ANNALES DE L'I. H. P., SECTION A

S. JANECZKO

Coisotropic varieties and their generating families

Annales de l'I. H. P., section A, tome 56, nº 4 (1992), p. 429-441
<http://www.numdam.org/item?id=AIHPA_1992__56_4_429_0>

Ann. Inst. Henri Poincaré,

Vol. 56, n° 4, 1992, p. 429-441. *Physique théorique*

Coisotropic varieties and their generating families

by

S. JANECZKO

Institute of mathematics, Technical University of Warsaw,
Pl. Jednośći Robotniczej 1, 00-661 Warsaw, Poland

ABSTRACT. — Generating families for coisotropic varieties are introduced. The local structure of singularities of coisotropic varieties and reduced symplectic structures is studied. The relationship with a previously proposed caustic equivalence of unfoldings is pointed out.

RÉSUMÉ. — Nous introduisons des familles génératrices pour des variétés coisotropes et nous étudions la strucutre locale des singularités ainsi que la structure symplectique réduite. Nous montrons les relations avec une notion d'équivalence de caustique des déploiements.

1. INTRODUCTION

The most important objects in symplectic geometry, after the symplectic manifolds themselves, are the isotropic, lagrangian and coisotropic submanifolds. If W is a submanifold of a symplectic manifold (X, ω) then W is isotropic, Lagrangian or coisotropic if for every $x \in W$ the orthogonal space $W_x^\perp$ defined in (2.1) below, contains $T_x W$, is equal to $T_x W$ or is contained in $T_x W$, respectively. All types of these submanifolds appeared in many branches of mathematics and physics, e.g. in holonomic differential systems [P] and geometrical diffraction theory [K], [AG], in a multitude of ways in the symplectic interpretations of geometric phenomena, in classical and quantum physics etc. Coisotropic submanifolds are usually exploited to generate new symplectic manifolds, reduced ones [AM], [AVG], which are suitable candidates for phase spaces of physical systems. The main tool used in local investigations of lagrangian submanifolds of the cotangent bundle, say $(T^* M, \omega_M)$, is the notion of generating family ([We], Section 6) $F: M \times R^K \to R$, which defines a Lagrangian submanifold $L \subset T^* M$ by the following equations:

$$L = \left\{ (p, q);\ \text{there exists } \lambda \in R^K \text{ such that } p = \frac{\partial F}{\partial q}(q, \lambda),\ 0 = \frac{\partial F}{\partial \lambda}(q, \lambda) \right\}.$$

An idea of generating family for isotropic varieties was introduced by F. Pham [P]. In fact one can look at the isotropic variety as an intersection of several Lagrangian submanifolds. Let $F: M \times R \times R^K \to R$ define two generating families, the original one F and also $F_1: M \times \{0\} \times R^K \to R$. The isotropic variety generated by F is the intersection of the respective two Lagrangian varieties and given by the formulae:

$$I = \left\{ (p, q);\ \text{there exists } \lambda \in R^K \text{ such that } \right.$$
$$\left. p = \frac{\partial F}{\partial q}(q, 0, \lambda),\ 0 = \frac{\partial F}{\partial \lambda}(q, 0, \lambda),\ 0 = \frac{\partial F}{\partial \mu}(q, 0, \lambda) \right\}.$$

The main aim of this paper is to provide an analogous generating family setting for the local description of the coisotropic varieties and, in consequence, the various reduced symplectic structures. In contrast to the idea of intersection for isotropic varieties, the coisotropic varieties should be obtained as families of Lagrangian varieties; this suggests an appropriate notion of generating family for coisotropic variety introduced in Section 3. The various examples encountered in geometrical optics are presented in Section 4. A kind of relative Darboux theorem concerning local geometry of singular coisotropic varieties is obtained in Section 5. As a further application of the generating family approach, in Section 6 and 7 one can find the construction of generating families for Hamiltonian actions of Lie groups, and the local classification of prenormal generic forms for coisotropic varieties.

2. Reduced symplectic structures

Let (X, ω) be a symplectic manifold (*cf.* [AM]). Let $C \subset X$ be an embedded, connected, coisotropic submanifold of X, *i.e.* for every $x \in C$

(2.1) $C_x^{\perp} = \{ v \in T_x X;\ \langle v \wedge u, \omega \rangle = 0, \quad \text{for every } u \in T_x C \} \subset T_x C.$

Thus we see that $\dim C_x^{\perp} = \operatorname{codim} C$. $\{C_x^{\perp}\}$ forms the characteristic distribution of $\omega|_C$ defined by $d(\omega|_C)=0$. Thus the distribution $D = \bigcup_{x \in C} C_x^{\perp}$ is involutive. Maximal connected integral manifolds of D are called characteristics. They form the characteristic foliation of C (*cf.* [BT]). D represents the generalized hamiltonian system with Hamiltonian C.

Let Y be the set of characteristics of C. Let $\rho : C \to Y$ be a canonical projection along characteristics. Now, instead of ρ, we consider rather graph ρ,

$$R_C = \operatorname{graph} \rho = \{(x, y) \in X \times Y;\ y = \rho(x),\ x \in C\}$$

If Y admits a differentiable structure and ρ is a submersion, then there is a unique symplectic structure β on Y such that

$$\rho^* \beta = \omega|_C \qquad (cf.\ [AM]).$$

Thus we deduce that R_C is a Lagrangian submanifold of $X \times Y$ endowed with the symplectic form $\Omega = \pi_2^* \beta - \pi_1^* \omega$, where $\pi_i : X \times Y \to X\,(Y)$, $i = 1, 2$, are the natural projections. In fact R_C is isotropic, *i.e.* $\Omega|_{R_C} = \rho^* \beta - \omega|_C = 0$ and its dimension is equal to $\frac{1}{2}(\dim X + \dim Y)$.

We need to investigate the local properties of coisotropic varieties in general. Reversing back the above construction, we can start with the Lagrangian varieties in $(X \times Y, \Omega)$.

3. GENERATING FAMILIES FOR COISOTROPIC VARIETIES

We are interested in local properties of the reduced symplectic structure (Y, β) (*cf.* [We]). We can assume that $X \cong T^* M$, $Y \cong T^* N$, $M \cong R^m$, $N \cong R^n$. They are endowed with the Liouville symplectic structures ω_M and β_N respectively.

DEFINITION 3.1. – *The smooth function* (*germ*) $F : M \times N \times R^K \to R$ *is called a* C-*generating family if the smooth map* $(q, \alpha, \lambda) \to \frac{\partial F}{\partial \lambda}(q, \alpha, \lambda)$ *of* $M \times N \times R^k$ *to the fibre* R^K *is nonsingular on the stationary set*

$$\Sigma_F = \left\{(q, \alpha, \lambda);\ \frac{\partial F}{\partial \lambda}(q, \alpha, \lambda) = 0\right\}$$

and the smooth map $(q, \alpha, \lambda) \to \left(\frac{\partial F}{\partial \alpha}(q, \alpha, \lambda), \alpha\right)$ restricted to the stationary set Σ_F is a surjective map.

Let the above introduced condition hold on a collection of smooth components Σ'_F of Σ_F, *i.e.* hold on all Σ'_F. Σ'_F is then a smooth submanifold of $M \times N \times R^k$ of dimension $m+n$ and the image of $\Phi_F : \Sigma'_F \to T^* M$, $\Phi_F(q, \alpha, \lambda) = \left(\dfrac{\partial F}{\partial q}(q, \alpha, \lambda), q\right)$, defines a coisotropic variety generated by F;

$$(3.1)\quad C = \{(p, q) \in T^* M;\ \text{there exists } (\alpha, \lambda) \in N \times R^K \text{ such that } p_i = \partial F/\partial q_i (q, \alpha, \lambda),\ \partial F/\partial \lambda_k (q, \alpha, \lambda) = 0\}$$

$1 \leqq i \leqq m$, $1 \leqq k \leqq K$.

The coisotropic variety C of $(T^* M, \omega_M)$, generated by F, is provided by formula (3.1). Usually it is not a smooth submanifold and F is not a uniquely defined C-generating family. To describe the germs of coisotropic varieties we use the germs of C-generating families with minimal number of parameters $\{\lambda\}$. The germ $F : (M \times N \times R^K, (0, 0, 0)) \to R$ is called minimal if additionally

$$\left(\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}\right)(0, 0, 0) = 0, \qquad 1 \leqq i, \quad j \leqq K$$

DEFINITION 3.2. — *The germ of a C-generating family* $F : (M \times N \times R^K, (0, 0, 0)) \to R$ *is called a C-Morse family if the smooth map* $\Psi_F : (q, \alpha, \lambda) \to \left(\dfrac{\partial F}{\partial \alpha}(q, \alpha, \lambda), \alpha\right)$ *is regular on the stationary set* Σ_F.

If this condition holds on Σ'_F then the smooth map $\Phi_F : \Sigma'_F \to T^* M$ is an immersion of a coisotropic submanifold of $T^* M$. In this sense we say that an immersion $i : C \to T^* M$ of a coisotropic submanifold C is defined by F.

PROPOSITION 3.3. — *To each germ of a coisotropic (immersed) submanifold* $(C, 0) \subset T^* M$, $(M \cong R^m)$, *there exists a germ of a C-Morse family* $F : (M \times N \times R^K, (0, 0, 0)) \to R$, *such that* (C, 0) *is defined by* (3.1).

Proof. — By the standard lines of symplectic geometry (*cf.* [We] p. 26) R_C is a Lagrangian submanifold of $\Xi = (T^* M \times T^* N;\ \pi_2^* \beta_N - \pi_1^* \omega_M)$. It is generated, at least locally, by a Morse family, say $F : M \times N \times R \to R$, $K \leq \dim M + \dim N$, *i. e.*

$$R_C = \left\{((p, q), (\xi, \alpha)) \in T^* M \times T^* N;\ \text{there exists } \lambda \in R^K \text{ such that } p_i = \frac{\partial F}{\partial q_i}(q, \alpha, \lambda),\ \xi_j = \frac{\partial F}{\partial \alpha_j}(q, \alpha, \lambda),\ \frac{\partial F}{\partial \lambda_k}(q, \alpha, \lambda) = 0\right\},$$

$$1 \leqq i \leqq \dim M = m, \qquad 1 \leqq j \leqq \dim N = n, \qquad 1 \leqq k \leqq K.$$

F satisfies an extra condition

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial \lambda \partial \lambda}, \frac{\partial^2 F}{\partial \lambda \partial \alpha}, \frac{\partial^2 F}{\partial \lambda \partial q}\right) = K.$$

This completes the proof of the proposition. □

4. EXAMPLES

Example 4.1. – The space of optical rays.

Let (W, g) be a Riemannian manifold endowed with the metric tensor $g=n^2\left(\sum_{i=1}^{n} dx_i^2\right)$. Let X_g be the geodesic flow of g on (T^*W, ω_W). X_g is Hamiltonian vector field with Hamiltonian

$$H_g : T^*W \to R,\ H_g(p)=\frac{1}{2}\langle p, p\rangle_g,$$

where $\langle\ .,\ .\ \rangle_g$ is the inner product on T^*W induced by g, *i.e.*

$$\omega_W(X_g, .) = -dH_g(.).$$

If $\hat{s}:[a, b]\to T^*W$ is an integral curve of X_g then $s=\pi_W\circ\hat{s}:[a, b]\to W$ is a geodesic on (W, g). The space of all such projections of integral curves of X_g is called the space of optical rays. The space of optical rays is defined by a coisotropic level set of the Hamiltonian H_g.

4.1.1. $n : V\cong R^3\to R$, $n\equiv 1$; free particle space.

In the case $H_g : T^*V\to R, H_g(p, q)=\frac{1}{2}\|p\|^2$ and

$$C=\left\{(p, q)\in T^*V;\ H_g(p, q)=\frac{1}{2}\right\}.$$

The characteristics of C form the space of all oriented lines in V. It is easy to check that the C-Morse families for appropriate charts on C can be written in the form

$$F(q, \alpha)=q_1\alpha_1+q_2\alpha_2\pm q_3(1-\alpha_1^2-\alpha_2^2)^{1/2}$$

up to change of order of $\{q_i\}$.

4.1.2. $n : V\cong R^3\to R^1$, $n(q_1, q_2, q_3)=1/(1+q_1^2+q_2^2+q_3^2)$.

This system is called Maxwell's fisheye. Here we have

$$C=\{(p, q)\in T^*V : p_1^2+p_2^2+p_3^2=1/(1+q_1^2+q_2^2+q_3^2)^2\}.$$

Without loss of generality we can limit the investigations to rays which lie in the $q_1 q_2$-plane. In polar coordonates $q_1=r\cos\theta$, $q_2=r\sin\theta$, we find the C-Morse family for C in the form

$$F(r, \theta, \alpha)=\alpha\theta\pm\int_{r_0}^{r}(n^2-\alpha^2/r^2)^{1/2}\,dr$$

where we assumed that n is a continuous function of $r=(q_1^2+q_2^2)^{1/2}$.The Maxwell's fisheye is formed by rays which go through two points on opposite ends of a diameter of the unit circle (*cf.* [L]). The elements of this new symplectic space can be written down in the following way:

$$(q_1-\sqrt{R^2-1}\cos\beta)^2+(q_2-\sqrt{R^2-1}\sin\beta)^2=R^2, \qquad R\geqq 1$$

One can notice that the Riemannian metric, with refractive index $n=1/(1+r^2)$, is given by the stereographic projection of the line element of the sphere onto the $q_1 q_2$-plane. This fact explains the perfectness of the Maxwell's fisheye optical system.

Example 4.2. – Most of the coisotropic varieties provided by generating families tend to be singular. They are usually given parametrically. Let us consider the coisotropic variety generated by the C-generating family

$$F(q_1, q_2, \alpha, \lambda_1, \lambda_2, \lambda_3)=\lambda_1\lambda_2\lambda_3+\lambda_1 q_1+\lambda_2 q_2+\lambda_3\alpha.$$

Obviously this is not a C-Morse family. The coisotropic variety defined by it is given parametrically in the following form:

$$\begin{aligned} x_1 &= -\lambda_2\lambda_3,\\ x_2 &= -\lambda_1\lambda_3,\\ p_1 &= \lambda_1,\\ p_2 &= \lambda_2, \qquad (\lambda_1, \lambda_2, \lambda_3)\in R^3. \end{aligned}$$

The symplectic space of characteristics is parametrized by λ_3 and α with an additional equation $\dfrac{\partial F}{\partial\lambda_3}=\lambda_1\lambda_2+\alpha=0$.

Example 4.3. – Nested coisotropic hypersurfaces.

Let A be a hypersurface in a symplectic manifold (M, ω). Let Z be a symplectic manifold of bicharacteristics of A, $\pi_A: A\to Z$ is a natural projection. Let G be an another hypersurface of M meeting transversally A. We denote $J=G\cap A$. The inclusion of $J\subset A$ gives the projection map:

$$\pi_J := \pi_A|_J : J\to Z,$$

associating to each point of J the local bicharacteristic on A through it. Let $p\in A$, then there are Darboux coordinates at p, on M, in which

$$(*) \qquad \omega=\sum_{i=0}^{n} dp_i\wedge dq_i, \qquad A=\{p\in M;\ q_0=0\}$$

Now one can seek to reduce J to a simple form in local Darboux coordinates in which (*) holds. The tangency type of the Hamilton foliation of A to J is a natural invariant of this problem. Following [A], [Me], one can distinguish the four initial steps in the classification of normal form of J in Darboux coordinates, in which (*) holds,

0. $J_0=\{p\in M;\ q_0=0,\ p_0=0\}$,
1. $J_1=\{p\in M;\ q_0=0,\ p_1+p_0^2=0\}$,
2. $J_2=\{p\in M;\ q_0=0,\ -p_1+q_1 p_0+p_0^3=0\}$,
3. $J_3=\{p\in M;\ q_0=0,\ p_2+q_2 p_0+p_1 p_0^2+p_0^4=0\}$.

In these four cases the map π_J is a local diffeomorphism, has a fold singularity or a cusp singularity or a swallowtail singularity. The representative example of an application of these hypersurfaces is a description of geodesics on a Riemannian manifold with boundary where these four normal forms gives the complete generic classification [A]. In all the above listed cases $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ form Darboux coordinates on Z. The corresponding normal forms of nested hypersurfaces in Z, defined as a critical values $\Gamma\pi_J$ of π_J are listed as follows:

1. $\Gamma\pi_J = \{p_1 = 0\} \subset Z$,
2. $\Gamma\pi_J = \{(1/27)\, q_1^3 + (1/4)\, p_1^2 = 0\} \subset Z$,
3. $\Gamma\pi_J = \{(p, q) \in Z;\ \lambda^4 + p_1\lambda^2 + q_2\lambda + p_2$ has a multiple real root $\lambda\}$ $\subset \{(p, q) \in Z;\ 16\, p_1^4 p_2 - 4\, p_1^3 q_2^2 - 128\, p_1^2 p_2^2 + 144\, p_1 q_2^2 p_2 + 256\, p_2^3 - 27\, q_2^4 = 0\} \subset Z$.

The generating families for these three, singular, coisotropic hypersurfaces in local Darboux coordinates are given in the following forms,

1. $F(q, \alpha) = q_2\alpha_1 + \ldots + q_n\alpha_{n-1}$,
2. $F(\xi, q, \alpha) = (9/10)\,\xi^5 + q_1\xi^3 + (1/2)\, q_1^2\xi + \alpha_1 q_2 + \ldots + \alpha_{n-1} q_n$,
3. $F(\xi, q, \alpha) = (48/7)\,\xi^7 + (32/5)\,\alpha_1\xi^5 + (4/3)\,\alpha_1^2\xi^3 + q_2(3\xi^4 + \alpha_1\xi^2) + \alpha_1 q_1 + \alpha_2 q_3 + \ldots + \alpha_{n-1} q_n$.

Remark 4.4. – To each germ of a coisotropic submanifold in (T^*X, ω_X) there exists the corresponding germ of a C-Morse family (Proposition 3.3). It is an open question, whether to each germ of a coisotropic semialgebraic (or analytic in the complex case) variety in (T^*M, ω_M) there exists a germ of C-generating family.

5. THE GEOMETRY OF A COISOTROPIC VARIETY

Let (X, ω) be a complex $(\cong C^{2n})$ or real $(\cong R^{2n})$ symplectic manifold, $\omega = d\alpha$. Let $C \subset X$ be a coisotropic variety and $\rho: C \to Y$ be its canonical projection. Differential forms on X vanishing on all fibres $\rho^{-1}(y)$, $y \in Y$ form a subcomplex in the de Rham complex of X. The respective cochains we denote by $C^*(X, C_\rho)$, $\bar{d}: C^k(X, C_\rho) \to C^{k+1}(X, C_\rho)$. We define the factor complex,

$$(\Omega^*(C), \bar{d}) = (C^*(X)/C^*(X, C_\rho), \bar{d})$$

Here we have

$$\alpha \in C^1(X)/C^1(X, C_\rho)$$

and $\alpha \in \operatorname{Ker} \bar{d}$, so we can define the characteristic class of ω,

$$[\alpha] \in H^1(C).$$

By S_C we denote the space of all symplectic structures on X for which C is a coisotropic variety. We say that $\omega_1, \omega_2 \in S_C$ are equivalent iff there exists a diffeomorphism $\varphi:(X, C) \to (X, C)$, such that $\varphi^* \omega_1 = \omega_2$ and $\rho \circ \varphi|_C = \bar{\varphi} \circ \rho$, for some diffeomorphism $\bar{\varphi}: Y \to Y$. We consider the local stability of elements of S_C as a stability of the corresponding germs [GT]. Let ω be a germ of a symplectic structure on X. We write $\omega \in S_C$ iff it has a representative belonging to S_C.

Now we show the following stability result (*cf.* [G]).

PROPOSITION 5.1. — *Let* $\omega_1, \omega_2 \in S_C$, *be two close germs with the same characteristic class. Then* ω_1, ω_2 *are equivalent.*

Proof. — By differentiating the one-parameter family

$$\omega_t = (1-t)\,\omega_1 + t\,\omega_2, \qquad t \in [0, 1]$$

such that $g_t^* \omega_t = \omega_1$, we obtain the following equation

$$\mathscr{L}_{V_t} \omega_t + \omega_2 - \omega_1 = 0,$$

where g_t is the sought-for family of germs of diffeomorphisms preserving C, $g_0 = \mathrm{id}$, and V_t is its vector field. But $\omega_1 - \omega_2 = d\beta$, $\beta \in C^1(X, C_\rho)$ because ω_1, ω_2 have the same characteristic classes. So we have an equivalent equation

$$V_t \lrcorner\, \omega_t = \beta.$$

Now the vector field V_t is determined from this linear equation; it is uniquely defined because we assumed ω_t to be nondegenerate (*cf.* [Mo], p. 293). Existance of the solution of this equation is implied by the fact that the mapping $V \to V \lrcorner\, \omega$ is an isomorphism of the space of vector fields tangent to the fibration ρ onto the space $C^1(X, C_\rho)$. □

6. GENERATING FAMILIES FOR THE HAMILTONIAN ACTIONS

Let $\kappa: G \times M \to M$ be an action of a Lie group G on a symplectic manifold M. Let κ be a symplectic action, $\kappa_g^* \omega = \omega$, for each $g \in G$. A momentum mapping corresponding to κ is a differentiable, Ad*-equivariant mapping $J: M \to \mathfrak{g}^*$, such that, for every $X \in \mathfrak{g}$ ($\mathfrak{g}$ is a Lie algebra of G) $X_\kappa \lrcorner\, \omega = -dJ_X$, where $J_X: M \to R: M \to \langle X, J(p) \rangle$, and by X_κ we denote the infinitesimal generator of the action κ corresponding to $X \in T_e G$, $X_\kappa: M \to TM$, $M \to T\kappa_p(X)$, $\kappa_p: G \to M$; $g \to \kappa(g, p)$ (*cf.* [AM]).

For a momentum mapping J we have that $C = J^{-1}(0)$ is a coisotropic submanifold of (M, ω) provided 0 is a regular value of J. C is invariant under the action κ and characteristics of C are orbits of κ. Let κ

be a canonical lift of an action $\rho : G \times Q \to Q$, $(M, \omega) \cong (T^* Q, \omega_Q)$. The mapping $J_\rho : T^* Q \to \mathfrak{g}^*$, $J_\rho(p) = (T_e \rho_q)^*(p)$, $q = \pi_Q(p)$, where $(T_e \rho_q)^* : T_q^* M \to T_e^* G$ is the dual of the tangent mapping $T_e \rho_q$, is a momentum mapping for κ. We see that the submanifold

$$\Psi = \{(h, p, p') \in T^* G \times T^* Q \times T^* Q;\ \kappa(g, p) = p',\ g = \pi_G(h),\ J_G(h) - J_\rho(p) = 0\},$$

where $J_G : T^* G \to \mathfrak{g}^*$ denotes a momentum mapping for the right action of G on itself: $G \times G \to G : (g, g') \to g' g$, is Lagrangian in $(T^* G \times T^* Q \times T^* Q,\ \omega_Q \ominus \theta\, \omega_G \ominus \omega_Q)$. Thus we have obtained the following result,

PROPOSITION 6.1. — *A Morse family corresponding to* Ψ *with extra Morse parameters* $g \in G$ *forms a generating family for a coisotropic variety* $J_\rho^{-1}(0)$.

In the local coordinate representation we have rather more explicit formulae. Let (g_α, h_β) be the canonical coordinates on $T^* G$. Let $X_\alpha = \partial / \partial g_\alpha |_e$ be the natural basis of $\mathfrak{g}$ and (μ_β) its dual in $\mathfrak{g}^*$, $\langle X_\alpha, \mu_\beta \rangle = \delta_{\alpha\beta}$. Let (q_i, p_j) be local canonical coordinates on $T^* Q$. So we have $J_G(g_\alpha, h_\beta) = \sum_{i,j} J_{ij}(g_\alpha) h_i \mu_j$, $J_\rho(q_i, p_j) = \sum_k H_k(q_i, p_j) \mu_k$.

Remark 6.2. — The submanifold Ψ is defined for κ not necessarily a symplectic lifting. Thus the Proposition 6.1 is true in this general formulation as well, with an extra assumption about the special cotangent bundle structure existing on (M, ω). In the case of symplectic liftings of actions on Q we have an explicit normal form of generating family for $C \subset (M, \omega)$, namely $F(\lambda, g, q, \alpha) = \sum_{i=1}^{\dim Q} \lambda_i (\alpha_i - \rho_i(g, q))$.

Example 6.3. — Let us consider the space of binary forms

$$T^* Q = M^4 \ni f(x, y) = q_0 \frac{x^3}{3!} + q_1 \frac{x^2 y}{2!} - p_1 \frac{x y^2}{1!} + p_0 y^3, \quad [G],$$

endowed with the unique, up to constant multiples, $Sl_2(R)$-invariant symplectic structure $\omega = dp_0 \wedge dq_0 + dp_1 \wedge dq_1$. Let $G = R$ be a one parameter subgroup of $Sl_2(R)$ with $\kappa : R \times M^4 \to M^4$ a symplectic R-action generating translations along the variable x; $G \ni \bar{g} = \begin{pmatrix} 1, & t \\ 0, & 1 \end{pmatrix}$. We have $\mathfrak{g} = R$. The infinitesimal generator X_κ of κ corresponding to $X = 1 \in \mathfrak{g}$ is the hamiltonian vector field with Hamiltonian

$$H : T^* Q \to R, \qquad H(p, q) = q_0 p_1 + \frac{1}{2} q_1^2$$

(*see* e. g. [J], p. 20). H is the Hamiltonian momentum mapping for κ. The momentum mapping for the left/right (G is abelian) G-action on itself has the form $J_G : T^* G = R^2 \to g^* = R$, $J_G(t, h) = h$. Thus the Lagrangian submanifold $\Psi \subset (T^* G \times T^* Q \times T^* Q', \omega' \ominus \omega_G \ominus \omega)$, $\omega' = d\bar{p}_0 \wedge d\bar{q}_0 + d\bar{p}_1 \wedge d\bar{q}_1$, $\omega_G = dh \wedge dt$, is given by the following equations: $\kappa(t, p, q) = (\bar{p}, \bar{q})$, $J_G(t, h) - H(p, q) = 0$. By straightforward if messy calculations we find its Morse family

$$F(q, \bar{q}, t, \lambda_1, \lambda_2)$$
$$= \lambda_1 (q_0 - \bar{q}_0) + \lambda_2 (q_1 - t\bar{q}_0 - \bar{q}_1) - \frac{1}{6} t^3 \bar{q}_0^2 - \frac{1}{2} t^2 \bar{q}_0 \bar{q}_1 - \frac{1}{2} t \bar{q}_1^2.$$

Thus the coisotropic submanifold

$$\left\{ ((t, h), (q, p)) \in T^* G \times T^* Q;\ h - q_0 p_1 - \frac{1}{2} q_1^2 = 0 \right\}$$

is generated by the following family

$$F'(q, t, \alpha, \lambda)$$
$$= \lambda_1 (q_0 - \alpha_0) + \lambda_2 (q_1 - t\alpha_0 - \alpha_1) - \frac{1}{6} t^3 \alpha_0^2 - \frac{1}{2} t^2 \alpha_0 \alpha_1 - \frac{1}{2} t \alpha_1^2.$$

Also the corresponding generating family for the zero-level set of the momentum mapping H, which is a singular coisotropic variety, is the following

$$F''(q, \alpha, \lambda)$$
$$= \lambda_1 (q_0 - \alpha_0) + \lambda_2 (q_1 - \lambda_3 \alpha_0 - \alpha_1) - \frac{1}{6} \lambda_3^3 \alpha_0^2 - \frac{1}{2} \lambda_3^2 \alpha_0 \alpha_1 - \frac{1}{2} \lambda_3 \alpha_1^2.$$

This provides the local generalized complete solution of the Hamilton-Jacobi equation

$$q_0 \frac{\partial S}{\partial q_1} + \frac{1}{2} q_1^2 = 0.$$

Generalization of this result for the Hamilton-Jacobi equation of the form:

$$\sum_{r=1}^{n} q_{r-1} \frac{\partial S}{\partial q_r} + \frac{1}{2} q_n^2 = 0$$

gives the following generating family:

$$F(q, \alpha, \lambda) = \sum_{i=0}^{n} \lambda_i \left(q_i - \sum_{k=0}^{i} \frac{1}{k!} \lambda_{n+1}^k \alpha_{i-k} \right) - \frac{1}{2} \int_0^{\lambda_{n+1}} \left(\sum_{k=0}^{n} \frac{1}{k!} s^k \alpha_{n-k} \right)^2 ds$$

7. LOCAL EQUIVALENCES OF THE REDUCED SYMPLECTIC STRUCTURES

Following the theory of singular Lagrangian varieties [Giv], we are able to define the basic notions; versality, stability and pre/normal form for a coisotropic variety.

Let $C \subset T^* M$ be a coisotropic variety, and let $l: R_C \to (T^* M \times T^* N; \pi_2^* \beta_N - \pi_1^* \omega_M)$ be a Lagrangian variety corresponding to C. We define a local algebra Q_C of C as an algebra of functions (smooth, analytic) on $R_C \cap \pi_{M \times N}^{-1}(0)$, $\pi_{M \times N}: T^*(M \times N) \to M \times N$,

$$Q_C = \mathcal{O}_{R_C} / \mathcal{O}_{R_C} (\pi_{M \times N} l)^* m_{M \times N}.$$

DEFINITION 7.1. — *The germ of a coisotropic variety* C *is called versal iff its local algebra* Q_C *is generated by linear inhomogeneous functions on the affine space* $\pi_{M \times N}^{-1}(0)$. *Obviously if* C *is versal then it is finite*, i.e. $\dim Q_C < \infty$. *We say that* C *is stable iff the corresponding lagrangian variety* R_C *is stable in the usual sense*, i.e. C *is versal and* $H^1(R_C) = 0$ [G].

By the Malgrange Preparation Theorem [Ma], we have the classification of pre-normal forms of generating families for C (*cf.* [AVG]), namely

$$F(q, \alpha, \lambda) = f(\lambda) + \sum_{i=1}^{\mu-1} \varphi_i(q, \alpha) e_i(\lambda),$$

where

$$\{e_1, \ldots, e_{\mu-1}\} = m_\lambda \Big/ \left\langle \frac{\partial f}{\partial \lambda} \right\rangle,$$

and $\Phi = (\varphi_1, \ldots, \varphi_{\mu-1}): M \times N \to R^{\mu-1}$ is a smooth inducing mapping [Ma].

Let C be a coisotropic variety in $T^* M$. It defines an associated reduced symplectic structure. We see that the classification of local reduced symplectic structures needs a modification of the standard equivalence of Lagrange projections $\pi_{M \times N}: R_C \to M \times N$.

Let F, $G: (M \times N \times R^k, (0, 0, 0)) \to R$ be two germs of C-generating families. By C(F), C(G) we denote the germs of coisotropic varieties generated by F and G respectively.

DEFINITION 7.2. — *We say that* C(F) *and* C(G) *are C-equivalent, or their corresponding symplectic structures are C-equivalent, iff there exist germs of diffeomorphisms* $\Phi: (M \times N \times R^k, (0, 0, 0)) \to (M \times N \times R^k, (0, 0, 0))$,

$\varphi:(M\times N),(0,0))\to(M\times N,(0,0))$, $g:(N,0)\to(N,0)$, *such that the following diagram commutes*

$$\begin{array}{ccccc} M\times N\times R^k & \xrightarrow{\pi} & M\times N & \xrightarrow{\bar\pi} & N \\ \downarrow \Phi & & \downarrow \varphi & & \downarrow g \\ M\times N\times R^k & \xrightarrow{\pi} & M\times N & \xrightarrow{\bar\pi} & N \end{array} \quad \pi,\ \bar\pi\text{-}projections$$

and

$$F(q,\alpha,\lambda)=\Lambda(G\circ\Phi(q,\alpha,\lambda),q,\alpha),$$

for some $\Lambda:(R\times M\times N,(0,0,0))\to(R,0)$, $\Lambda(.,0,0)\in \mathrm{Diff}\,R$.

One can see that C-equivalence of the coisotropic varieties is not equivalent to the Lagrange equivalence of the corresponding Lagrangian submanifolds $R_{C(F)}$, $R_{C(G)}$. In fact if Λ is not the family of identity mappings then C-equivalnce is only caustic equivalence in $T^*(M\times N)$ (*cf.* [Z]). However, in our approach to determining the typical Lagrange varieties in the reduced symplectic structures this notion is quite adequate.

Remark 7.3. — C-equivalence of C-generating families is equivalent to (r, s)-equivalence of unfoldings [Wa], with an extra assumption $r>s$, and Definition 3.1 defining the restricted space of unfoldings. By the transversality theorem [Ma] this space is still an open and dense subset of the initial space of (r, s)-unfoldings. If Λ is the identity mapping then C-equivalence preserves the space of C-Morse families.

Now we have come to a recognition problem for the local models of C-generating families. This problem can be immediately reduced to the classification of stable (r, s)-unfoldings. Using [Wa], (Theorem 5.2), and methods presented there, by generic modification of local normal forms for (3,1)-stable unfoldings we obtain the following.

PROPOSITION 7.4. — *The germs of stable C-generating families* $F:(R^3\times R\times R^K,(0,0,0))\to R$ *can be reduced to the following normal forms:*

1. $F(q,\alpha,\lambda)=\alpha q_1$
2. $F(q,\alpha,\lambda)=\lambda^3+\lambda q_1+\alpha q_2$
3. $F(q,\alpha,\lambda)=\lambda^3+\alpha\lambda+(\pm q_1^2\pm q_2^2\pm q_3^2)\lambda$
4. $F(q,\alpha,\lambda)=\lambda^4+\lambda^2 q_1+\lambda q_2+\alpha q_3$
5. $F(q,\alpha,\lambda)=\lambda^4+\alpha\lambda+\lambda^2 q_1+(q_1\pm q_2^2\pm q_3^2)\lambda$
6. $F(q,\alpha,\lambda)=\lambda^5+\lambda^3 q_1+\lambda^2 q_2+(\lambda+\alpha)q_3$
7. $F(q,\alpha,\lambda)=\lambda_1^3+\lambda_2^3+\lambda_1\lambda_2 q_1+\lambda_1 q_2+\lambda_2 q_3+\alpha q_1$
8. $F(q,\alpha,\lambda)=\lambda_1^3-\lambda_1\lambda_2^2+q_1(\lambda_1^2+\lambda_2^2)+\lambda_1 q_2+\lambda_2 q_3+\alpha q_1$