic single particle linear operator, its action on a sequence $\Phi = \{\varphi_j\}_{j \in \mathbf{N}}$ is given by $A\Phi = \{A\varphi_j\}_{j \in \mathbf{N}}$ with the obvious condition that if A is unbounded $A\Phi$ is defined for all those Φ such that any φ_j belongs to the domain of A.

Written in such a form (0.1) looks like a nonlinear Schrödinger equation for a single particle and our work is a generalization of the tools that in the last years have been used in the study of it to Hartree-Fock equations, we refer to [2] to [5] and in particular to [11]. By this way we are able to improve previous results about time dependent Hartree-Fock equations which can be found in [6] to [9]. As in [11] we deal with L^2-solutions and with H^1 solutions ($L^2 = L^2(\mathbf{R}^n)$, $H^1 = H^1(\mathbf{R}^n)$, usual Sobolev space) with the modification realized by substituting these spaces respectively by $l^2(L^2)$ and by $l^2(H^1)$, where, in general, for a Banach space X $l^2(X)$ is the Banach space of sequences $\Phi = \{\varphi_j\}_{j \in \mathbf{N}}$ such that any φ_j belongs to X and $\|\Phi, l^2(X)\|^2 \overset{\text{def}}{=} \sum_{j \in \mathbf{N}} \|\varphi_j\|_X^2$ is finite. The choice of H^1 is due to the fact that in this space conservation of energy finds a natural expression, while L^2 is the natural space in which quantum mechanical problems are formulated.

Our purpose is to establish existence and uniqueness of solutions for the Cauchy problem associated to (0.2) written in integral form under suitable conditions on V and on v. Now we briefly anticipate the content of the paper. The first result we state is about H^1 solutions: under the assumption

$$V \in L^{q_1} + L^{q_2}, \qquad \frac{n}{q_i} < 2,$$
$$v \in L^{p_1} + L^{p_2}, \qquad \frac{n}{p_i} < 4,$$

($f \in L^{r_1} + L^{r_2}$ means $f = f_1 + f_2$ with $f_i \in L^{r_i}$ and, in general, $L^r = L^r(\mathbf{R}^n)$, $r \geqq 1$) we prove local existence and uniqueness. The proof is made via an approximating procedure and relies crucially on certain properties of the free evolution group $e^{-it\Delta}$ that we will resume in the following section. The approximation procedure is performed by cutting the potentials V and v. This allows the use of a result about abstract equation in Banach spaces that one can find in [10]. After having proved existence and uniqueness of solutions we derive conservation of $l^2(\mathrm{L}^2)$-norm and of the energy and under the further assumption that it is

$$v_- \in \mathrm{L}^{\bar{p}_1} + \mathrm{L}^{\bar{p}_2}, \qquad \frac{n}{\bar{p}_i} < 2,$$

where $v_- = \max\{0, -v\}$ is the negative part of the two-body interaction potential, we prove that solutions can be prolonged up to infinity. The second result regards L^2 solutions and the proof of local existence and uniqueness requests

$$\mathrm{V} \in \mathrm{L}^{q_1} + \mathrm{L}^{q_2}, \qquad \frac{n}{q_i} < 2$$

$$v \in \mathrm{L}^{p_1} + \mathrm{L}^{p_2}, \qquad \frac{n}{p_i} < 2,$$

and is a classical contracting mapping argument. Then by a density argument and using the previous result we show that solutions are actually global. For the sake of simplicity in the whole proving frame we write $\mathrm{V} \in \mathrm{L}^q$ and $v \in \mathrm{L}^p$, $v_- \in \mathrm{L}^{\bar{p}}$ omitting the decomposition in two addenda; one can easily convince himself that the proof in the general case would request only a heavier notation.

The paper is organized as follows. Section 1 contains some well known results about basic tools in the study of nonlinear Schrödinger equations together with the definition of the quantities and of the notation which will be used in the paper. In Section 2 we simply list some inequalities used in section 3 and 4 that we have collected for compactness. Section 3 is about approximating solutions, their existence and properties, while section 4 contains the results stated in a correct form.

1. PRELIMINARIES AND NOTATIONS

We start with the definition of a series of those quantities useful to write and to study equation (0.2) in the way specified in the introduction.

DEFINITION 1.1. – Let $\mathrm{V} : \mathbf{R}^n \to \mathbf{R}$, $v : \mathbf{R}^n \to \mathbf{R}$ even. For the sequences $\Phi = \{\varphi_j\}_{j \in \mathbf{N}}$, $\Psi = \{\psi_j\}_{j \in \mathbf{N}}$, $\Omega = \{\omega_j\}_{j \in \mathbf{N}}$. $\Lambda = \{\lambda_j\}_{j \in \mathbf{N}}$, we set:

(i) $\tilde{f}_{v,j}(\Phi, \Psi, \Omega) = \sum_{k \in \mathbf{N}} (\varphi_j v * (\psi_k \bar{\omega}_k) - \psi_k v * (\varphi_j \bar{\omega}_k))$,

(ii) $\tilde{\mathrm{F}}_v(\Phi, \Psi, \Omega) = \{\tilde{f}_{v,j}(\Phi, \Psi, \Omega)\}_{j \in \mathbf{N}}$,

(iii) $F_v(\Phi)=\tilde{F}_v(\Phi, \Phi, \Phi)$,

(iv) $\tilde{p}_{v,j}(\Phi, \Psi, \Omega, \Lambda)=\frac{1}{2}\sum_{k\in\mathbf{N}}((\varphi_j\bar{\psi}_j)v*(\omega_k\bar{\lambda}_k)-(\bar{\psi}_j\omega_k)v*(\varphi_j\bar{\lambda}_k))$,

(v) $\tilde{P}_v(\Phi, \Psi, \Omega, \Lambda)=\{\tilde{p}_{v,j}(\Phi, \Psi, \Omega, \Lambda)\}_{j\in\mathbf{N}}$,

(vi) $p_{v,j}(\Phi)=\tilde{p}_{v,j}(\Phi, \Phi, \Phi, \Phi)$,

(vii) $P_v(\Phi)=\tilde{P}_v(\Phi, \Phi, \Phi, \Phi)$,

(viii) we define the energy:

$$E(\Phi)=\sum_{j\in\mathbf{N}}\left[\int|\nabla\varphi_j|^2+\int V|\varphi_j|^2+\int p_{v,j}(\Phi)\right].$$

We make also the convention that $\Phi.\Psi=\{\varphi_j\psi_j\}_{j\in\mathbf{N}}$, $|\Phi|^2=\{|\varphi_j|^2\}$.

It is worth remarking that whenever they exist $E(\Phi)$ is real and that $v\geqq 0$ implies $\sum_{j\in\mathbf{N}} p_{v,j}(\Phi)\geqq 0$.

With these positions the Cauchy problem is written in the following form:

$$i\frac{d\Phi}{dt}=-\Delta\Phi+V\Phi+F_v(\Phi), \qquad \Phi(0)=\Phi_0. \tag{1.1}$$

We now introduce the basic notations used in the paper. We denote by n the dimension of the space assuming $n\geqq 2$ and with $\|.\|_p$ the L^p norm ($L^p=L^p(\mathbf{R}^n)$); for $r\geqq 1$ we write $r'=\frac{r}{r-1}$ and for any $k\in\mathbf{Z}$. $H^k=H^k(\mathbf{R}^n)$ is the usual Sobolev's space and we remind that setting $2^*=\frac{2n}{n-2}$ for any $r\in[2, 2^*[$ and for any $\varphi\in H^1$ it holds true that

$$\|\varphi\|_r\leqq C_{r,n}\|\varphi\|_2^{1-\delta(r)}\|\nabla\varphi\|_2^{\delta(r)}, \tag{1.2}$$

where $\delta(r)=\frac{n}{2}-\frac{n}{r}$; and also is valid the continuous embedding:

$$L^{r'}\hookrightarrow H^{-1}. \tag{1.3}$$

For I interval of $\mathbf{R}$ and X Banach (or Hilbert) space we call $L^p(I, X)$ the Banach space of functions $f: I\to X$ strongly measurable with the condition $\|f, L^p(I, X)\|^p\overset{\text{def}}{=}\int_I\|f(t)\|_X^p\,dt<+\infty$. Moreover $l^p(X)$ is the Banach space of sequences $\Phi=\{\varphi_j\}_{j\in\mathbf{N}}$ such that $\varphi_j\in X$ for any $j\in X$ for any $j\in\mathbf{N}$ and $\|\Phi, l^p(X)\|^p\overset{\text{def}}{=}\sum_{j\in\mathbf{N}}\|\varphi_j\|_X^p<+\infty$ (with the obvious definition of the sum and of external product). We remark that $l^2(X)$, eventually endowed with the inner product $(\Phi, \Psi)_{l^2(X)} \stackrel{\text{def}}{=} \sum_{j \in \mathbf{N}} (\varphi_j, \psi_j)_X$, $l^2(X)$ is a Hilbert space.

For the sake of simplicity we set:

$$\|\Phi, l^2(L^2)\| = \|\Phi\|,$$

and

$$(\Phi, \Psi)_{l^2(L^2)} = (\Phi, \Psi).$$

As consequences of (1.2) and of (1.3) the following assertions are true:

– For any $r \in [2, 2^*[$ and for any $\Phi \in l^2(H^1)$:

$$\|\Phi, l^2(L^r)\| \leqq C_{r,n} \|\Phi\|^{1-\delta(r)} \|\nabla\Phi\|^{\delta(r)}. \tag{1.4}$$

– For any $r \in [2, 2^*[$ and for any $\Phi \in l^2(L^{r'})$:

$$\|\Phi, l^2(H^{-1})\| \leqq C_{r,n} \|\Phi, l^2(L^{r'})\|. \tag{1.5}$$

Introducing the free evolution group $S(t)$, formally defined by $e^{-it\Delta}$, the Cauchy problem (0, 1) is written in the integral form:

$$\Phi(t) = S(t)\Phi_0 - i \int_0^t S(t-\tau)[V\Phi(\tau) + F_v(\Phi(\tau))]\, d\tau.$$

Setting

$$(U\Psi)(t) = -i \int_0^t S(t-\tau)\Psi(\tau)\, d\tau,$$

we study the equation

$$\Phi = S(.)\Phi_0 + U[V\Phi + F_v(\Phi)]. \tag{1.6}$$

Now we give properties of $S(t)$ and of U.

LEMMA 1.1. – *Let* $r_1, r_2 \in [2, 2^*[$; $\rho_i = \frac{2}{\delta(r_i)}$, I *interval of* **R**. *Then there exists a positive constant* C, *such that, for any* $\Phi \in L^{\rho_2'}(I, l^2(L^{r_2'}))$:

$$\|U\Phi, L^{\rho_1}(I, l^2(L^{r_1}))\| \leqq C \|\Phi, L^{\rho_2'}(I, l^2(L^{r_2'}))\| \tag{1.7}$$

LEMMA 1.2. – *Let* $r \in [2, 2^*[$, $\rho = \frac{2}{\delta(r)}$, *I interval of* **R**. *Then there exists a positive constant* C *such that, for any* $\Phi \in l^2(L^2)$:

$$\|S(.)\Phi, L^\rho(I, l^2(L^r))\| \leqq C \|\Phi\| \tag{1.8}$$

The proofs of these two lemmas are based on the inequality:

$$\|S(t)\Phi, l^2(L^r)\| \leqq C |t|^{-\delta(r)} \|\Phi, l^2(L^{r'})\|,$$

see [11].

Now we define the main spaces used in the paper. I denotes the interval $[0, T]$, unless otherwise specified, we take $r \in [2, 2^*[$, $\rho = \dfrac{2}{\delta(r)}$, $T > 0$ and set:

$$X_r(T) = L^\infty(I, l^2(L^2)) \cap L^\rho(I, l^2(L^r)),$$

equipped with the intersection norm:

$$\|., X_r(T)\| = \|., L^\infty(I, l^2(L^2))\| + \|., L^\rho(I, l^2(L^r))\|;$$

and

$$Y(T) = L^\infty(I, l^2(H^1)).$$

By virtue of Holder inequality and of Sobolev embedding (1.4) it is easy to prove the following

LEMMA 1.3. – (i) *For any* $s \in [2, 2^*[$, $\sigma = \dfrac{2}{\delta(s)}$, *for any* $T > 0$, *for any* $\Phi \in Y(T)$:

$$\Phi \in L^\sigma(I, l^2(L^s))$$

and

$$\|\Phi, L^\sigma(I, l^2(L^s))\| \leqq T^{1/\sigma} \|\Phi, Y(T)\|. \tag{1.9}$$

(ii) *Given* $r \in [2, 2^*[$, $\rho = \dfrac{2}{\delta(r)}$, *for any* $s \in [2, r]$, $\sigma = \dfrac{2}{\delta(s)}$, *for any* $T > 0$ *and for any* $\Phi \in X_r(T)$:

$$\Phi \in L^\sigma(I, l^2(L^s))$$

and

$$\|\Phi, L^\sigma(I, l^2(L^s))\| \leqq \|\Phi, X_r(T)\|. \tag{1.10}$$

2. A PRIORI ESTIMATES

In this section we collect a series of inequalities used in the following. The most important device in proving them is property (1.7) of the free Schrödinger evolution group.

We assume:

$$V: \mathbf{R}^n \to \mathbf{R},$$
$$v: \mathbf{R}^n \to \mathbf{R} \quad \textit{even},$$
$$r \in [2, 2^*[.$$

We say that

- (V, r) satisfies H 1 if $V \in L^q$ and $\frac{n}{q} \leqq 2\delta(r)$,
- (v, r) satisfies H 2 if $v \in L^p$ and $\frac{n}{p} < 3\delta(r)+1$,
- (v, r) satisfies H 3 if $v \in L^p$, $\frac{n}{p} < 2$ *and* $\frac{n}{p} < 3\delta(r)+1$.

All over the section C denotes a positive constant, and $I=[0, T]$, $T \in \mathbf{R}^+$.

LEMMA 2.1. – (i) *Let* (V, r) satisfy H 1 *and let* $s_1 \in [2, r]$ *such that* $\frac{n}{q} = 2\delta(s_1)$. *Then, for any* $\Phi \in l^2(L^{s_1})$:

$$\| V\Phi, l^2(L^{s_1'}) \| \leqq \| V \|_q \| \Phi, l^2(L^{s_1}) \|. \tag{2.1}$$

(ii) *Let* (v, r) *satisfy* H 2 *and* $r_2 \in [2, 2^*[$ *and* $s_2 \in [2, r]$, *such that* $\frac{n}{p} = \delta(r_2) + 3\delta(s_2)$. *Then, for any* $\Phi, \Psi, \Omega \in l^2(L^{s_2})$:

$$\| \tilde{F}_v(\Phi, \Psi, \Omega), l^2(L^{r_2'}) \| \leqq 2 \| v \|_p \| \Phi, l^2(L^{s_2}) \| \| \Psi, l^2(L^{s_2}) \| \| \Omega, l^2(L^{s_2}) \| \tag{2.2}$$

(iii) *Let* (V, r) *and* (W, r) *satisfy* H 1 *with the same* q. *Let* $T>0$. *Then there exist* $C>0$ *and* $\mu>0$ *such that, for any* $\Phi, \Psi \in X_r(T)$, *and for any* $a \in [2, 2^*[$:

$$\| U[V\Phi - W\Psi], L^\alpha(I, l^2(L^a)) \| \leqq CT^\mu \| W \|_q \| \Phi - \Psi, X_r(T) \| + CT^\mu \| V - W \|_q \| \Phi, X_r(T) \|. \tag{2.3}$$

where $\alpha = \frac{2}{\delta(a)}$.

(iv) *Let* (v, r) *and* (w, r) *satisfy* H 2 *with the same* p. *Let* $T>0$. *Then there exist* $C>0$, $\nu>0$ *and* $\nu_0>0$ *such that for any* $\Phi, \Psi \in Y(T)$ *and for any* $a \in [2, 2^*[$:

$$\begin{aligned} \| U[F_v(\Phi) - F_w(\Psi)], L^\alpha(I, l^2(L^a)) \| &\leqq CT^\nu \| v \|_p [\| \Phi, Y(T) \|^2 + \| \Phi, Y(T) \| \| \Psi, Y(T) \| + \| \Psi, Y(T) \|^2] \\ &\quad \times \| \Phi - \Psi, X_r(T) \| + CT^{\nu_0} \| v - w \|_p \| \Psi, Y(T) \|^3. \end{aligned} \tag{2.4}$$

where $\alpha = \frac{2}{\delta(a)}$.

(v) *Let* (v, r) *satisfy* H 3. *Let* $T>0$. *Then there exist* $C>0$ *and* $\nu>0$ *such that for any* $\Phi, \Psi \in X_r(T)$ *and for any* $a \in [2, 2^*[$:

$$\| U[F_v(\Phi)-F_v(\Psi)], L^\alpha(I, l^2(L^a)) \| \\ \leqq CT^\nu \| v \|_p [\| \Phi, X_r(T) \|^2 + \| \Phi, X_r(T) \| \| \Psi, X_r(T) \| \\ + \| \Psi, X_r(T) \|^2] \| \Phi - \Psi, X_r(T) \|. \quad (2.5)$$

where $\alpha = \dfrac{2}{\delta(a)}$.

(vi) *Let* $V \in L^q$ *and* $\dfrac{n}{q} = 2\delta(s_3)$ *with* $s_3 \in [2, 2^*[$. *Then for any* Φ, $\Psi \in l^2(L^{s_3})$:

$$\| V(|\Phi|^2 - |\Psi|^2), l^1(L^1) \| \\ \leqq C \| V \|_q [\| \Phi, l^2(L^{s_3}) \| + \| \Psi, l^2(L^{s_3}) \|] \| \Phi - \Psi, l^2(L^{s_3}) \|. \quad (2.6)$$

(vii) *Let* $v \in L^p$ *and* $s_4 \in [2, 2^*[$ *such that* $\dfrac{n}{p} = 4\delta(s_4)$. *Then, for any* Φ, $\Psi \in l^2(L^{s_4})$:

$$\| P_v(\Phi) - P_v(\Psi), l^1(L^1) \| \\ \leqq C \| v \|_p . [\| \Phi, l^2(L^{s_4}) \|^3 + \| \Phi, l^2(L^{s_4}) \|^2 \| \Psi, l^2(L^{s_4}) \| \\ + \| \Phi, l^2(L^{s_4}) \| \| \Psi, l^2(L^{s_4}) \|^2 + \| \Psi, l^2(L^{s_4}) \|^3] . \| \Phi - \Psi, l^2(L^{s_4}) \|. \quad (2.7)$$

Proof: First we notice that $\delta(.)$ is a continuous increasing function from $[2, 2^*[$ to $[0,1[$ so the expressions chosen for $\dfrac{n}{q}$ and $\dfrac{n}{p}$ can be satisfied; then (i), (ii), (vi) are straightforward applications of Hölder's, Young's and Schwartz's inequalities.

(iii) Applying inequality (1.7) we have:

$$\| U[V\Phi - W\Psi], L^\alpha(I, l^2(L^a)) \| \leqq C \| V\Phi - W\Psi, L^{\sigma_1'}(I, l^2(L^{s_1'})) \|,$$

where s_1 is chosen in such a way that $\dfrac{n}{q} = 2\delta(s_1)$ for some $s_1 \in [2, r]$ $\left(\text{as usual } \sigma_1 = \dfrac{2}{\delta(s_1)}\right)$; then we get the result by (2.1), by Hölder's inequality and by (1.10) setting $\mu = 1 - \delta(s_1)$.

(iv) Again by (1.7):

$$\| U[F_v(\Phi) - F_w(\Psi), L^\alpha(I, l^2(L^a)) \| \leqq \| F_v(\Phi) - F_w(\Psi), L^{\rho_2'}(I, l^2(L^{r_2'})) \|,$$

where $r_2 \in [2, 2^*[$ is chosen in such a way that

$$\frac{n}{p} = \delta(r_2) + 3\delta(s_2); \qquad \left(\rho_2 = \frac{2}{\delta(r_2)}\right);$$

then (2.4) is a consequence of (1.9), of (1.10), of Hölder's inequality and of the decomposition:

$$F_v(\Phi)-F_w(\Psi)=\tilde{F}_v(\Phi-\Psi,\Phi,\Phi)+\tilde{F}_v(\Psi,\Phi-\Psi,\Phi)$$
$$+\tilde{F}_v(\Psi,\Psi,\Phi-\Psi)+\tilde{F}_{v-w}(\Psi,\Psi,\Psi),$$

setting $\nu=1-\frac{\delta(r_2)}{2}-\frac{\delta(s_2)}{2}$ and $\nu_0=1-\frac{\delta(r_2)}{2}$.

(v) The proof of (2.5) is analogous to that one of (2.4) and makes use of (1.10).

(vii) Start with the decomposition:

$$P_v(\Phi)-P_v(\Psi)=\tilde{P}_v(\Phi-\Psi,\Phi,\Phi,\Phi)$$
$$+\tilde{P}_v(\Psi,\Phi-\Psi,\Phi,\Phi)+\tilde{P}_v(\Psi,\Psi,\Phi-\Psi,\Phi)+\tilde{P}_v(\Psi,\Psi,\Psi,\Phi-\Psi),$$

then the thesis follows by Hölder's, Young's and Schwartz's inequalities.

We end this section with a

Remark 2.1. — Inequalities (2.1), (2.2), (2.6) and (2.7) hold true when $V\in L^\infty$ and $v\in L^\infty$, implying in particular that the map

$$\Phi\to V\Phi+F_v(\Phi)$$

is locally lipschitzian in $l^2(L^2)$.

3. APPROXIMATE SOLUTIONS

We consider in this section approximate solutions of the equation (1.6). The approximation is realized by cutting the potentials V and v. The purpose is then to take the limit of these solutions as the cut potentials grow to the original ones.

DEFINITION 3.1. — For $V\in L^q$, $q\geqq 1$; $v\in L^p$, $p\geqq 1$ and for $m\in\mathbf{N}$, we set:

$$\begin{aligned} V_m &= m && \text{if } V\geqq m\\ &\quad V && \text{if } |V|\leqq m\\ &\quad -m && \text{if } V\leqq -m,\\ v_m &= m && \text{if } v\geqq m\\ &\quad v && \text{if } |v|\leqq m\\ &\quad -m && \text{if } v\leqq -m,\end{aligned}$$

and we write:

$$F_{v_m}(\Phi)=F_m(\Phi),$$
$$p_{v_{m_j}}(\Phi)=p_{m,j}(\Phi),$$
$$P_{v_m}(\Phi)=P_m(\Phi),$$
$$E_m(\Phi)=\sum_{j\in\mathbf{N}}\left[\int|\nabla\varphi_j|^2+\int V_m|\varphi_j|^2+\int p_{m,j}(\Phi)\right].$$

We study the equation:

$$\Phi = S(.)\,\Phi_0 + U[V_m \Phi + F_m(\Phi)] \tag{3.1 m}$$

in $L^\infty(0, T, l^2(L^2))$.

LEMMA 3.1. – *Let* $V \in L^q$, $q \geqq 1$; $v \in L^p$, $p \geqq 1$; $m \in \mathbf{N}$ *and* $\Phi_0 \in l^2(H^2)$. *Then there exists a unique* $\Phi \in C^1(]0, +\infty[, l^2(L^2)) \cap C([0, +\infty[, l^2(H^2))$, *such that:*

(i) Φ *is a solution of* (3.1 *m*),
(ii) $\|\Phi(t)\| = \|\Phi_0\|$, $\forall t \geqq 0$,
(iii) $E_m(\Phi(t)) = E_m(\Phi_0)$, $\forall t \geqq 0$,
(iv) *Given* $T > 0$ *there exists a* $C > 0$ *such that if* $\Psi_0 \in l^2(H^2)$ *and*

$$\Psi \in C^1(]0, +\infty[, l^2(L^2)) \cap C([0, +\infty[, l^2(H^2))$$

is solution of $\Psi = S(.)\,\Psi_0 + U[V_m \Psi + F_m(\Psi)]$, *it holds true that:*

$$\|\Phi - \Psi, C([0, T], l^2(L^2))\| \leqq C \|\Phi_0 - \Psi_0\|$$

(*Continuous dependence from initial data*).

Proof. – By remark (2.1) the map

$$\Phi \to V_m \Phi + F_m(\Phi)$$

is locally lipschitzian from $l^2(L^2)$ into itself, then there exist $T_1 > 0$, depending only on $\|\Phi_0\|$, and a unique

$$\Phi \in C^1(]0, T_1[, l^2(L^2)) \cap C([0, T_1], l^2(H^2))$$

solution of (3.1 *m*) depending continuously on initial data and solution moreover of the classical Cauchy problem associated to (0.1) with $\varphi_j(0) = \varphi_j^{(0)}$ where $\Phi_0 = \{\varphi_j^{(0)}\}_{j \in \mathbf{N}}$. For a proof *see* [10]. Multiplying (0.1) by $\bar{\varphi}_l$ and the conjugate of (0.1) with subscript l by φ_j and subtracting one gets that

$$\frac{d}{dt}(\varphi_j(t), \varphi_l(t))_{L^2} = 0, \tag{3.2}$$

therefore the solution preserves $l^2(L^2)$-norm and then it can be prolonged up to infinity (*see* [10] again). Statement (iv) easily follows because the length of the interval of existence depens only on the norm of initial datum. To end the proof we notice that the map $t \to \|\nabla \varphi_j(t)\|^2$ is differentiable with respect to t and, for any j, it is:

$$\frac{d}{dt}\|\nabla \varphi_j\|_{L^2}^2 = -\operatorname{Re}\left(\frac{\partial \varphi_j}{\partial t}, \Delta \varphi_j\right)_{L^2},$$

then standard computation ensures energy conservation.

Remark 3.1. – It is of physical interest to notice that by virtue of (3.2), if Φ_0 is a sequence of orthonormal functions $\Phi(t)$ is still so.

LEMMA 3.2. — *Let* $V \in L^q$, $q \geqq 1$; $v \in L^p$, $p \geqq 1$; $m \in \mathbf{N}$; $T > 0$ *and* $\Phi_0 \in l^2(H^1)$. *Then there exist a unique* $\Phi \in C([0, T], l^2(H^1))$ *such that:*

(i) Φ *is solution of* (3.1 m) *in* $C([0, T], l^2(L^2))$,

(ii) $\|\Phi(t)\| = \|\Phi_0\|$, $\forall t \in [0, T]$,

(iii) $E_m(\Phi(t)) = E_m(\Phi_0)$, $\forall t \in [0, T]$.

Proof. — Let $\{\Phi_0^{(l)}\}_{l \in \mathbf{N}}$ be a sequence in $l^2(H^2)$ converging to Φ_0 in $l^2(H^1)$. For any l consider

$$\Phi^{(l)} \in C^1(]0, T[, l^2(L^2)) \cap C([0, T], l^2(H^2))$$

solution of (3.1 m) as ensured by previous lemma. By continuous dependence from initial data if follows that $\Phi^{(l)}$ is a Cauchy sequence in $C(I, l^2(L^2))$ $(I = [0, T])$ and one gets immediately that the limit $\Phi \in C(I, l^2(L^2))$ is solution of (3.1 m) and preserves $l^2(L^2)$-norm. By energy conservation, (2.6), (2.7) with $q = p = +\infty$ and by $l^2(L^2)$ norm conservation we have:

$$\|\nabla \Phi^{(l)}(t)\|^2 \leqq \|\nabla \Phi_0^{(l)}\|^2 + 4\|V_m\|_\infty \|\Phi_0^{(l)}\|^2 + 8\|v_m\|_\infty \|\Phi_0^{(l)}\|^4, \quad (3.3)$$

As a consequence of the uniform bound (3.3) Φ belongs to $L^\infty(I, H^1)$, is weakly continous in $l^2(H^1)$ and, by (1.4), $\Phi^{(l)} \overset{l \to \infty}{\to} \Phi$ in $C(I, l^2(L^k))$ for any $k \in [2, 2^*[$. Using again (2.6) and (2.7), it is easy to see that

$$E_m(\Phi_0^{(l)}) \overset{l \to \infty}{\to} E_m(\Phi_0), \quad (3.4)$$

and that

$$\left| \sum_{j \in \mathbf{N}} \left[\int V_m |\varphi_j^{(l)}(t)|^2 + \int p_{m,j}(\Phi^{(l)}(t)) \right] - \sum_{j \in \mathbf{N}} \left[\int V_m |\varphi_j(t)|^2 + \int p_{m,j}(\Phi(t)) \right] \right| \overset{l \to \infty}{\to} 0; \quad (3.5)$$

(3.4), (3.5) and weak lower semicontinuity of $l^2(H^1)$-norm imply

$$E_m(\Phi(t)) \leqq E_m(\Phi_0)$$

and then by a time reversal argument we finally get

$$E_m(\Phi(t)) = E_m(\Phi_0), \qquad \forall t \in [0, T].$$

To end the proof of the lemma we see that (2.6), (2.7) and continuity in $l^2(L^k)$, $k \in [2, 2^*[$ imply also that $t \to \|\nabla \Phi(t)\|^2$ is a continuous function and this fact, joined with weak continuity of Φ, ensures that Φ belongs to $C(I, l^2(H^1))$.

LEMMA 3.3. — *Let* $V \in L^q, \dfrac{n}{q} < 2$; $v \in L^p, \dfrac{n}{p} < 4$ *and* $\Phi_0 \in l^2(H^1)$. *Then there exist* $T > 0$ *and* $K > 0$, *depending only on* $\|\Phi_0, l^2(H^1)\|$, *such that if* $\Phi_m \in C([0, T], l^2(H^1))$ *is the solution of* (3.1 m) *ensured by lemma* (3.2), *then*

$$\|\Phi_m,\ L^\infty(0,\ T,\ l^2(H^1))\| \leqq K, \qquad \forall m \in \mathbf{N}.$$

Proof. – We choose a $r \in [2, 2^*[$ in such a way that (V, r) satisfies H 1 and (v, r) satisfies H 2, by virtue of energy conservation of (2.6), of (2.7) and by Sobolev inequality (1.4) there exist $C>0$, $\delta_1, \delta_2 \in [0, 1[$ such that

$$\begin{aligned}\|\nabla\Phi_m\|^2 \leqq \|\nabla\Phi_0\|^2 + C\|V\|_q.[\|\Phi_0\|^{1-\delta_1}\|\nabla\Phi_0\|^{\delta_1}\| \\ + \|\Phi_m(t)\|^{1-\delta_1}\|\nabla\Phi_m(t)\|^{\delta_1}].\|\Phi_m(t)-\Phi_0\|^{1-\delta_1}\|\nabla(\Phi_0-\Phi_m(t))\|^{\delta_1} \\ + C\|v\|_p[(\|\Phi_0\|^{1-\delta_2}\|\nabla\Phi_0\|^{\delta_2})^3 \\ + (\|\Phi_0\|^{1-\delta_2}\|\nabla\Phi_0\|^{\delta_2})^2(\|\Phi_m(t)\|^{1-\delta_2}\|\nabla\Phi_m(t)\|^{\delta_2}) \\ + (\|\Phi_0\|^{1-\delta_2}\|\nabla\Phi_0\|^{\delta_2})(\|\Phi_m(t)\|^{1-\delta_2}\|\nabla\Phi_m(t)\|^{\delta_2})^2 \\ + (\|\Phi_m(t)\|^{1-\delta_2}\|\nabla\Phi_m(t)\|^{\delta_2})^3]\|\Phi_m(t) \\ -\Phi_0\|^{1-\delta_2}\|\nabla(\Phi_0-\Phi_m(t))\|^{\delta_2}.\end{aligned} \tag{3.6}$$

Then, defining

$$b(t) = 1 + \|\nabla\Phi_m(t)\|^2$$

and reminding $l^2(L^2)$-norm conservation of solution, we get:

$$\begin{aligned}b(t) \leqq b(0) + C\,b(t)^{\delta_1}\|\Phi_m(t)-\Phi_0\|^{1-\delta_1} \\ + C\,b(t)^{2\delta_2}\|\Phi_m(t)-\Phi_0\|^{1-\delta_2},\end{aligned} \tag{3.7}$$

where C now depends on $\|\Phi_0,\ l^2(H^1)\|$, $\|v\|_p$ and on $\|V\|_q$. We now notice that the equation

$$\Phi_m = S(.)\Phi_0 + U[V_m\Phi + F_m(\Phi_m)]$$

holds true in $C(I, l^2(L^2))$ and also in $C(I, l^2(H^{-2}))$; local lipschitzianity of $\Phi \to V_m\Phi + F_m(\Phi)$ in $l^2(L^2)$ and the regularity theorem expressed in [10] imply:

$$i\frac{d\Phi_m}{dt} = -\Delta\Phi_m + V_m\Phi_m + F_m(\Phi_m) \tag{3.8}$$

in $l^2(H^{-2})$. By virtue of embedding (1.3) and of inequalities (2.1) and (2.2) all terms in the right hand side of equation (3.8) lie in $l^2(H^{-1})$ and moreover

$$\begin{aligned}\left\|\frac{d}{dt}\Phi_m,\ l^2(H^{-1})\right\| \leqq \|\Phi_m(t),\ l^2(H^1)\| + C\|V\|_q\|\Phi_m(t),\ l^2(H^1)\| \\ + C\|v\|_p\|\Phi_m(t),\ l^2(H^1)\|^3 \leqq C(1+\|\nabla\Phi_m(t)\|^2)^{3/2} = C\,b(t)^{3/2}.\end{aligned} \tag{3.9}$$

Now by $l^2(H^1) - l^2(H^{-1})$ duality:

$$\begin{aligned}\|\Phi_m(t)-\Phi_0\|^2 \leqq \|\Phi_m(t)-\Phi_0,\ l^2(H^1)\|\,\|\Phi_m(t)-\Phi_0,\ l^2(H^{-1})\| \\ \leqq \|\Phi_m(t)-\Phi_0,\ l^2(H^1)\|\int_0^t\left\|\frac{d}{ds}\Phi(s),\ l^2(H^{-1})\right\|ds \\ \leqq C\,b(t)^{1/2}\big(\sup_{s\in[0,t]} b(s)\big)^{3/2}\,t.\end{aligned} \tag{3.10}$$

Inserting (3.9) and (3.10) in (3.7) we have

$$b(t) \leqq b(0) + C\, b(t)^{(3\delta_1+1)/4}\, t^{(1-\delta_1)/2} \big(\sup_{s \in [0,t]} b(s)\big)^{3(1-\delta_1)/4}$$
$$+ C\, b(t)^{(7\delta_2+1)/4}\, t^{(1-\delta_1)/2} \big(\sup_{s \in [0,t]} b(s)\big)^{3(1-\delta_2)/4}. \quad (3.11)$$

Introducing

$$B(t) = \sup_{s \in [0,t]} b(s)$$

(3.11) becomes

$$B(t) \leqq B(0) + C\,(t^{(1-\delta_1)/2} + t^{(1-\delta_2)/2})\, B(t)^{\beta},$$

for some positive β, and choosing T in such a way that

$$C\,(T^{(1-\delta_1)/2} + T^{(1-\delta_2)/2})\,(2\,B(0))^{\beta} < B(0)$$

we finally get that $B(t) \leqq 2\,B(0)$ for any $t \in [0, T]$; hence the result.

4. RESULTS

We are ready to give the main results of the paper which are about equation (1.6).

PROPOSITION 4.1. – *Let* $V \in L^q$ *with* $\frac{n}{q} < 2$, $v \in L^p$ with $\frac{n}{p} < 4$, $\Phi_0 \in l^2(H^1)$. *Then there exist* $T^* \in]0, +\infty]$ *and a* $\Phi \in C([0, T^*[, l^2(H^1))$ *such that:*

(i) Φ *is solution of* (1.6) *in* $C([0, T^*[, l^2(L^2))$;

(ii) Φ *is unique in* $L^\infty([0, T_0], l^2(H))$ *for any* $T_0 \in]0, T^*[$;

(iii) $\|\Phi(t)\| = \|\Phi_0\|$ *for any* $t \in [0, T^*[$;

(iv) $E(\Phi(t)) = E(\Phi_0)$ *for any* $t \in [0, T^*[$;

(v) *One of the following properties holds true:*

1. $T^* = +\infty$

2. $T^* < \infty$ *and* $\lim_{t \nearrow T^*} \|\Phi(t), l^2(H^1)\| = +\infty$

(vi) *If moreover* $v_- = \max\{0, -v\} \in L^{\bar{p}}$ *with* $\frac{n}{\bar{p}} < 2$ *then* $T^* = +\infty$.

Proof: Choose $r \in [2, 2^*[$ in such a way that (V, r) satisfies H 1 and (v, r) satisfies H 2 and consider the sequence of approximate solutions $\{\Phi_m\}_{m \in \mathbf{N}}$ in $C([0, T], l^2(H^1))$ with the same initial value Φ_0 whose existence was proved in lemma 3.2. By lemma 3.3 if T is small enough

$$K = \sup_{m \in \mathbf{N}} \{\|\Phi_m, Y(T)\|\} < +\infty. \quad (4.1)$$

From (1.7), (1.9), (2.3), and (2.4), for any $a \in [2, 2^*[\left(\alpha = \frac{2}{\delta(a)}\right)$ it is:

$$\begin{aligned}
&\| \Phi_m - \Phi_l, \mathrm{L}^\alpha(\mathrm{I}, l^2(\mathrm{L}^a)) \| \\
&\quad = \| \mathrm{U}[\mathrm{V}_m \Phi_m - \mathrm{V}_l \Phi_l] + \mathrm{U}[\mathrm{F}_m(\Phi_m) - \mathrm{F}_l(\Phi_l)], \mathrm{L}^\alpha(\mathrm{I}, l^2(\mathrm{L}^a)) \| \\
&\quad \leqq \mathrm{CT}^\mu \| \mathrm{V} \|_q \| \Phi_m - \Phi_l, \mathrm{X}_r(\mathrm{T}) \| + \mathrm{CT}^\mu (1 + \mathrm{T}^{1/\rho}) \mathrm{K} \| \mathrm{V}_m - \mathrm{V}_l \|_q \\
&\qquad + 3\, \mathrm{CT}^\nu \| v \|_p \mathrm{K}^2 \| \Phi_m - \Phi_l, \mathrm{X}_r(\mathrm{T}) \| + \mathrm{CT}^{\nu_0} \mathrm{K}^3 \| v_m - v_l \|_p, \qquad (4.2)
\end{aligned}$$

$\left(\rho = \frac{2}{\delta(r)}\right)$ by setting in (4.2) first $a=2$ and then $a=r$ and imposing T such that:

$$2\,\mathrm{C} \| \mathrm{V} \|_q \mathrm{T}^\mu + 6\, \mathrm{CK}^2 \mathrm{T}^\nu \| v \|_p < 1 \qquad (4.3)$$

we get that $\{\Phi_m\}$ is a Cauchy sequence in $\mathrm{X}_r(\mathrm{T})$. As in lemma 3.2, by (4.1) the limit Φ belongs to $\mathrm{L}^\infty(\mathrm{I}, l^2(\mathrm{H}^1))$, is weakly continuous in $l^2(\mathrm{H}^1)$ and $\Phi_m \overset{m \to \infty}{\to} \Phi$ in $\mathrm{C}(\mathrm{I}, l^2(\mathrm{L}^k))$ for any $k \in [2, 2^*[$.

Using again (1.7), (1.9), (2.3) and (2.4):

$$\begin{aligned}
&\| \Phi - \mathrm{S}(.)\Phi_0 - \mathrm{U}[\mathrm{V}\Phi + \mathrm{F}_v(\Phi)], \mathrm{X}_r(\mathrm{T}) \| \\
&\qquad \leqq \| \Phi - \Phi_m, \mathrm{X}_r(\mathrm{T}) \| + 2\, \mathrm{CT}^\mu \| \mathrm{V} \|_q \| \Phi - \Phi_m, \mathrm{X}_r(\mathrm{T}) \| \\
&\quad + 2\, \mathrm{CT}^\mu (1 + \mathrm{T}^{1/\rho}) \| \mathrm{V}_m - \mathrm{V} \|_q + 6\, \mathrm{CT}^\nu \| v \|_p \mathrm{K}^2 \| \Phi - \Phi_m, \mathrm{X}_r(\mathrm{T}) \| \\
&\qquad\qquad + 2\, \mathrm{CT}^{\nu_0} \mathrm{K}^3 \| v_m - v \|_p \overset{m \to \infty}{\to} 0,
\end{aligned}$$

so Φ is solution of (1.6) in $\mathrm{X}_r(\mathrm{T})$.

Obviously $\| \Phi(t) \| = \| \Phi_0 \|$ for any $t \in \mathrm{I}$ and by virtue of (2.6), (2.7) by convergence in $\mathrm{C}(\mathrm{I}, l^2(\mathrm{L}^k))$, $k \in [2, 2^*[$ and by weak lower semicontinuity of $l^2(\mathrm{H}^1)$-norm:

$$\mathrm{E}(\Phi_0) \geqq \mathrm{E}(\Phi(t)).$$

Then energy conservation follows from a time reversal argument.

It is still a consequence of (2.6) and of (2.7) that the map:

$$t \to \sum_{j \in \mathbf{N}} \left[\int \mathrm{V} | \varphi_j(t) |^2 + \int p_j(\Phi(t)) \right]$$

is continuous, and this implies that $t \to \| \Phi(t), l^2(\mathrm{H}^1) \|$ is continuous. Finally, reminding that Φ is weakly continuous in $l^2(\mathrm{H}^1)$, we have:

$$\Phi \in \mathrm{C}([0, \mathrm{T}], l^2(\mathrm{H}^1)).$$

Properties (i), (ii), (iv) are proved on an interval of length T depending on

$$\| \Phi_0, l^2(\mathrm{H}^1) \|.$$

Assuming $\Phi(T)$ as initial datum the solution can be prolonged up to a T^* such that

$$\lim_{t \nearrow T^*} \|\Phi(t), l^2(H^1)\| = +\infty,$$

otherwise $T^* = +\infty$.

To prove uniqueness we take a $T_0 \in]0, T^*[$ and let $\Phi, \Psi \in L^\infty(0, T_0, l^2(H^1))$ be two solution of (1.6); we set

$$M = \max\{\|\Phi, L^\infty(0, T_0, l^2(H^1))\|, \|\Psi, L^\infty(0, T_0, l^2(H^1))\|\};$$

if $T_1 < T_0$ (2.3) with $V = W$ and (2.4) with $v = w$ imply:

$$\|\Phi - \Psi, X_r(T_1)\| \leqq 2CT_1^\mu \|V\|_q \|\Phi - \Psi, X_r(T_1)\| + 6CT_1^\nu \|v\|_p M^2 \|\Phi - \Psi, X_r(T_1)\|,$$

taking T_1 small enough we get $\Phi = \Psi$ on $[0, T_1]$ then, by iteration, $\Phi = \Psi$ on $[0, T_0]$.

To prove the last assertion we notice that by (2.6) and (2.7) with $\Psi = 0$, by conservation laws and by (1.4) we get:

$$\begin{aligned}\|\nabla\Phi(t)\|^2 &= E(\Phi_0) - \sum_{j \in \mathbf{N}} \int [V|\varphi_j|^2 + p_{v,j}(\Phi(t))] \\ &\leqq E(\Phi_0) + \|V|\Phi(t)|^2, l^1(L^1)\| + \|P_{v_-}(\Phi(t)), l^1(L^1)\| \\ &\leqq E(\Phi_0) + C\|V\|_q \|\Phi(t)\|^{2-(n/q)} \|\nabla\Phi(t)\|^{n/q} \\ &\quad + C\|v_-\|_{\bar p} \|\Phi(t)\|^{4-(n/\bar p)} \|\nabla\Phi(t)\|^{n/\bar p}; \qquad (4.4)\end{aligned}$$

setting $b(t) = 1 + \|\nabla\Phi(t)\|^2$ (4.4) becomes

$$b(t) \leqq C_1 + C_2 b(t)^\theta$$

with $C_1, C_2 > 0$ depending on $\|\Phi_0\|$, $\|V\|_q$, and $\|v_-\|_{\bar p}$, and $\theta \in]0, 1[$. Then $\|\nabla\Phi(t)\|$ is bounded and hence, by (v) $T^* = +\infty$.

This concludes the proof.

We consider now L^2-solutions of (1.6). We set up a contracting mapping procedure to state existence and uniqueness on a small interval $[0, T]$ whose length depends only on the $l^2(L^2)$-norm of intial datum. Then globality of the H^1-solutions ensured by previous Proposition imply, without any further conditions, prolongability of L^2-solutions up to infinity via a density argument. We start with a

DEFINITION 4.1. – Let $R > 0$, $T > 0$; we set

$$\tilde{X}_r(T, R) = \{\Phi \in X_r(T) : \|\Phi, X_r(T)\| \leqq R\}.$$

$\tilde{X}_r(T, R)$ equipped with the distance induced by $X_r(T)$-norm is a complete metric space.

For $\Phi_0 \in l^2(L^2)$ and $\Phi \in X_r(T)$ we set:

$$Q_{\Phi_0}(\Phi) = S(.) + U[V\Phi + F_v(\Phi)].$$

LEMMA 4.1. – *Let* (V, r) *satisfy* H 1 *and* (v, r) *satisfy* H 3. *Then, given* D>0, *there exist* R>0 *and* T>0 *such that, for any* $\Phi_0 \in l^2(L^2)$ *with* $\|\Phi_0\| \leqq D$, Q_{Φ_0} *is a contraction on* $\tilde{X}_r(T, R)$. *Moreover continuous dependence on initial data still holds, that is to say that there exists a positive constant* C *such that, given* $\Psi_0 \in l^2(L^2)$ *with* $\|\Psi_0\| \leqq D$ *and denoting by* Φ *and* Ψ *the fixed points of* Q_{Φ_0} *and of* Q_{Ψ_0} *respectively, it is:*

$$\|\Phi-\Psi, X_r(T)\| \leqq C \|\Phi_0-\Psi_0\|. \tag{4.5}$$

Proof. – We take $a \in [2, 2^*[$, $\alpha=\dfrac{2}{\delta(a)}$, T>0, Φ, $\Psi \in X_r(T)$, using (1.7), (1.8), (2.3) with W=0, and (2.5) we get:

$$\begin{aligned}\|Q_{\Phi_0}(\Phi)-Q_{\Psi_0}(\Psi), L^\alpha(I, l^2(L^a))\| \\ \leqq C\|\Phi_0-\Psi_0\|+C\|V\|_q T^\mu \|\Phi-\Psi, X_r(T)\| \\ +C\|v\|_p T^\nu [\|\Phi, X_r(T)\|^2+\|\Phi, X_r(T)\|\,\|\Psi, X_r(T)\| \\ +\|\Psi, X_r(T)\|^2]\,\|\Phi-\Psi, X_r(T)\|. \end{aligned}\tag{4.6}$$

Now we take (4.6), inserting $\Phi \in \tilde{X}_r(T, R)$, $\Phi_0 \in l^2(L^2)$ with $\|\Phi_0\| \leqq D$, $\Psi=0$, $\Psi_0=0$, first with $a=2$ and subsequently with $a=r$ and summing, we get

$$\|Q_{\Phi_0}(\Phi), X_r(T)\| \leqq 2CD+2C\|V\|_q T^\mu R+2C\|v\|_p T^\nu R^3, \tag{4.7}$$

then putting in (4.6) $\Psi_0=\Phi_0$ and $\Psi \in \tilde{X}_r(T, R)$, $a=2$ and $a=r$ and summing:

$$\begin{aligned}\|Q_{\Phi_0}(\Phi)-Q_{\Phi_0}(\Psi), X_r(T)\| \leqq (2C\|V\|_q T^\mu \\ +6C\|v\|_p T^\nu R^2)\,\|\Phi-\Psi, X_r(T)\|; \end{aligned}\tag{4.8}$$

then, choosing R and T in such a way that

$$2CD+2C\|V\|_q T^\mu R+2C\|v\|p T^\nu R^3 \leqq R \tag{4.9}$$
$$2C\|V\|_q T^\mu+6C\|v\|_p T^\nu R^2<1, \tag{4.10}$$

(4.7) and (4.8) imply that Q_{Φ_0} is a contracting map on $\tilde{X}_r(T, R)$.

Let now Φ and Ψ be fixed points of Q_{Φ_0} and of Q_{Ψ_0}, consider again (4.6) with $a=2$ and $a=r$, it is:

$$\begin{aligned}\|\Phi-\Psi, X_r(T)\|=\|Q_{\Phi_0}(\Phi)-Q_{\Psi_0}(\Psi), X_r(T)\| \\ \leqq 2C\|\Phi_0-\Psi_0\|+2C\|V\|_q T^\mu \|\Phi-\Psi, X_r(T)\| \\ +6C\|v\|_p T^\nu R^2 \|\Phi-\Psi, X_r(T)\|. \end{aligned}\tag{4.11}$$

(4.10) and (4.11) imply continuous dependence (4.5).

PROPOSITION 4.2. – *Let* $V \in L^q$ *with* $\dfrac{n}{q}<2$, $v \in L^q$ *with* $\dfrac{n}{p}<2$. *Let* $\Phi_0 \in l^2(L^2)$. *Then there exists a* $\Phi \in C([0, +\infty[, l^2(L^2))$ *such that:*

(i) Φ *is solution of* (1.6) *in* $C([0, +\infty[, l^2(L^2))$;

(ii) $\Phi \in L^\alpha(0, T, l^2(L^a))$, $\forall T>0$, $\forall a \in [2, 2^*[\left(\alpha=\dfrac{2}{\delta(a)}\right)$;

(iii) Φ *is unique in* $L^\infty(0, T, l^2(L^2)) \cap L^\rho(0, T, l^2(L^r))\ \forall T>0$, *where* $r \in [2, 2^*[$ *is such that* (V, r) *satisfy* H 1 *and* (v, r) *satisfy* H 2 $\left(\rho = \dfrac{2}{\delta(r)}\right)$;

(iv) $\|\Phi(t)\| = \|\Phi_0\|,\ \forall t \geqq 0$.

Proof. – We choose r in such a way that (V, r) satisfy H 1 and (v, r) satisfy H 3, by lemma 4.2 there exists a solution of (1.6) on an interval $[0, T]$ sufficiently small, first we prove that it belongs to $C([0, T], l^2(L^2))$ and that it preserves $l^2(L^2)$-norm. By (1.3), (2.1), (2.2) and absolute continuity of the integral $\Phi \in C([0, T], l^2(H^{-1}))$, so Φ is weakly continuous in $l^2(L^2)$. Let $\{\Phi_0^{(l)}\}_{l \in N}$ be a sequence in $l^2(H^1)$ such that

$$\|\Phi_0^{(l)}\| \leqq \|\Phi_0\| \tag{4.12}$$

and

$$\Phi_0^{(l)} \overset{l \to \infty}{\to} \Phi_0$$

in $l^2(L^2)$. By virtue of Proposition 4.1, and reminding that $\dfrac{n}{p} < 2$ for any $l \in N$ it exists

$$\Phi^{(l)} \in C([0, +\infty[, l^2(H^1))$$

solution of (1.6) preserving $l^2(L^2)$-norm. Inequality (4.12) implies that Φ and the restriction of $\Phi^{(l)}$ are fixed points of Q_{Φ_0} and of $Q_{\Phi_0^{(l)}}$ respectively on the same closed ball of $X_r(T)$; then by continuous dependence from initial data (4.5), from (4.13) and passing to the limit $l \to \infty$, $\|\Phi(t)\| = \|\Phi_0\|$ for any $t \in [0, T]$. So continuity in $l^2(L^2)$) and prolungability are proved. Statements (ii) and (iii) are obvious.

Remark 4.1. – By remark 3.1 and by the features of the construction made up in Proposition 4.1 and in proposition 4.2 [*i.e.* $C(I, l^2(L^2))$-convergence] both L^2-solutions and H^1-solutions maintain the property expressed in remark 3.1.

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