ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 57, nº 1 (1992), p. 3-26

<http://www.numdam.org/item?id=AIHPA_1992__57_1_3_0>

Ann. Inst. Henri Poincaré,

Vol. 57, n° 1, 1992, p. 3-26. *Physique théorique*

Higher order graded and Berezinian Lagrangian densities and their Euler-Lagrange equations

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ABSTRACT. – We obtain the Euler-Lagrange equations for higher order graded Lagrangian densities. As in the first order case, some of these equations are non standard and they can be interpreted as constraints. If the graded Lagrangian density comes from a Berezinian Lagrangian density we obtain then the standard Euler-Lagrange equations on the graded ring for the Berezinian Lagrangian densities with both even and odd coordinates.

Key words : Variational calculus, graded manifolds, Berezinian densities, Euler-Lagrange equations.

RÉSUMÉ. – Les équations d'Euler-Lagrange pour les densités Lagrangiennes graduées d'ordre supérieur sont obtenues. Aussi dans ce cas-ci, quelques-unes d'entre elles ne sont pas standard et on peut bien les interpréter comme des contraintes. Si le Lagrangien provient d'une densité Lagrangienne Berezinienne alors on obtient les équations d'Euler-Lagrange, pour la densité Lagrangienne Berezinienne, standard dans l'anneau gradué, avec les coordonnées paires et impaires.

Classification A.M.S. : 58 E 30, 58 A 50, 58 A 20.

This research has been partially supported by the CICYT grant n° PS87 – 0115 – G03 – 01. I want to express my gratitude to Jaime Muñoz Masqué, who proposed me to do this work, for his help and encouragement.

INTRODUCTION

Dealing with graded manifolds one notes that there are two kinds of integration: the graded integration, defined by means of the natural morphism of a graded manifold [6], and the Berezin integration [7]. Each one produces different variational problems.

In this paper we first deduce the variational equations for graded Lagrangian densities of any order. The definitions of graded jet manifolds, and of graded Lagrangian densities, and the deduction of the first order variational equations can be found in [2], [3].

As it is well known, in classical variational calculus, it is equivalent to take as algebra of variations the subalgebra of vertical vector fields or the subalgebra that commutes with the subalgebra of horizontal vector fields. The resulting variational equations are the same: the Euler-Lagrange equations. However, in the graded case the situation is a little bit different. Each subalgebra produces different sets of variational equations. To be more precise, the second subalgebra gives more equations than the first one. Therefore, we will take as algebra of variations the subalgebra of prolongations of vertical vector fields.

The deduction of the variational equations is made by computing the first variation of the functional defined by a graded Lagrangian density with respect to a vertical vector field and removing the terms that are total differentials. The equations that appear are equations on the ring of differentiable functions of the underlying manifold, but not on the graded ring.

Let us suppose that the variational problem is defined on the submersion of graded manifolds $p : (\mathrm{Y}, \mathrm{B}) \to (\mathrm{X}, \mathrm{A})$ with graded dimensions $\dim(\mathrm{X}, \mathrm{A}) = (m, n)$ and $\dim(\mathrm{Y}, \mathrm{B}) = (m+m_1, n+n_1)$.

The total number of equations for a general graded Lagrangian is $(m_1+n_1).2^n$. But if the order of the Lagrangian is $r<n$ then the number of nontrivial equations is $(m_1+n_1)\sum_{i=0}^{r}\binom{n}{r}$.

A subset of the variational equations can be seen as the image by the natural morphism of the graded manifold (X, A) of the standard Euler-Lagrange equations for the even coordinates. In [3], for first order graded Lagrangians, the other equations are considered as previous constraints and the authors give an example with physical meaning. We have thus deduced here the whole set of constraints for a graded Lagrangian of arbitrary order.

In the second part we define, following [4] and [5], the concept of Berezinian Lagrangian density. The main fact of this part establishes the local equivalence between graded and Berezinian variational problems with respect to the algebra of prolongations of vertical graded vector fields. The Berezinian variational problems are defined by means of the Berezin integral. The computation of such integral implies, as a first step, the differentiation of the expression under the sign of the Berezin integral with respect to all the odd variables. With variational problems, these derivatives are transformed into total derivatives, and thus, r-order Berezinian variational problems are equivalent to graded variational problems of order $r+n$, where n is the odd dimension of the base manifold.

This is the reason why it is necessary to deal with graded Lagrangian densities of arbitrary order: even for 0-order Berezinian Lagrangian densities, the equivalent graded Lagrangian densities are of order n.

When we compute the variational equations corresponding to graded Lagrangian densities coming from Berezinian Lagrangian densities all the $(m_1+n_1)\,2^n$ scalar equations can be expressed as m_1+n_1 equations on the graded ring, and these equations are the Euler-Lagrange equation for all the variables, even and odd.

Finally, we study a curious example. It is possible to speak of variational problems when the underlying manifolds are reduced to a point. In this case, the local sections are determined by a set of real numbers and thus the Euler-Lagrange equations become relations between those real numbers.

1. PRELIMINARIES AND NOTATIONS

Let (X, A) be a graded manifold of graded dimension $\dim(\mathrm{X}, \mathrm{A})=(m, n)$ [6], in the following sense

– X is a $\mathscr{C}^\infty$ real manifold of dimension m.

– A is a sheaf of $\mathbb{Z}_2$ graded commutative algebras such that

(1) there exists a surjective sheaf morphism

$$\begin{aligned} \mathrm{A}(\mathrm{X}) &\to \mathscr{C}^\infty(\mathrm{X}) \\ f &\to f^\sim \end{aligned}$$

often called the natural morphism,

(2) there exists an open covering $\{\mathrm{U}_i\}_{i\in\mathrm{I}}$ of X and sheaf isomorphisms

$$\mathrm{A}(\mathrm{U}_i) \to \Lambda\,\mathbb{R}^n(\mathrm{U}_i).$$

Let $p=(q, q^*) : (\mathrm{Y}, \mathrm{B}) \to (\mathrm{X}, \mathrm{A})$ be a submersion of graded manifolds ([2], Def. 11) with graded dimensions $\dim(\mathrm{X}, \mathrm{A})=(m, n)$ and $\dim(\mathrm{Y}, \mathrm{B})=(m+m_1, n+n_1)$. Let us denote by $\{x^i, s^\mathrm{I}\}$, $(i=1, \ldots, m;$ $\mathrm{I}=1, \ldots, n)$ the graded A-coordinates for the coordinate open (U, A (U)), and by $\{x^i, y^j, s^\mathrm{I}, t^\mathrm{J}\}$, $(j=1, \ldots, m_1;\ \mathrm{J}=1, \ldots, n_1)$ the graded B-coordinates for the coordinate open (V, B (V)) with a suitable $\mathrm{V}\subset q^{-1}(\mathrm{U})$ and with degrees

$$(|x^i|=|y^j|=0, |s^I|=|t^J|=1).$$

Let $J^k(B/A)$ be the graded k-jet bundle of graded sections of p ([2], Def. 17). Let us denote by X^k the base manifold and by $\mathscr{A}^k$ the graded sheaf of rings of the graded manifold $J^k(B/A)$. Let us consider the projections (*see* [2] for the definitions)

$$p_{k,l}: \quad J^k(B/A) \to J^l(B/A), \qquad \forall k \geqq l \geqq 0,$$

that satisfy the relations

$$p_{l,m} \circ p_{k,l} = p_{k,m}, \qquad \forall k \geqq l \geqq m \geqq 0.$$

By the very definition of a graded morphism, there are canonic projections between the base manifolds

$$q_{k,l}: \quad X^k \to X^l, \qquad \forall k \geqq l \geqq 0,$$

satisfying the relations

$$q_{l,m} \circ q_{k,l} = q_{k,m}, \qquad \forall k \geqq l \geqq m \geqq 0;$$

and there are canonic projections between the graded sheafs of rings

$$q^*_{k,l}: \mathscr{A}^l \to \mathscr{A}^k, \qquad \forall k \geqq l \geqq 0,$$

with relations

$$q^*_{k,l} \circ q^*_{l,m} = q^*_{k,m}, \qquad \forall k \geqq l \geqq m \geqq 0.$$

Let X^∞ be the projective or inverse limit of the projective system

$$((X^k)_{K \in \mathbb{N}}, (q_{k,l})_{(k,l) \in \mathbb{N} \times \mathbb{N}}).$$

The induced projections will be denoted by $q_{\infty,k}$, q_∞.

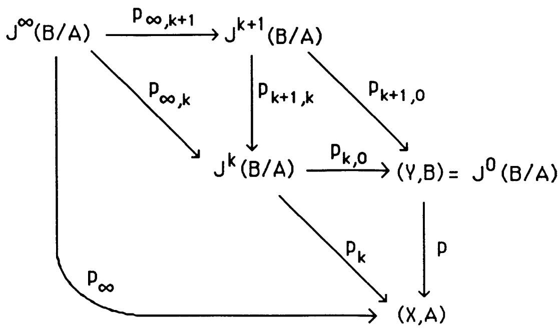
Given an open subset U of X^∞, let $\mathscr{A}^\infty(U)$ be the inductive or direct limit of the inductive system

$$((\mathscr{A}^k(U))_{k \in \mathbb{N}}, (q^*_{k,l})_{(k,l) \in \mathbb{N} \times \mathbb{N}}).$$

The induced projections will be denoted by $q^*_{\infty,k}$, q^*_∞.

Let $J^\infty(B/A) = (X^\infty, \mathscr{A}^\infty)$ be the graded ∞-jet bundle of graded sections of p [4]. The graded canonic projections are denoted by $p_{\infty,k} = (q_{\infty,k}, q^*_{\infty,k})$ and $p_\infty = (q_\infty, q^*_\infty)$.

The description of all these projections above can be visualized in the following diagram:



Let $\Gamma(p)$ be the space of local sections of p. If

$$\sigma=(\tau, \tau^*) : (X, A) \rightarrow (Y, B)$$

is a local section of p, then $j^\infty(\sigma) : (X, A) \rightarrow J^\infty(B/A)$ is the limit of the morphisms $(j^k(\sigma))_{k \in \mathbb{N}}$:

$$j^\infty(\sigma)=(\lim_{\leftarrow} \tau_k, \lim_{\rightarrow} \tau_k^*)=(j^\infty(\tau), j^\infty(\tau)^*).$$

In order to denote the coordinate functions of $J^\infty(B/A)$ we will need to define two kinds of multiindexes. An even multiindex α will be an element of $\mathbb{N}^m$, *i.e.*, $\alpha=(\alpha_1, \ldots, \alpha_m)$, with $\alpha_i \in \mathbb{N}$, $i=1, \ldots, m$. We will call them even multiindexes because they will only appear with derivatives with respect to the even coordinates, $\{x^i\}$. $|\alpha|$ stands for $\sum_{i=1}^{m} \alpha_i$, the length of α. Let us denote by $\varnothing$ the even multiindex $(0, 0, \ldots, 0)$.

For $f \in A(U)$, we set

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha}=\frac{\partial^{\alpha_1}}{\partial (x^1)^{\alpha_1}} \circ \ldots \circ \frac{\partial^{\alpha_m}}{\partial (x^m)^{\alpha_m}} f.$$

An odd multiindex β will be an ordered subset of $\{1, 2, \ldots, n\}$, *i.e.*, $\beta=\{\beta_1<\beta_2<\ldots<\beta_l\}$, $1 \leqq \beta_i \leqq n$, $i=1, \ldots, l$. We will call them odd multiindexes because it will only appear with derivatives with respect to the odd coordinates, $\{s^I\}$. Let us denote by $d(\beta)$ the cardinal of β.

For $f \in A(U)$, we set

$$\frac{\partial^{d(\beta)} f}{\partial s^\beta}=\frac{\partial}{\partial s^{\beta_1}} \circ \ldots \circ \frac{\partial}{\partial s^{\beta_{d(\beta)}}} f.$$

Let $W \subset q_\infty^{-1}(U)$ be an open subset of X^∞. The graded coordinates for $(W, \mathscr{A}^\infty(W))$, suitable reorganized, are

$$\{x^i, y^j, s^I, t^J, y^j_{\alpha,\beta}, t^J_{\alpha,\beta}\},$$

where α is an even multiindex and β is an odd one, with degrees

$$|x^i|=|y^j|=0, \qquad |s^I|=|t^J|=1,$$
$$|y^j_{\alpha,\beta}|=d(\beta)\,(\mathrm{mod}\ 2), \qquad |t^J_{\alpha,\beta}|=1+d(\beta)\,(\mathrm{mod}\ 2),$$

and where the last coordinate functions are defined by

$$j^\infty(\sigma)^*(y^j_{\alpha,\beta})=\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{d(\beta)}}{\partial s^\beta}(\sigma^*(y^j)),$$
$$j^\infty(\sigma)^*(t^J_{\alpha,\beta})=\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{d(\beta)}}{\partial s^\beta}(\sigma^*(t^J)),$$

with $\sigma\in\Gamma(p)$.

Remarks:

(1) We will identify $y^j_{\varnothing,\varnothing}$ (resp. $t^J_{\varnothing,\varnothing}$) with y^j (resp. t^J).

(2) Unless otherwise stated, subindexes will denote derivatives.

2. ALGEBRAS OF VECTOR FIELDS ON $J^\infty(B/A)$

A graded vector field on a graded manifold (X, A) is an $\mathbb{R}$-derivation of the sheaf A, *i. e.*, an element of $\mathrm{Der}_{\mathbb{R}}(A)$.

A vector field Z on (Y, B) is said to be *p-projectable* if there exists a vector field $p(Z)$ on (X, A) such that $Z\circ p^*=p^*\circ p(Z)$.

Horizontal vector fields [4]

Let Z be a vector field on the graded manifold (X, A). We define the total or horizontal graded lifting Z^H of Z as the vector field on $J^\infty(B/A)$ such that for every open subset W of X^∞, for every $f\in\mathscr{A}^\infty(W)$, and every local section $\sigma=(\tau,\tau^*):(U, A(U))\to(V, B(V))$ of p, with suitable open subsets U, V such that $j^\infty(\tau)^*(f)\in A(U)$, we have

$$j^\infty(\tau)^*(Z^H(f))=Z(j^\infty(\tau)^*(f)).$$

Note that Z^H is a p_∞-projectable vector field with $p_\infty(Z^H)=Z$.

A graded vector field on $J^\infty(B/A)$, W, is called horizontal if there exists a graded vector field, Z, on (X, A) such that $W=Z^H$. Note that $[Z^H_1, Z^H_2]=[Z_1, Z_2]^H$. Thus we have a subalgebra of vector fields, the subalgebra of horizontal vector fields.

Let us take graded A-coordinates $\{x^i, s^I\}$ for the coordinate open (U, A(U)) and graded B-coordinates $\{x^i, y^j, s^I, t^J\}$ for (V, B(V)) such that $V\subset q^{-1}(U)$:

$$(|x^i|=|y^j|=0,\ |s^I|=|t^J|=1).$$

Let us denote the horizontal graded liftings of $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial s^{\mathrm{I}}}\right\}$ by $\left\{\frac{d}{dx^i}, \frac{d}{ds^{\mathrm{I}}}\right\}$. Their local expression are given by

$$\left(\frac{\partial}{\partial x^i}\right)^{\mathrm{H}} = \frac{d}{dx^i} = \frac{\partial}{\partial x^i} + y^j_{\alpha+i,\,\beta} \frac{\partial}{\partial y^j_{\alpha,\,\beta}} + t^{\mathrm{J}}_{\alpha+i,\,\beta} \frac{\partial}{\partial t^{\mathrm{J}}_{\alpha,\,\beta}},$$

$$\left(\frac{\partial}{\partial s^{\mathrm{I}}}\right)^{\mathrm{H}} = \frac{d}{ds^{\mathrm{I}}} = \frac{\partial}{\partial s^{\mathrm{I}}} + (-1)^{sg(\beta,\,\mathrm{I})} y^j_{\alpha,\,\beta+\mathrm{I}} \frac{\partial}{\partial y^j_{\alpha,\,\beta}} + (-1)^{sg(\beta,\,\mathrm{I})} t^{\mathrm{J}}_{\alpha,\,\beta+\mathrm{I}} \frac{\partial}{\partial t^{\mathrm{J}}_{\alpha,\,\beta}},$$

where the following conventions are used: If $\alpha = (\alpha_1, \ldots, \alpha_m)$ then $\alpha + i$ is the even multiindex $(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_m)$. Note that i is a Latin index.

In the second formula, the index β runs over the subsets to which I does not belong. If $\beta = \{\beta_1 < \beta_2 < \ldots < \beta_l\}$ and $\mathrm{I} \notin \beta$ then $\beta + \mathrm{I}$ is the odd multiindex $\{\beta_1 < \beta_2 < \ldots < \beta_k < \mathrm{I} < \beta_{k+1} < \ldots < \beta_l\}$, *i.e.*, the ordered subset $\beta \cup \{\mathrm{I}\}$. Thus, $sg(\beta, \mathrm{I}) = k$.

Vertical vector fields

A graded vector field Z on (Y, B) is said to be vertical if for all $f \in \mathrm{A}(\mathrm{U})$ we have that $\mathrm{Z}(q^*(f)) = 0$. Analogously, a graded vector field Z on $\mathrm{J}^\infty(\mathrm{B}/\mathrm{A})$ is said to be vertical if for all $f \in \mathrm{A}(\mathrm{U})$ we have that $\mathrm{Z}(q^*_\infty(f)) = 0$. The bracket of two vertical vector fields is another vertical vector field. Then we can define the subalgebra of vertical vector fields.

Obviously, we have that if Z is a p_∞- projectable vector field on $\mathrm{J}^\infty(\mathrm{B}/\mathrm{A})$ then $\mathrm{Z} - p_\infty(\mathrm{Z})^{\mathrm{H}}$ is a vertical vector field, an thus Z can be uniquely decomposed as a sum of a horizontal vector field and a vertical vector field. This spliting is only possible in $\mathrm{J}^\infty(\mathrm{B}/\mathrm{A})$.

Graded infinitesimal contact transformations [2]

In the classical calculus of variations the elements of the subalgebra of infinitesimal contact transformations play the role of tangent vectors of the allowed variations [8]. There are other characterizations of this subalgebra; for ∞-jets, it is possible to define it as the subalgebra that commutes with the subalgebra of horizontal vector fields. We will generalize this construction for graded manifolds.

Let $\mathscr{W}$ be the subalgebra of vector fields on $\mathrm{J}^\infty(\mathrm{B}/\mathrm{A})$ that commutes with the subalgebra of horizontal graded vector fields. That is, W belongs to $\mathscr{W}$ if and only if $\forall$ Z horizontal graded vector field we have that [W, Z] is again a horizontal graded vector field. It is not hard to see that if the local expression of W is

$$X^i \frac{\partial}{\partial x^i} + S^I \frac{\partial}{\partial s^I} + U^j_{\alpha,\beta} \frac{\partial}{\partial y^j_{\alpha,\beta}} + V^J_{\alpha,\beta} \frac{\partial}{\partial t^J_{\alpha,\beta}},$$

then the following recurrence relations hold:

$$U^l_{\alpha+i,\beta} = \frac{dU^l_{\alpha,\beta}}{dx^i} - y^l_{\alpha+j,\beta} \frac{dX^j}{dx^i} - (-1)^{sg(\beta, I)} y^l_{\alpha,\beta+I} \frac{dS^I}{dx^i},$$

$$V^J_{\alpha+i,\beta} = \frac{dV^J_{a,\beta}}{dx^i} - t^J_{\alpha+j,\beta} \frac{dX^j}{dx^i} - (-1)^{sg(\beta, I)} t^J_{\alpha,\beta+I} \frac{dS^I}{dx^i},$$

$$U^l_{\alpha,\beta+I} = \frac{dU^l_{\alpha,\beta}}{ds^I} - y^l_{\alpha+j,\beta} \frac{dX^j}{ds^I} - (-1)^{sg(\beta, L)} y^l_{\alpha,\beta+L} \frac{dS^L}{ds^I},$$

$$V^J_{\alpha,\beta+I} = \frac{dV^J_{\alpha,\beta}}{ds^I} - t^J_{\alpha+j,\beta} \frac{dX^j}{ds^I} - (-1)^{sg(\beta, L)} t^J_{\alpha,\beta+L} \frac{dS^L}{ds^I}.$$

These equations are useful to define the following lifting. For every graded vector field Z on (Y, B) there exists a unique graded infinitesimal contact transformation, Z^∞ on $J^\infty(B/A)$ which projects onto Z. Z^∞ is called the prolongation of Z to $J^\infty(B/A)$.

We have the following

PROPOSITION 2.1 [4]. — *Let* W *be a vector field on* (X, A) *and let* Z *be a p-projectable vector field on* (Y, B). *Then* $[Z^\infty, W^H] = [p(Z), W]^H$.

In the deduction of the Euler-Lagrange equations for the standard variational calculus on manifolds, the subalgebra of variations is the subalgebra of prolongations of vertical vector fields, and it is well known that if the choice of the subalgebra of variations is the subalgebra that commutes with the subalgebra of horizontal vector fields, then the resulting equations are the same. It is equivalent to take any of the two subalgebras. But this is not the case for graded manifolds. To enlarge the subalgebra of vertical vector fields with the horizontal vector fields produce new equations. (*See* remark 5 of theorem 4.2.) Then we will restrict the subalgebra of variations to just that of vertical vector fields.

3. DIFFERENTIAL FORMS ON $J^\infty(B/A)$

Let $\Omega^r(B/A)$ be the space of differential *r*-forms on $J^\infty(B/A)$. We put $H^l_k(B/A)$ to denote the module of $(k+l)$-forms on $J^\infty(B/A)$ that are *k* times horizontal and *l* times vertical, *i.e.*, such that they vanish when acting on more than *l* vertical vector fields or more than *k* horizontal vector fields.

Let *d* be the exterior differentiation ([6], proposition 4.2.3) and let $D : H^l_k(B/A) \to H^l_{k+1}(B/A)$ be the horizontal differential and let

$\partial : \mathrm{H}^l_k(\mathrm{B/A}) \to \mathrm{H}^{l+1}_k(\mathrm{B/A})$ be the vertical differential. We have

$$d = \mathrm{D} + \partial, \qquad \text{and} \qquad \mathrm{D}^2 = \partial^2 = \mathrm{D}\partial + \partial\mathrm{D} = 0.$$

We can make a local refinement of the bigraduation. Let $(\mathrm{W}, \mathscr{A}^\infty(\mathrm{W}))$ be a coordinate open set of $\mathrm{J}^\infty(\mathrm{B/A})$. Since $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial s^\mathrm{I}}\right\}$ are a basis of vector fields, then we can define $\mathrm{H}^l_{p,q}(\mathrm{W})$ as the submodule of differential forms of $\mathrm{H}^l_{p+q}(\mathrm{W})$ such that they vanish when acting on more than p vector fields of $\left\{\frac{\partial}{\partial x^i}\right\}$ or more than q vector fields of $\left\{\frac{\partial}{\partial s^\mathrm{I}}\right\}$.

Therefore

$$\mathrm{H}^l_k(\mathrm{W}) = \bigoplus_{p+q=k} \mathrm{H}^l_{p,q}(\mathrm{W})$$

with projections $\pi_{p,q} : \mathrm{H}^l_k(\mathrm{W}) \to \mathrm{H}^l_{p,q}(\mathrm{W})$. Considering the action of D on a fixed $\mathrm{H}^l_{p,q}(\mathrm{W})$, we define $\mathrm{D}_0 := \pi_{p+1,q} \circ \mathrm{D}$ and $\mathrm{D}_1 = \mathrm{D} - \mathrm{D}_0$.

If Z is a vector field on $\mathrm{J}^\infty(\mathrm{B/A})$ of degree $|\mathrm{Z}|$, then the insertion operator $i(\mathrm{Z})$, defined in [6], prop. 4.2.3, is a derivation of bidegree $(-1, |\mathrm{Z}|)$.

We can compute, by means of the insertion operator with respect to suitable vector fields, the following local expressions

$$\begin{gathered}
\mathrm{D}y^j_{\alpha,\beta} = \mathrm{D}_0 y^j_{\alpha,\beta} + \mathrm{D}_1 y^j_{\alpha,\beta} = dx^i\, y^j_{\alpha+i,\beta} + (-1)^{sg(\beta,\mathrm{I})}\, ds^\mathrm{I}\, y^j_{\alpha,\beta+\mathrm{I}},\\
\mathrm{D}t^\mathrm{J}_{\alpha,\beta} = \mathrm{D}_0 t^\mathrm{J}_{\alpha,\beta} + \mathrm{D}_1 t^\mathrm{J}_{\alpha,\beta} = dx^i\, t^\mathrm{J}_{\alpha+i,\beta} + (-1)^{sg(\beta,\mathrm{I})}\, ds^\mathrm{I}\, t^\mathrm{J}_{\alpha,\beta+\mathrm{I}},\\
\partial y^j_{\alpha,\beta} = dy^j_{\alpha,\beta} - \mathrm{D}y^j_{\alpha,\beta},\\
\partial t^\mathrm{J}_{\alpha,\beta} = dt^\mathrm{J}_{\alpha,\beta} - \mathrm{D}t^\mathrm{J}_{\alpha,\beta}.
\end{gathered}$$

Let us fix a global section $\eta : \mathscr{C}^\infty(\mathrm{X}) \to \mathrm{A}(\mathrm{X})$ of the natural morphism, $\mathrm{A}(\mathrm{X}) \to \mathscr{C}^\infty(\mathrm{X})$ [1]. A coordinate system, $\{x^i, s^\mathrm{I}\}$, on an open subset $(\mathrm{U}, \mathrm{A}(\mathrm{U}))$, is said to be compatible with the section η if there is a coordinate system, $\{z^i\}$, on the open subset U, such that $x^i = \eta(z^i)$.

The section η induces a morphism $\Lambda^* \Omega(\mathrm{X}) \to \Omega(\mathrm{X}, \mathrm{A})$ and thus, every ordinary differential form can be considered as a graded differential form. Let us assume that X is oriented by a volume element $\tau \in \Lambda^m \Omega(\mathrm{X})$ and let us also denote by τ the induced graded m-form on $\Lambda^m \Omega(\mathrm{X}, \mathrm{A})$ and the induced graded m-form on $\mathrm{H}^0_m(\mathrm{B/A})$ by the map p_∞.

DEFINITION [3]. — A graded Lagrangian density on the submersion p is an element $\lambda \in \mathrm{H}^0_m(\mathrm{B/A})$ such that there exists $\mathrm{L} \in \mathscr{A}^\infty_{\mathrm{}}$ satisfying $\lambda = \mathrm{L}\,\tau$.

Locally, if $(\mathrm{W}, \mathscr{A}^\infty(\mathrm{W}))$ is a coordinate open of $\mathrm{J}^\infty(\mathrm{B/A})$, a graded Lagrangian density is written as

$$\lambda = \mathrm{L}\, dx^1 \wedge \ldots \wedge dx^m, \qquad \text{with} \quad \mathrm{L} \in \mathscr{A}^\infty(\mathrm{W}),$$

thus $\lambda \in \mathrm{H}^0_{m,0}(\mathrm{W})$.

If (X, A) is the standard linear graded manifold $(\mathbb{R}^m, C^\infty(\mathbb{R}^m)\otimes\Lambda\,\mathbb{R}^n)$, and σ is a local section then

$$\int_U [j^\infty(\sigma)^*\lambda]^\sim = \int_U L_\varnothing(x)\,dx^1\wedge\ldots\wedge dx^m,$$

by the definition of the natural morphism [6], where $L_\varnothing(x)$ is the first or independent coefficient of the development of $j^\infty(\sigma)^* L \in C^\infty(\mathbb{R}^m)\otimes\Lambda\,\mathbb{R}^n$ as products of the s^J's, $\sum L_\beta(x)\,s^\beta$.

4. EULER-LAGRANGE EQUATIONS FOR GRADED LAGRANGIAN DENSITIES

Let us state the following variational principle:

A local section of p, $\sigma:(X, A)\to(Y, B)$ is a graded critical section for the graded variational problem defined by the graded Lagrangian density λ if for every p-projectable vertical vector field Z on (Y, B) whose support has compact image on the domain of σ, we have that

$$\int_X [j^\infty(\sigma)^*\mathcal{L}_{Z^\infty}\lambda]^\sim = 0.$$

We will need the following

LEMMA 4.1. – *Let* $(W, \mathcal{A}^\infty(W))$ *be a coordinate open of* $J^\infty(B/A)$. *Let* $\Omega\in H^1_{m-1,0}(W)$ *an let* Z *be a vertical vector field on* $(p_{\infty,0}(W), B(p_{\infty,0}(W)))$, *then*

$$i(Z^\infty)\,D_0\Omega = -D_0\,i(Z^\infty)\Omega, \qquad \forall\sigma\in\Gamma(p).$$

Proof. – Both expressions are elements of $H^0_{m,0}(W)$, it is thus enough to check that they agree acting on $\left\{\frac{d}{dx^1},\ldots,\frac{d}{dx^m}\right\}$.

$$D_0\Omega\left(Z^\infty,\frac{d}{dx^1},\ldots,\frac{d}{dx^m}\right) = d\Omega\left(Z^\infty,\frac{d}{dx^1},\ldots,\frac{d}{dx^m}\right)$$
$$= -\sum_{i=1}^m(-1)^i\frac{d}{dx^i}\left(\Omega\left(Z^\infty,\frac{d}{dx^1},\ldots,\widehat{\frac{d}{dx^i}},\ldots,\frac{d}{dx^m}\right)\right).$$

Now it is easy to see that this expression is the same as

$$-D_0(i(Z^\infty)\Omega)\left(\frac{d}{dx^1},\ldots,\frac{d}{dx^m}\right) \quad \blacksquare$$

NOTATION. – In order to avoid irrelevant repetitions, we will denote by z either the variable y or t. Thus $z^j_{\alpha,\beta}$ will be either $y^j_{\alpha,\beta}$ or $t^J_{\alpha,\beta}$. The only difference between y and t is the parity, thus, $sg(z)$ will denote the parity of the variable, *i.e.*, $sg(z)=0$ if $z=y$ and $sg(z)=1$ if $z=t$.

The following theorem presents the deduction of the variational equations associated to the variational problem defined by a graded Lagrangian density. The proof follows closely the methods of [9] in the sense that we look for a local source form defined by the Lagrangian.

THEOREM 4.2. — *A local section of* p, $\sigma:(X, A)\to(Y, B)$ *is a graded critical section for the graded variational problem defined by the Lagrangian* λ *if and only if for each regular coordinate open* $(W, \mathscr{A}^\infty(W))$ *of* $J^\infty(B/A)$ *such that*

$$\lambda = L\, dx^1 \wedge \ldots \wedge dx^m, \qquad L\in\mathscr{A}^\infty(W)$$

the following equations hold

$$\left[(j^\infty(\sigma))^*\left(\sum_\alpha(-1)^{|\alpha|}\frac{d^{|\alpha|}}{dx^\alpha}\left(\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right)\right)\right]^\sim=0,$$
$$\forall\beta,\ 0\leqq d(\beta)\leqq n, \qquad \forall z^j=y^1,\ \ldots,\ y^{m_1};\ t^1,\ \ldots,\ t^{n_1}.$$

Proof. — Let $(U, A(U))$ be a coordinate open set such that $W\subset q_\infty^{-1}(U)$.

Let us suppose that Z is a vertical vector field on (Y, B) such that $Z^\infty|_{j^\infty(\sigma)^*(\partial U)}=0$. Then $i(Z^\infty)\lambda=0$, and thus

$$[j^\infty(\sigma)^*\mathscr{L}_{Z^\infty}\lambda]^\sim=[j^\infty(\sigma)^*\, i(Z^\infty)\, d\lambda]^\sim.$$

In the coordinate open $(W, \mathscr{A}^\infty(W))$ we have

$$D\lambda=D_1\lambda=ds^I\frac{dL}{ds^I}\wedge dx^1\wedge\ldots\wedge dx^m$$
$$=ds^I\left(\frac{\partial L}{\partial s^I}+(-1)^{sg(\beta,I)}z^j_{\alpha,\beta+I}\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right)\wedge dx^1\wedge\ldots\wedge dx^m,$$
$$\partial\lambda=d\lambda-D\lambda=\left(dz^j_{\alpha,\beta}\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right)\wedge dx^1\wedge\ldots\wedge dx^m$$
$$-ds^I\left((-1)^{sg(\beta,I)}z^j_{\alpha,\beta+I}\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right)\wedge dx^1\wedge\ldots\wedge dx^m.$$

Let us note that $i(Z^\infty)D\lambda=0$ because Z^∞ is vertical and $D\lambda\in H^0_{m,1}$. Therefore

$$i(Z^\infty)\,d\lambda=i(Z^\infty)\left[dz^j_{\alpha,\beta}\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right]\wedge dx^1\wedge\ldots\wedge dx^m.$$

We define two differential forms on $(W, \mathscr{A}^\infty(W))$, $\omega \in H^1_{m,0}$ and $\Omega \in H^1_{m-1,0}$ as follows:

$$\omega = \left(\sum_\beta \sum_\alpha (-1)^{|\alpha|} \partial z^j_{\varnothing,\beta} \frac{d^{|\alpha|}}{dx^\alpha}\left(\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right)\right) \wedge dx^1 \wedge \ldots \wedge dx^m$$
$$= (\partial z^j_{\varnothing,\beta}\, \omega^j_{\varnothing,\beta}) \wedge dx^1 \wedge \ldots \wedge dx^m.$$

This form plays the role of source form in the statement of [9]. To define the second one we need to make the following conventions:

If $\alpha > \varnothing$ is the even multiindex $(\alpha_1, \ldots, \alpha_m)$ and if $1 \leqq i \leqq m$ and $1 \leqq k \leqq \alpha_i$ then let $\alpha^{i,k}$ be the even multiindex $(\alpha_1, \ldots, \alpha_{i-1}, k, 0, \ldots, 0)$.

$$\Omega := \sum_{\beta,\, \varnothing < \alpha} \left[\sum_{i=1}^{m} \sum_{k=1}^{\alpha_i} (-1)^{i+|\alpha - \alpha^{i,k}|} \partial z^j_{\alpha^{i,k-1},\beta} \frac{d^{|\alpha-\alpha^{i,k}|}}{dx^{\alpha-\alpha^{i,k}}}(\mu^j_{\alpha,\beta}) \right] \wedge dx^1 \wedge \ldots \widehat{dx^i} \ldots \wedge dx^m,$$

where

$$\mu^j_{\alpha,\beta} = \frac{\partial L}{\partial z^j_{\alpha,\beta}}.$$

Now, let us check that

$$\omega + D_0 \Omega = \left(\partial z^j_{\alpha,\beta} \frac{\partial L}{\partial z^j_{\alpha,\beta}} \right) \wedge dx^1 \wedge \ldots \wedge dx^m.$$

To see this, let us compute, for each β and $\alpha > \varnothing$, the partial term

$$D_0\left(\sum_{i=1}^{m} \left[\sum_{k=1}^{\alpha_i} (-1)^{i+|\alpha-\alpha^{i,k}|} \partial z^j_{\alpha^{i,k-1},\beta} \frac{d^{|\alpha-\alpha^{i,k}|}}{dx^{\alpha-\alpha^{i,k}}}(\mu^j_{\alpha,\beta}) \right] \wedge dx^1 \wedge \ldots \widehat{dx^i} \ldots \wedge dx^m \right)$$
$$= \left(\sum_{i=1}^{m} \sum_{k=1}^{\alpha_i} (-1)^{|\alpha-\alpha^{i,k}|} \partial z^j_{\alpha^{i,k-1},\beta} \frac{d^{|\alpha-\alpha^{i,k-1}|}}{dx^{\alpha-\alpha^{i,k-1}}}(\mu^j_{\alpha,\beta}) \right.$$
$$\left. + \sum_{i=1}^{m} \sum_{k=1}^{\alpha_i} (-1)^{\alpha-\alpha^{i,k}|} \partial z^j_{\alpha^{i,k},\beta} \frac{d^{|\alpha-\alpha^{i,k}|}}{dx^{\alpha-\alpha^{i,k}}}(\mu^j_{\alpha,\beta}) \right) \wedge dx^1 \wedge \ldots \wedge dx^m,$$

where the following equation has been used

$$D_0 (\partial z^j_{\alpha^{i,k-1},\beta} \wedge dx^1 \wedge \ldots dx^i \ldots \wedge dx^m) = -(-1)^{i-1} \partial z^j_{\alpha^{i,k},\beta} \wedge dx^1 \wedge \ldots \wedge dx^m.$$

The term of the second sum with indexes $((i, k), k < \alpha_i)$ is the same, but with opposite sign, as the term of the first sum with indexes $(i, k+1)$. The term of the second sum with indexes $((i, k = \alpha_i), i < m)$ is the same, but with opposite sign, as the term of the first sum with indexes $(i+1, k=1)$. The only terms that left are that of the second sum with index (m, α_m) and that of the first with index $(1,1)$.

Thus, simplifying this equation we get, for each β, $\alpha > \varnothing$,

$$\left(-(-1)^{|\alpha|}\partial z^j_{\varnothing,\beta}\frac{d^{|\alpha|}}{dx^\alpha}\left(\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right)+\partial z^j_{\alpha,\beta}\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right)\wedge dx^1\wedge\ldots\wedge dx^m.$$

We get thus the desired result, and therefore

$$i(Z^\infty)\left[dz^j_{\alpha,\beta}\frac{\partial L}{\partial z^j_{\alpha,\beta}}\right]\wedge dx^1\wedge\ldots\wedge dx^m = i(Z^\infty)\,\omega + i(Z^\infty)\,D_0\,\Omega.$$

But according to lemma 4.1 we have that

$$-\int_U [j^\infty(\sigma)^*\, i(Z^\infty)\, D_0\,\Omega]^\sim = \int_U [j^\infty(\sigma)^*\, D_0\, i(Z^\infty)\,\Omega]^\sim$$

$$= \int_U [j^\infty(\sigma)^*\, D i(Z^\infty)\,\Omega]^\sim = \int_U [dj^\infty(\sigma)^*\, i(Z^\infty)\,\Omega]^\sim$$

by [6] Prop. 4.6.1,

$$= \int_U d[j^\infty(\sigma)^*\, i(Z^\infty)\,\Omega]^\sim = \int_{\partial U} [j^\infty(\sigma)^*\, i(Z^\infty)\,\Omega]^\sim = 0.$$

Let us compute the other term. If $Z = Z^j\dfrac{\partial}{\partial y^j} + Z^J\dfrac{\partial}{\partial t^J}$, then

$$Z^\infty = Z^j_{\alpha,\beta}\frac{\partial}{\partial y^j_{\alpha,\beta}} + Z^J_{\alpha,\beta}\frac{\partial}{\partial t^J_{\alpha,\beta}},$$

where $Z^j_{\alpha,\beta} = \dfrac{d}{dx^\alpha}\dfrac{d}{ds^\beta}Z^j$ and the same for Z^J.

Thus the integrand becomes

$$[j^\infty(\sigma)^*(Z^j_{\varnothing,\beta}\,\omega^j_{\varnothing,\beta} + Z^J_{\varnothing,\beta}\,\omega^J_{\varnothing,\beta})]^\sim.$$

Let us suppose that $Z^J = 0$ and, given β fixed, $Z^j = fs^\beta$, $f \in A(U)$, then $Z^j_{\varnothing,\beta} = f$ and therefore the integrand is $f^\sim\,[j^\infty(\sigma)^*\,\omega^j_{\varnothing,\beta}]^\sim$.

Since f is arbitrary we get the equation $[j^\infty(\sigma)^*\,\omega^j_{\varnothing,\beta}]^\sim = 0$. We can deduce the other equation in a similar way, $[j^\infty(\sigma)^*\,\omega^J_{\varnothing,\beta}]^\sim = 0$. ■

Remarks. – (1) The deduction of these equations seems to consider the coordinates $y^j_{\varnothing,\beta}$, $t^J_{\varnothing,\beta}$ as fibre coordinates of a fibre bundle on X, and to identify $(y^j_{\varnothing,\beta})_\alpha$ with $y^j_{\alpha,\beta}$. Thus the deduction is just a copy, for each β, of the classical deduction of the Euler-Lagrange equations.

(2) The total number of the variational equations for a general graded Lagrangian is $(m_1+n_1)\,2^n$. But if the order of the Lagrangian is $r<n$ then the number of nontrivial equations is $(m_1+n_1)\sum_{i=0}^{r}\binom{n}{r}$.

(3) For first order graded Lagrangian densities, the equations agree with equations of [3]. Indeed, if $L \in \mathscr{A}^1(U)$, where U is an open subset of X^1, then the equations are:

$$\beta=\varnothing, \quad \left[(j^\infty(\sigma))^*\left(\frac{\partial L}{\partial y^j}-\frac{d}{dx^i}\left(\frac{\partial L}{\partial y^j_{i,\varnothing}}\right)\right)\right]^\sim=0, \qquad \forall j=1,\ldots,m_1,$$

$$\beta=\varnothing, \quad \left[(j^\infty(\sigma))^*\left(\frac{\partial L}{\partial t^J}-\frac{d}{dx^i}\left(\frac{\partial L}{\partial t^J_{i,\varnothing}}\right)\right)\right]^\sim=0, \qquad \forall J=1,\ldots,n_1,$$

$$\beta=I, \quad \left[(j^\infty(\sigma))^*\left(\frac{\partial L}{\partial y^j_{\varnothing,I}}\right)\right]^\sim=0, \qquad \forall j=1,\ldots,m_1,$$

$$\beta=I, \quad \left[(j^\infty(\sigma))^*\left(\frac{\partial L}{\partial t^J_{\varnothing,I}}\right)\right]^\sim=0, \qquad \forall J=1,\ldots,n_1.$$

For β such that $d(\beta)\geqq 2$ the equations are trivial.

(4) In [3] this last set of equations are considered as previous constraints and the authors give an example with physical meaning. With the equations of the theorem we have now the whole set of constraints for a graded Lagrangian of arbitrary order.

(5) If the subalgebra of variations chosen is the subalgebra that commutes with the subalgebra of horizontal vector fields, then a new set of equations appears:

$$\left[j^\infty(\sigma)^*\left(\frac{\partial L}{\partial s^J}\right)\right]^\sim=0, \qquad \forall J=1,\ldots,n_1.$$

This fact points out a difference with the classical variational calculus.

The decomposition of the part in $H^1_{m,0}$ of $\partial\lambda$ as the sum $\omega+D_0\Omega$ is, in a certain sense, unique. For each coordinate open compatible with the global section $\eta:\mathscr{C}^\infty(X)\to A(X)$ it is possible to state an analogous of lemma 5.5, p. 560 of [9].

COROLLARY 4.3. – *Let* $(W, \mathscr{A}^\infty(W))$ *be a coordinate open of* $J^\infty(B/A)$. *Let* μ *be a differential form in* $H^1_{m,0}(W)$, *then* μ *can be uniquely written as* $\omega+D_0\Omega$ *where* ω *is a form in* $H^1_{m,0}(W)$ *generated by*

$$\{\partial y^j_{\varnothing,\beta}\wedge dx^1\wedge\ldots\wedge dx^m,\ \partial t^J_{\varnothing,\beta}\wedge dx^1\wedge\ldots\wedge dx^m\}$$

and $\Omega\in H^1_{m-1,0}(W)$.

5. BEREZINIAN LAGRANGIAN DENSITIES

Let us recall the intrinsic construction of the Berezinian shearf given in [5]. The Berezinian sheaf can be globally described as follows: Let (X, A) be a graded manifold of graded dimension (m, n) and let $P^k(A)$

be the sheaf of k-order differential operators of A. This sheaf has two essentially different structures of A-module: The left structure is given by $(f.\mathrm{P})(g)=f.\mathrm{P}(g)$, and the right structure is given by $(\mathrm{P}.f)(g)=\mathrm{P}(f.g)$ (over every open subset). One has that if $\{x^i, s^{\mathrm{I}}\}$ are A-coordinates in an open subset $\mathrm{U}\subset\mathrm{X}$, then $\mathrm{P}^k(\mathrm{A}(\mathrm{U}))$ is free with basis

$$\left\{\frac{\partial^{|\alpha|}}{\partial x^\alpha}\circ\frac{\partial^{d(\beta)}}{\partial s^\beta};\ \forall\,\alpha,\ \beta \text{ such that } |\alpha|+d(\beta)\leqq k\right\}$$

for both structures of A-module.

Let Ω_{A}^m be the module of m-forms on (X, A).

Let us consider the sheaf $\mathrm{P}^k(\mathrm{A}, \Omega_{\mathrm{A}}^m)=\Omega_{\mathrm{A}}^m\otimes_{\mathrm{A}}\mathrm{P}^k(\mathrm{A})$ of m-form valued k-order differential operators and for every open subset $\mathrm{U}\subset\mathrm{X}$, let $\mathrm{K}_n(\mathrm{U})$ be the set of operators $\mathrm{P}\in\mathrm{P}^n(\mathrm{A}(\mathrm{U}), \Omega^m(\mathrm{U}, \mathrm{A}))$ such that for every $f\in\mathrm{A}(\mathrm{U})$ with compact support, there exists an ordinary $(m-1)$-form of compact support, ω fulfilling $d\omega=\mathrm{P}(f)^\sim$. Hence K_n is a submodule of $\mathrm{P}^n(\mathrm{A}, \Omega_{\mathrm{A}}^m)$ for its right structure and one obtains the following description of the Berezinian sheaf (*see* [5] th. 2.2):

$$\mathscr{B}(\mathrm{A})=\frac{\mathrm{P}^n(\mathrm{A}, \Omega_{\mathrm{A}}^m)}{\mathrm{K}_n}.$$

According to this description a local basis of $\mathscr{B}(\mathrm{A})$ can be given explicitly: If $\{x^i, s^{\mathrm{I}}\}$ are A-coordinates in an open subset $\mathrm{U}\subset\mathrm{X}$, then

$$\Gamma(\mathrm{U}, \mathscr{B}(\mathrm{A}))=\left[dx^1\wedge\ldots\wedge dx^m\otimes\frac{\partial}{\partial s^1}\circ\ldots\circ\frac{\partial}{\partial s^n}\right].\mathrm{A}(\mathrm{U}),$$

where [ ] stands for the equivalence class modulo K_n.

Now the Berezin integral can be defined over the sections with compact support of the Berezinian sheaf by means of the formula:

$$\int_{\mathrm{Ber}}:\Gamma_c(\mathrm{X}, \mathscr{B}(\mathrm{A}))\to\mathbb{R}$$
$$\int_{\mathrm{Ber}}[\mathrm{P}]=\int_{\mathrm{X}}\mathrm{P}(1)^\sim,$$

where X is assumed to be oriented and the right hand side integral is taken with respect to this orientation.

If (X, A) is the standard linear graded manifold $(\mathbb{R}^m, \mathrm{C}^\infty(\mathbb{R}^m)\otimes\Lambda\,\mathbb{R}^n)$ and $f\in\mathrm{C}^\infty(\mathbb{R}^m)\otimes\Lambda\,\mathbb{R}^n$, then

$$\int_{\mathrm{Ber}}\left[dx^1\wedge\ldots\wedge dx^m\otimes\frac{\partial}{\partial s^1}\circ\ldots\circ\frac{\partial}{\partial s^n}\right].f$$
$$=(-1)^{\binom{n}{2}}\int_{\mathrm{U}}f_{\{1,2,\ldots,n\}}\,dx^1\wedge\ldots\wedge dx^m,$$

where $f_{\{1,2,\ldots,n\}}$ is the last coefficient in the development $f=\sum_{\beta} f_\beta . s^\beta$ in products of the s^J's.

Let us remark that

$$\frac{\partial}{\partial s^1}\circ\ldots\circ\frac{\partial}{\partial s^n} f=(-1)\binom{n}{2} f_{\{1,2,\ldots,n\}}.$$

Note that this definition is fully opposed to the definition of the graded integral. The Berezinian integral is defined by means of the last (or higher) coefficient of the development, instead of the graded integral, defined by means of the first (or independent) coefficient.

Infinite order Berezinian sheaf

Let $p:(Y, B)\to(X, A)$ be a submersion of graded manifolds with graded dimensions $\dim(X, A)=(m, n)$ and $\dim(Y, B)=(m+m_1, n+n_1)$. Given $P\in P^k(A, \Omega^m_A)$ let $P^H:\mathscr{A}^\infty\to H^0_m$ be k-order operator defined by

$$j^\infty(\sigma)^* P^H f=P j^\infty(\sigma)^* f, \qquad \forall f\in\mathscr{A}^\infty, \quad \forall \sigma\in\Gamma(p).$$

We will call P^H the total or horizontal lifting of P. Let us denote by $PH^k(\mathscr{A}^\infty, H^0_m)$ [resp. $KH_n(\mathscr{A}^\infty)$] the sheaf consisting of those operators of $P^k(\mathscr{A}^\infty, H^0_m)$ that are horizontal liftings of operators of $P^k(A, \Omega^m_A)$ [resp. $K_n(A)$]. Then, the infinite order Berezinian sheaf is defined as

$$\mathscr{B}^\infty(A^\infty)=\frac{PH^n(\mathscr{A}^\infty, H^0_m)}{KH_n(\mathscr{A}^\infty)}\otimes\mathscr{A}^\infty.$$

According to this description a local basis of $\mathscr{B}^\infty(A^\infty)$ can be given explicitly: If $\{x^i, s^I\}$ are the graded A-coordinates for the coordinate open $(U, A(U))$, and $\{x^i, y^j, s^I, t^J\}$ are the graded B-coordinates for the coordinate open $(V, B(V))$ with a suitable $V\subset q^{-1}(U)$ then if W is an open subset of X^∞ such that $W\subset q_\infty^{-1}(U)$ we have

$$\Gamma(W, \mathscr{B}^\infty(A^\infty))=\left[dx^1\wedge\ldots\wedge dx^m\otimes\frac{d}{ds^1}\circ\ldots\circ\frac{d}{ds^n}\right].A^\infty(W),$$

where [ ] stands for the equivalence class modulo KH_n.

DEFINITION. — *A Berezinian Lagrangian density is an element*

$$[P^H].f\in\mathscr{B}^\infty(\mathscr{A}^\infty).$$

Let $\sigma:(X, A)\to(Y, B)$ be a local section of p. Let us define

$$j^\infty(\sigma)^*([P^H].f):=[P].(j^\infty(\sigma)^* f)\in\mathscr{B}(A).$$

Note that $j^\infty(\sigma)^* f\in A$, and that if $P^H\in KH_n(\mathscr{A}^\infty)$ then, by definition, $P\in K_n(A)$.

We will need the following lemma that relates Lie derivatives with differential operators of $\mathrm{PH}^n(\mathscr{A}^\infty, \mathrm{H}^0_m)$.

LEMMA 5.1. — *Let* Z *be a vertical vector field on* (Y, B) *and let* $\mathrm{P^H} \in \mathrm{PH}^n(\mathscr{A}^\infty, \mathrm{H}^0_m)$. *Then, as operators acting on* $\mathscr{A}^\infty$,

$$[\mathscr{L}_{\mathrm{Z}^\infty}, \mathrm{P^H}] = 0.$$

Proof. — The question is local and linear. Let us suppose that locally $\mathrm{P^H}$ is written as

$$dx^\alpha \wedge ds^\beta \otimes \frac{d}{dx^\gamma} \circ \frac{d}{ds^\delta} . g, \qquad g \in \mathrm{A},$$

such that $|\alpha| + d(\beta) = m$ and $|\gamma| + d(\delta) \leqq n$, where we identify $g \in \mathrm{A}$ and $q^*_\infty g \in \mathscr{A}^\infty$. Then, for $f \in \mathscr{A}^\infty$,

$$\mathrm{P^H}(f) = \frac{d}{dx^\gamma} \circ \frac{d}{ds^\delta}(gf)\, dx^\alpha \wedge ds^\beta = \mathscr{L}_{d/dx^\gamma} \circ \mathscr{L}_{d/ds^\delta}(gf)\, dx^\alpha \wedge ds^\beta.$$

Note that by lemma 2.1, for vertical and homogeneous Z, we have

$$\mathscr{L}_{\mathrm{Z}^\infty} \mathscr{L}_{d/dx^i} = \mathscr{L}_{d/dx^i} \mathscr{L}_{\mathrm{Z}^\infty}, \qquad \text{and} \qquad \mathscr{L}_{\mathrm{Z}^\infty} \mathscr{L}_{d/ds^\mathrm{I}} = (-1)^{|\mathrm{Z}|} \mathscr{L}_{d/ds^\mathrm{I}} \mathscr{L}_{\mathrm{Z}^\infty},$$

and also

$$\mathscr{L}_{\mathrm{Z}^\infty}(dx^\alpha \wedge ds^\beta) = 0.$$

Thus

$$\mathscr{L}_{\mathrm{Z}^\infty} \circ \mathrm{P^H}(f) = (-1)^{d(\delta)\,|\mathrm{Z}|} \mathscr{L}_{d/dx^\gamma} \circ \mathscr{L}_{d/ds^\delta} \circ \mathscr{L}_{\mathrm{Z}^\infty}(gf)\, dx^\alpha \wedge ds^\beta.$$

Finally, for vertical Z, $\mathrm{Z}^\infty(g) = 0$ because $g \in \mathrm{A}$, and thus

$$\begin{aligned} \mathscr{L}_{\mathrm{Z}^\infty} \circ \mathrm{P^H}(f) &= (-1)^{(d(\delta) + |g|)\,|\mathrm{Z}|} \mathscr{L}_{d/dx^\gamma} \circ \mathscr{L}_{d/ds^\delta}(g\, \mathscr{L}_{\mathrm{Z}^\infty}(f))\, dx^\alpha \wedge ds^\beta \\ &= (-1)^{|\mathrm{P}|\,|\mathrm{Z}|} \mathrm{P^H} \circ \mathscr{L}_{\mathrm{Z}^\infty}(f). \end{aligned}$$ ■

This lemma suggests to give the following definition of Lie derivative of Berezinian Lagrangian densities with respect to the prolongations of vertical vector fields.

DEFINITION. — Let Z be a vertical vector field on (Y, B) and let $[\mathrm{P^H}].f \in \mathscr{B}^\infty(\mathscr{A}^\infty)$ be a Berezinian Lagrangian density. The Lie derivative of $[\mathrm{P^H}].f$ with respect to Z^∞ is defined, for homogeneous P and Z, by

$$\mathscr{L}_{\mathrm{Z}^\infty}([\mathrm{P^H}].f) = (-1)^{|\mathrm{P}|\,|\mathrm{Z}|}[\mathrm{P^H}].(\mathscr{L}_{\mathrm{Z}^\infty} f) = (-1)^{|\mathrm{P}|\,|\mathrm{Z}|}[\mathrm{P^H}].\mathrm{Z}^\infty(f).$$

We can now define the Berezinian variational principle. A local section of p, $\sigma : (\mathrm{X}, \mathrm{A}) \to (\mathrm{Y}, \mathrm{B})$ is a Berezinian critical section for the functional defined by the Berezinian Lagrangian density $[\mathrm{P^H}].f \in \mathscr{B}^\infty(\mathscr{A}^\infty)$ if for every Z vertical vector field on (Y, B), whose support has compact image on the domain of σ, we have that

$$\int_{\text{Ber}} j^{\infty}(\sigma)^* \mathscr{L}_{Z^{\infty}}([P^H].f)=0.$$

Let $(W, \mathscr{A}^{\infty}(W))$ be a coordinate open of $J^{\infty}(B/A)$ with coordinate functions $\{x^i, y^j, s^I, t^J\}$.

If $[P^H].f \in \mathscr{B}^{\infty}(\mathscr{A}^{\infty}(W))$ is a Berezinian Lagrangian density defined on $(W, \mathscr{A}^{\infty}(W))$ then we can define the graded Lagrangian density $P^H(f) \in H^0_{m,0}(W)$. This association is, in a certain sense, unique. Two graded Lagrangian densities are equivalent if they define the same functional and the same graded variational problem. Let us suppose that $P \in KH_n(\mathscr{A}^{\infty}(W))$ and that $f \in \mathscr{A}^{\infty}(W)$, then the graded Lagrangian $P^H(f)$ defines a trivial functional and a trivial graded variational problem. Indeed, if σ is a local section of p whose support has compact image on $p_{\infty}(W)$, then

$$\int_X j^{\infty}(\sigma)^* P^H(f) = \int_X P(j^{\infty}(\sigma)^* f),$$

and, by definition of $K(A)$, this expression is equal to $\int_X d\omega$ for an $(m-1)$-form, ω, on (X, A) whose support thas compact image on $p_{\infty}(W)$. Then the integral vanishes and the functional is trivial.

Note that if $[P^H].f$ is a k-order Berezinian Lagrangian density, then $P^H(f)$ is an $(n+k)$-order graded Lagrangian density. The order grows as many times as indicated by the odd dimension of the base manifold.

THEOREM 5.2 (Higher Order Comparison Theorem). — *Let* $(W, \mathscr{A}^{\infty}(W))$ *be a coordinate open of* $J^{\infty}(B/A)$ *with coordinate functions* $\{x^i, y^j, s^I, t^J\}$ *and let* $[P^H].f \in \mathscr{B}^{\infty}(\mathscr{A}^{\infty}(W))$ *be a Berezinian Lagrangian density defined on* $(W, \mathscr{A}^{\infty}(W))$. *Then,* $P^H(f) \in H^0_{m,0}(W)$ *is a graded Lagrangian density defined on* $(W, \mathscr{A}^{\infty}(W))$ *that defines an equivalent variational problem because they have the same critical sections*, i.e., *a local section* σ *is a Berezinian critical section for the Berezinian variational problem defined by* $[P^H].f$ *if and only if it is a graded critical section for the graded variational problem defined by* $P^H(f)$.

Moreover, if

$$[P^H].f = \left[dx^1 \wedge \ldots \wedge dx^m \otimes \frac{d}{ds^1} \circ \ldots \circ \frac{d}{ds^n} \right].f, \qquad f \in A^{\infty}(W),$$

then

$$P^H(f) = \frac{d}{ds^1} \circ \ldots \circ \frac{d}{ds^n}(f)\, dx^1 \wedge \ldots \wedge dx^m$$
$$= \mathscr{L}_{d/ds^1} \circ \ldots \circ \mathscr{L}_{d/ds^n}(f)\, dx^1 \wedge \ldots \wedge dx^m.$$

Proof. – By application of definitions and lemma 5.1.

$$\int_{\mathrm{Ber}} j^{\infty}(\sigma)^{*}\mathscr{L}_{Z^{\infty}}([\mathrm{P}^{\mathrm{H}}].f)=(-1)^{|\mathrm{P}|\,|\mathrm{Z}|}\int_{\mathrm{Ber}} j^{\infty}(\sigma)^{*}([\mathrm{P}^{\mathrm{H}}].\mathscr{L}_{Z^{\infty}}f)$$
$$=(-1)^{|\mathrm{P}|\,|\mathrm{Z}|}\int_{\mathrm{Ber}}([\mathrm{P}].j^{\infty}(\sigma)^{*}\mathscr{L}_{Z^{\infty}}f)=(-1)^{|\mathrm{P}|\,|\mathrm{Z}|}\int_{\mathrm{X}}(\mathrm{P}(j^{\infty}(\sigma)^{*}\mathscr{L}_{Z^{\infty}}f))^{\sim}$$
$$=(-1)^{|\mathrm{P}|\,|\mathrm{Z}|}\int_{\mathrm{X}}(j^{\infty}(\sigma)^{*}(\mathrm{P}^{\mathrm{H}}(\mathscr{L}_{Z^{\infty}}f)))^{\sim}=\int_{\mathrm{X}}(j^{\infty}(\sigma)^{*}(\mathscr{L}_{Z^{\infty}}\mathrm{P}^{\mathrm{H}}(f)))^{\sim}. \quad \blacksquare$$

6. EULER-LAGRANGE EQUATIONS FOR THE BEREZINIAN LAGRANGIAN DENSITIES

Let T the total odd multiindex $\mathrm{T}=\{1, 2, \ldots, n\}$.

We will study the Euler-Lagrange equations of theorem 4.2 for the graded Lagrangian densities that come from Berezinian Lagrangian densities, *i.e.*, graded Lagrangian densities that locally are written as

$$\frac{d^n}{ds^{\mathrm{T}}}(f)\,dx^1\wedge\ldots\wedge dx^m, \qquad f\in\mathscr{A}^{\infty}(\mathrm{W}),$$

for a suitable open subset $\mathrm{W}\subset\mathrm{X}^{\infty}$, or equivalently, that locally are written as

$$\mathscr{L}_{d/ds^1}\circ\ldots\circ\mathscr{L}_{d/ds^n}(f\,dx^1\wedge\ldots\wedge dx^m), \qquad f\in\mathscr{A}^{\infty}(\mathrm{W}),$$

where $\mathscr{L}_{\mathrm{Z}}$ denotes the Lie derivative with respect to the vector field Z.

By corollary 4.3 we know that, in a coordinate open, $(d\lambda)^1_{m,0}$ can be uniquely written as $\omega^{\lambda}+\mathrm{D}_0\Omega^{\lambda}$, where $(d\lambda)^1_{m,0}$ denotes the part in $\mathrm{H}^1_{m,0}$ of $d\lambda$.

Let us denote by $\mathscr{P}$ the operator, defined in a coordinate open, $\mathscr{L}_{d/ds^1}\circ\ldots\circ\mathscr{L}_{d/ds^n}$. Then, $\mathscr{P}\lambda$ is another graded Lagrangian density defined in the same coordinate open, and by the corollary 4.3, $(d\mathrm{P}\lambda)^1_{m,0}$ can be uniquely written as $\omega^{\mathscr{P}\lambda}+\mathrm{D}_0\Omega^{\mathscr{P}\lambda}$. But, by definition of $\mathscr{P}$, $d\mathscr{P}\lambda=\mathscr{P}d\lambda$, and then $(d\mathscr{P}\lambda)^1_{m,0}=(\mathscr{P}d\lambda)^1_{m,0}=\mathscr{P}\omega^{\lambda}+\mathscr{P}\mathrm{D}_0\Omega^{\lambda}$. $\mathscr{P}\mathrm{D}_0\Omega^{\lambda}$ is a form in the image of D_0 because, by definition of $\mathscr{P}$ and D_0, we have that $\mathscr{P}\mathrm{D}_0\Omega^{\lambda}=\mathrm{D}_0\mathscr{P}\Omega^{\lambda}$. And now, by uniqueness, $\omega^{\mathscr{P}\lambda}=\mathscr{P}\omega^{\lambda}$. The coefficients of such differential form will give the variational equations.

Let us note that the differential form $\mathscr{P}\omega^{\lambda}\in\mathrm{H}^1_{m,0}$ is D-closed. Indeed $\mathrm{D}(\mathscr{P}\omega^{\lambda})=\mathrm{D}_1(\mathscr{P}\omega^{\lambda})$ and this form is a sum of terms like $\mathscr{L}_{d/ds^{\mathrm{I}}}(\mathscr{P}\omega^{\lambda})\wedge ds^{\mathrm{I}}$. By definition of $\mathscr{P}$, $\mathscr{L}_{d/ds^{\mathrm{I}}}\circ\mathscr{P}=0$. Thus, $\mathrm{D}(\mathscr{P}\omega^{\lambda})=0$.

This property allows a kind of partial integration for the odd coordinates.

LEMMA 6.1. — *Let* $\mu \in H^1_{m,0}$. *Then* $D\mu = 0$ *if and only if for each coordinate open* $(W, \mathscr{A}^\infty(W))$ *in* $J^\infty(B/A)$ *there exists a differential form*

$$\gamma = \sum_{\beta, j} dx^1 \wedge \ldots \wedge dx^m \wedge \partial z^j_{\alpha, \varnothing} \gamma^j_\alpha \in H^1_{m,0}(W),$$

such that, in W, $\mu = \mathscr{P}\gamma$, *where* $\mathscr{P} = \mathscr{L}_{d/ds^1} \circ \ldots \circ \mathscr{L}_{d/ds^n}$.

Proof. — Let us suppose that $\mu = \sum_{\beta, j} dx^1 \wedge \ldots \wedge dx^m \wedge \partial z^j_{\alpha, \varnothing} \mu^j_{\alpha, \beta}$. (Subindexes of $\mu^j_{\alpha, \beta}$ and γ^j_α denotes components.)

Then

$$D_0(\mu) = D_1(\mu) = (-1)^m dx^1 \wedge \ldots \wedge dx^m \wedge \sum_{\beta, j} \sum_{I} \left(-(-1)^{sg(\beta, I)} ds^I \wedge \partial z^j_{\alpha, \beta + I} \mu^j_{\alpha, \beta} + (-1)^{d(\beta) + sg(z)} ds^I \wedge \partial z^j_{\alpha, \beta} \frac{d\mu^j_{\alpha, \beta}}{ds^I} \right).$$

This expression is equal to zero if and only if, for all I and for all β such that $I \in \beta$

$$\mu^j_{\alpha, \beta - I} = (-1)^{1 + sg(\beta - I, I) + sg(z) + d(\beta)} \frac{d\mu^j_{\alpha, \beta}}{ds^I}.$$

By induction, we can see that for all β

$$\mu^j_{\alpha, \beta} = (-1)^{C(\beta)} \frac{d\mu^j_{\alpha, T}}{ds^{T - \beta}},$$

where if $\beta = \{\beta_1, \ldots, \beta_k\}$, then $C(\beta) = \sum_{i=1}^{k} \beta_i + k\left(n + sg(z) + \frac{k-1}{2}\right)$.

Thus, if we put $\gamma^j_\alpha = \mu^j_{\alpha, T}$, then it is easy to see that $\mu = \mathscr{P}\gamma$. ■

Let us apply this lemma to

$$\omega^{P\lambda} = \partial z^j_{\alpha, \beta} \omega^j_{\alpha, \beta} \wedge dx^1 \wedge \ldots \wedge dx^m.$$

As we have seen above, $D\omega^{P\lambda} = 0$, then, its coefficients satisfy the following relations: the coefficients with $\beta < T$ are horizontal derivatives, with respect to $s^{T-\beta}$, of the coefficient with $\beta = T$.

$$\omega^j_{\alpha, \beta} = \frac{d\omega^j_{\alpha, T}}{ds^\beta}.$$

Let us remember that the variational equations of theorem 4.2 for the graded Lagrangian density $\omega^{P\lambda}$ are

$$[j^\infty(\sigma)^*(\omega^j_{\alpha, \beta})]^\sim = 0, \qquad [j^\infty(\sigma)^*(\omega^J_{\alpha, \beta})]^\sim = 0, \qquad \forall j, J, \beta.$$

Thus, substituting, we get

$$\left[j^\infty(\sigma)^* \left(\frac{d\omega^j_{\alpha, T}}{ds^\beta} \right) \right]^\sim = 0, \qquad \left[j^\infty(\sigma)^* \left(\frac{d\omega^J_{\alpha, T}}{ds^\beta} \right) \right]^\sim = 0, \qquad \forall j, J, \beta.$$

By definition of the horizontal lift, we have that

$$\frac{\partial}{\partial s^{\beta}}[j^{\infty}(\sigma)^{*}(\omega^{j}_{\alpha,\mathrm{T}})]^{\sim}=0, \qquad \frac{\partial}{\partial s^{\beta}}[j^{\infty}(\sigma)^{*}(\omega^{\mathrm{J}}_{\alpha,\mathrm{T}})]^{\sim}=0, \qquad \forall j,\ \mathrm{J},\ \beta.$$

But these scalar equations are equivalents to the following equations in the graded ring

$$j^{\infty}(\sigma)^{*}(\omega^{j}_{\alpha,\mathrm{T}})=0, \qquad j^{\infty}(\sigma)^{*}(\omega^{\mathrm{J}}_{\alpha,\mathrm{T}})=0, \qquad \forall j,\ \mathrm{J}.$$

Then the system of $(m_1+n_1)2^n$ scalar equations of theorem 4.2 has been reduced to just m_1+n_1 equations, one for each fibre coordinate, but, the equations are now defined on the graded ring. This fact is the cornerstone of the deduction of the variational equations of the graded Lagrangians that come from Berezinian Lagrangians.

Note that we need all the equations of theorem 4.2. Even for 0-order Berezinian Lagrangian densities, the associated graded Lagrangian densities are of order n.

To go on with the deduction we need to compute the brackets

$$\left[\frac{\partial}{\partial y^{j}_{\alpha,\beta}},\frac{d}{ds^{\mathrm{I}}}\right] \quad \text{and} \quad \left[\frac{\partial}{\partial t^{\mathrm{J}}_{\alpha,\beta}},\frac{d}{ds^{\mathrm{I}}}\right].$$

Let $h\in\mathscr{A}^{\infty}$, then,

$$\frac{\partial}{\partial y^{j}_{\alpha,\beta}}\frac{dh}{ds^{\mathrm{I}}}=\frac{\partial}{\partial y^{j}_{\alpha,\beta}}\left(\frac{\partial h}{\partial s^{\mathrm{I}}}+(-1)^{sg(\delta,\mathrm{I})}y^{k}_{\gamma,\delta+\mathrm{I}}\frac{\partial h}{\partial y^{k}_{\gamma,\delta}}+(-1)^{sg(\delta,\mathrm{I})}t^{\mathrm{K}}_{\gamma,\delta+\mathrm{I}}\frac{\partial h}{\partial t^{\mathrm{K}}_{\gamma,\delta}}\right)$$
$$=(-1)^{d(\beta)}\frac{d}{ds^{\mathrm{I}}}\frac{\partial h}{\partial y^{j}_{\alpha,\beta}}+(-1)^{sg(\beta-\mathrm{I},\mathrm{I})}\frac{\partial}{\partial y^{j}_{\alpha,\beta-\mathrm{I}}},$$

where the following convention is used: if $\mathrm{I}\notin\beta$ then the second term does not appear. The other bracket is computed in a similar way.

Hence we can state the following.

LEMMA 6.2. – *For all* $\mathrm{I}=1,\ldots,n$ *and for all* β, $0\leqq d(\beta)\leqq n$

$$\left[\frac{\partial}{\partial z^{j}_{\alpha,\beta}},\frac{d}{ds^{\mathrm{I}}}\right]=(-1)^{sg(\beta-\mathrm{I},\mathrm{I})}\frac{\partial}{\partial z^{j}_{\alpha,\beta-\mathrm{I}}}.$$

We can see by induction that

$$\frac{\partial}{\partial z^{j}_{\alpha,\mathrm{T}}}\frac{d^{n}}{ds^{\mathrm{T}}}=\sum_{\varnothing\leqq\beta\leqq\mathrm{T}}(-1)^{c(\beta)}\frac{d^{d(\mathrm{T}-\beta)}}{ds^{\mathrm{T}-\beta}}\frac{\partial}{\partial z^{j}_{\alpha,\mathrm{T}-\beta}},$$

where $c(\beta)=(n+1)d(\beta)+(n+d(\beta))(sg(z)+n)$.

Applying this lemma to the equations $j^{\infty}(\sigma)^{*}(\omega^{j}_{\alpha,\mathrm{T}})=0$, $j^{\infty}(\sigma)^{*}(\omega^{\mathrm{J}}_{\alpha,\mathrm{T}})=0$, we can state the following.

THEOREM 6.3. — *Let* (W, $\mathcal{A}^\infty$(W)) *be a coordinate open of* J^∞(B/A). *The variational equations for the graded Lagrangian density*

$$\frac{d^n}{ds^T}(f)\,dx^1 \wedge \ldots \wedge dx^m, \qquad f \in \mathcal{A}^\infty(W)$$

are

$$j^\infty(\sigma)^*\left(\sum_\alpha \sum_{\varnothing \leqq \beta \leqq T} (-1)^{|\alpha|+D(\beta)} \frac{d^{|\alpha|}}{dx^\alpha}\frac{d^{d(\beta)}}{ds^\beta}\frac{\partial f}{\partial z^j_{\alpha,\beta}}\right)=0,$$

for all $z^j = y^1, \ldots, y^{m_1}; t^1, \ldots, t^{n_1}$, and where $D(\beta) = d(\beta)(sg(z)+1)$.

Remarks. — (1) Note that the natural morphism does not appear. The equations live in the graded ring.

(2) These are the standard Euler-Lagrange equations one would expect for the Berezinian Lagrangian density $\left[dx^1 \wedge \ldots \wedge dx^m \otimes \frac{d}{ds^1} \circ \ldots \circ \frac{d}{ds^n}\right].f$ with $f \in \mathcal{A}^\infty$(W).

(3) If, $f \in \mathcal{A}^1$(W), the above equations are reduced to

$$j^\infty(\sigma)^*\left(\frac{\partial f}{\partial z^j} - \frac{d}{dx^i}\frac{\partial f}{\partial z^j_{i,\varnothing}} + (-1)^{sg(z)}\frac{d}{ds^l}\frac{\partial f}{\partial z^j_{\varnothing,l}}\right)=0,$$

for all $z^j = y^1, \ldots, y^{m_1}; t^1, \ldots, t^{n_1}$.

7. AN EXAMPLE

As a curious example let us study the variational equations when both graded manifolds are of zero even dimension, *i.e.*, when X and Y are reduced to a point. Thus

$$(X, A) = (., \Lambda^* \mathbb{R}^n) \quad \text{and} \quad (Y, B) = (., \Lambda^* \mathbb{R}^{n+n_1}) = (., \Lambda^* \mathbb{R}^n \otimes \Lambda^* \mathbb{R}^{n_1}).$$

A local section, σ, of $p : (X, A) \to (Y, B)$ induces a $\mathbb{Z}_2$-algebra homomorphism, σ^*, such that $\sigma^*(a \otimes 1) = a$. Then, a local section is determined by a $\mathbb{Z}_2$-algebra homomorphism from $\Lambda^* \mathbb{R}^{n_1}$ into $\Lambda^* \mathbb{R}^n$, and such homomorphism is determined by the image of the generators of $\Lambda^* \mathbb{R}^{n_1}$.

Let us fix the number of generators, $n=3$ and $n_1=1$, and let $\{s^1, s^2, s^3\}$ be a coordinate system of (X, A) and $\{s^1, s^2, s^3, t\}$ be a coordinate system of (Y, B). Thus a local section, σ, is determined by

$$\sigma^*(t) = q_1 s^1 + q_2 s^2 + q_3 s^3 + q_{123} s^1 s^2 s^3,$$

where $q_1, q_2, q_3, q_{123} \in \mathbb{R}$.

Let us consider the element of $\mathscr{A}^1, f=\frac{1}{3}\sum_{I=1\ldots 3} h^I(t_I)^3$, where t_I are the coordinate functions of the graded manifold of first order graded jets, and where $h^1, h^2, h^3 \in A$.

Note that, in this case, a graded Lagrangian density is a 0-form. The variational equations for the graded Lagrangian density $\frac{df}{ds^T}$ are

$$j^\infty(\sigma)^* \sum_{I=1\ldots 3} \frac{\partial h^I}{\partial s^I}(t_I)^2=0.$$

Substituting, we get

$$\frac{\partial h^1}{\partial s^1}(q_1+q_{123}s^2 s^3)^2+\frac{\partial h^2}{\partial s^2}(q_2-q_{123}s^1 s^3)^2+\frac{\partial h^3}{\partial s^3}(q_3+q_{123}s^1 s^2)^2=0.$$

Let us choice $h^I=c^I s^I$, with $c^I \in \mathbb{R}$. It is easy to see that the equation becomes equivalent to the following set of equations:

$$c^1(q_1)^2+c^2(q_2)^2+c^3(q_3)^2=0,$$
$$c^1 q_1 q_{123}=0, \qquad c^2 q_2 q_{123}=0, \qquad c^3 q_3 q_{123}=0.$$

Thus, the points of the conical surfaces $c^1(q_1)^2+c^2(q_2)^2+c^3(q_3)^2=0$ and $q_{123}=0$ give solutions of the variational problem. Taking, for instance, $c^1=1$, $c^2=1$, $c^3=-1$, we have that the graded Lagrangian density

$$\frac{1}{3}\frac{d}{ds^T}(s^1(t_1)^3+s^2(t_2)^3-s^3(t_3)^3)$$

achieves a minimum at sections of p given by

$$\sigma^*(t)=q_1 s^1+q_2 s^2+q_3 s^3,$$

where (q_1, q_2, q_3) is a point of the cone $x^2+y^2-z^2=0$.

Another possible choice for h^I would be $h^I=c^I s^I+d^I s^1 s^2 s^3$, with $c^I, d^I \in \mathbb{R}$ such that $\sum_{I=1\ldots 3}\frac{(c^I)^3}{(d^I)^2}=0$. The variational equations are now equivalent to the following set of equations:

$$c^1(q_1)^2+c^2(q_2)^2+c^3(q_3)^2=0,$$
$$2c^1 q_1 q_{123}+d^1(q_1)^2=0,$$
$$2c^2 q_2 q_{123}+d^2(q_2)^2=0,$$
$$2c^3 q_3 q_{123}+d^3(q_3)^2=0.$$

Solutions are now found with arbitrary q_{123} and $q_i=\frac{2c^i q_{123}}{d^i}$, $i=1, 2, 3$.

REFERENCES

[1] M. BATCHELOR, The Structure of Supermanifolds, *Trans. Amer. Math. Soc.*, Vol. **253**, 1979, pp. 329-338.

[2] D. HERNÁNDEZ RUIPÉREZ and J. MUÑOZ MASQUÉ, Global Variational Calculus on Graded Manifolds, I: Graded Jet Bundles, Structure 1-Form and Graded Infinitesimal Contact Transformations, *J. Math. Pures Appl.*, Vol. **63**, 1984, pp. 283-309.

[3] D. HERNÁNDEZ RUIPÉREZ and J. MUÑOZ MASQUÉ, Global Variational Calculus on Graded Manifolds, II, *J. Math. Pures Appl.*, Vol. **63**, 1985, pp. 87-104.

[4] D. HERNÁNDEZ RUIPÉREZ and J. MUÑOZ MASQUÉ, Variational Berezinian Problems and their Relationship with Graded Variational Problems, in "Proceedings of the Conference on Differential Geometric Methods in Mathematical Physics, Salamanca 1985, *Lect. Notes Math. Ser.*, No. 1251", Springer Verlag, 1987, pp. 137-149.

[5] D. HERNÁNDEZ RUIPÉREZ and J. MUÑOZ MASQUÉ, Construction intrinsèque du faisceau de Berezin d'une variété graduée, *C. R. Acad. Sci. Paris*, *t.* **301**, Series I, 1985, pp. 915-918.

[6] J. B. KOSTANT, Graded Manifolds, Graded Lie Theory and Prequantization, in "Proceedings of the Conference on Differential Geometric Methods in Mathematical Physics, Bonn 1975, *Lect. Notes Math., Ser.*, No. 570", Springer Verlag, 1977, pp. 177-306.

[7] D. A. LEITES, Introduction to the Theory of Supermanifolds, *Russian Math. Surveys*, Vol. **35**, 1, 1980, *Uspehi Math. Nauk.*, Vol. **35**, 1980, pp. 1-64.

[8] J. MUÑOZ MASQUÉ, Poincaré-Cartan forms in higher order variational problems, *Rev. Mat. Iberoramericana*, Vol. **1**, No. 20, 1985, pp. 85-126.

[9] F. TAKENS, A Global Version of the Inverse Problem of the Calculus of Variations, *J. Diff. Geom.*, **14**, 1979, pp. 543-562.

(*Manuscript received January* 2, 1991.)