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Potentials wells in high dimensions II, more about the one well case

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ABSTRACT. — This is a continuation of [S]. We obtain the asymptotics of the lowest eigenvalue of $-h^2\Delta+V$ when V has one point of minimum under some general assumptions. We also obtain a general estimate from below on the gap between the first and the second eigenvalue.

Key words : WKB, potential wells, high dimension.

RÉSUMÉ. — Ceci est une continuation de [S]. On obtient l'asymptotique de la première valeur propre de $-h^2\Delta+V$ quand V admet un point de minimum sous des hypothèses générales. On trouve aussi une minoration générale de l'écart entre la première et la seconde valeur propre.

0. INTRODUCTION

This paper is a continuation of [S]. The original motivation was the study of a model problem of a large number of interacting particles in a common exterior potential with wells, and for the full Schrödinger operator this leads to potential wells in very high dimensions. It turned out however

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that potential wells in high dimensions are of interest also in statistical mechanics and the present paper is to some extent the result of the reading of some notes of Bernard Helffer on models in statistical mechanics. We therefore generalized our WKB-constructions in order to apply to the kind of potential wells that appear in some of these models, and obtained as a result the determination of the bottom of the spectrum for the Dirichlet realization of $-\frac{1}{2}\Delta+V(x)$ in a box $[-r, r]^N$ modulo $\mathcal{O}(N h^\infty)$, uniformly with respect to the dimension, under assumptions on the potential which are somewhat more general than those of [S] and under the assumption that $r>0$ is sufficiently small. Another improvement concerns the bound from below on the gap between the first and the second eigenvalue, where we managed to simplify the treatement by a more efficient use of the creation and annihibition operators introduced in [S]. By using also an estimate of appendix B of Singer-Wong-Yau-Yau [SiWYY], essentially due to Brascamp-Lieb [BL], we were able to make the proof non-asymptotic, and obtain a more general result, where in particular, we have eliminated the assumption that the dimension grows at most as some power of $1/h$. In future works we hope to continue our original plan to study the tunnel effect. The main results of the present paper are given in section 3.

1. STUDY OF $V''(0)^{\pm 1/2}$

Let V be defined in $B(0, 1) \subset \mathbb{C}^N$ (where we use the l^∞-norms and distances if nothing else is indicated). We assume that V is realvalued on the real domain and that:

$$|\nabla V| = \mathcal{O}(1) \quad \text{in } B(0, 1), \tag{1.1}$$

with $|\,.\,|$ indicating the l^∞ norm on $\mathbb{C}^N$.

We adopt the convention that all estimates should be uniform with respect to the dimension N, and that constants are independent of N as long as nothing else is indicated. Our second assumption is:

$V''(0) = D + A$, where D is a diagonal matrix with $D \geqq r_0$, and the norm of A in $\mathcal{L}(l^\infty, l^\infty)$ is $\leqq r_1$. Here $0 < r_1 < r_0$ are constants (independent of N). (1.2)

A is symmetric, so by duality and interpolation, we have

$$\| A \|_{\mathcal{L}(l^p, l^p)} \leqq r_1, \tag{1.3}$$

for $1 \leqq p \leqq \infty$. In particular for $p=2$, we see that $V''(0) \geqq r_0 - r_1 > 0$ and hence $V''(0)^{\pm 1/2}$ are well defined positive symmetric matrices. Let C_0 be a positive constant such that $D \leqq C_0$. (The existence of such a bound follows from Proposition 1.1 in [S].) We then have:

$$V''(0)^{\pm 1/2} = (2\pi i)^{-1} \int_\gamma z^{\pm 1/2} (z - V''(0))^{-1} dz, \tag{1.4}$$

where γ is the positively oriented countour given by the points in $\mathbb{C}$ which are at the distance r_2 from the interval $[r_0, C_0]$. Here $r_2 \in]r_1, r_0[$.

Writing $(z - V''(0))^{-1} = (1 - (z - D)^{-1} A)^{-1} (z - D)^{-1}$, and using that $\| (z - D)^{-1} A \|_{\mathscr{L}(l^p, l^p)} \leqq r_1/r_2 < 1$, we see that $(z - V''(0))^{-1} = \mathcal{O}(1)$ in $\mathscr{L}(l^p, l^p)$ (uniformly with respect to N), and hence that the same conclusion holds for $V''(0)^{\pm 1/2}$.

The next problem is to estimate $\exp(-t V''(0)^{1/2})$ in $\mathscr{L}(l^p, l^p)$ when $t \to \infty$. This can also be done by using countour integrals, however, for later purposes we also need some more detailed information about $V''(0)^{1/2}$:

LEMMA 1.1. – *Under the assumptions above,*

$$V''(0)^{1/2} = \tilde{D} + \tilde{A}, \tag{1.5}$$

where $\tilde{D} = D^{1/2}$, *so that* $r_0^{1/2} \leqq \tilde{D} \leqq C_0^{1/2}$, *and where:*

$$\| \tilde{A} \|_{\mathscr{L}(l^p, l^p)} \leqq r_0^{1/2} - (r_0 - r_1)^{1/2}. \tag{1.6}$$

Proof. – We write first:

$$(z - V''(0))^{-1} = (z - D)^{-1} + (z - D)^{-1} A (z - (D + A))^{-1}, \tag{1.7}$$

so that by (1.4):

$$V''(0)^{1/2} = D^{1/2} + \tilde{A}, \tag{1.8}$$

with

$$\tilde{A} = (2\pi i)^{-1} \int_\gamma z^{1/2} (z - D)^{-1} A (z - V''(0))^{-1} dz. \tag{1.9}$$

The integrand behaves like $\mathcal{O}(|z|^{-3/2})$ near infinity, so we can replace γ by a new contour $\tilde{\gamma} =]-\infty + i0, i0[\cup \{ 0 \cdot e^{-it}; -\pi/2 < t < \pi/2 \} \cup]-i0, -\infty - i0[$, and more precisely, we get:

$$\tilde{A} = \pi^{-1} \int_{-\infty}^{0} (-x)^{1/2} (x - D)^{-1} A (x - V''(0))^{-1} dx. \tag{1.10}$$

Here

$$\| (x - D)^{-1} \| \leqq (r_0 - x)^{-1} \quad \text{in } \mathscr{L}(l^p, l^p),$$

and

$$(x - V''(0))^{-1} = (1 - (x - D)^{-1} A)^{-1} (x - D)^{-1},$$

so

$$\|(x-\mathrm{V}''(0))^{-1}\|_{\mathscr{L}(l^p,l^p)}\leqq(1-r_1/(r_0-x))^{-1}(r_0-x)^{-1}=(r_0-r_1-x)^{-1}.$$

It follows that:

$$\|\tilde{\mathrm{A}}\|_{\mathscr{L}(l^p,l^p)}\leqq(r_1/\pi)\int_{-\infty}^{0}(-x)^{1/2}(r_0-x)^{-1}(r_0-r_1-x)^{-1}\,dx. \tag{1.11}$$

The left hand side can be computed with residus and we get:

$$r_0^{1/2}-(r_0-r_1)^{1/2}. \quad \square \tag{1.12}$$

Using the lemma, we can prove:

PROPOSITION 1.2. — *Under the assumptions above, there is a constant* $\mathrm{C}>0$, *independent of the dimension, such that for all* $p\in[1,+\infty]$:

$$\|\exp(-t\mathrm{V}''(0)^{1/2}\|_{\mathscr{L}(l^p,l^p)}\leqq e^{-t/\mathrm{C}} \quad \textit{when } t\geqq 0, \tag{1.13}$$

$$\|\exp(-t\mathrm{V}''(0)^{1/2}\|_{\mathscr{L}(l^p,l^p)}\leqq e^{-\mathrm{C}t} \quad \textit{when } t\leqq 0. \tag{1.14}$$

Proof. — The estimate (1.14) follows from the fact that $\mathrm{V}''(0)^{1/2}$ is uniformly bounded in $\mathscr{L}(l^p, l^p)$. The estimate is easy to establish in the case $p=\infty$, simply by using Lemma 1.1, and since $\exp(-t\mathrm{V}''(0)^{1/2})$ is symmetric, we get the same estimate for $p=1$ by duality. The general case then follows by interpolation. $\square$

2. THE EICONAL EQUATION

Essentially as in [S], we put

$$x_\pm=2^{-1/2}(x\pm\mathrm{V}''(0)^{-1/2}\xi), \qquad \xi_\pm=2^{-1/2}(\pm\mathrm{V}''(0)^{1/2}x+\xi) \tag{2.1}$$

and we may remark that (x_+, ξ_-) and (x_-, ξ_+) are symplectic coordinates. Put $q_0=\frac{1}{2}\xi^2-\frac{1}{2}\mathrm{V}''(0)x.x$, $q=\frac{1}{2}\xi^2-\mathrm{V}(x)$. Let H_{q_0} and H_q denote the corresponding Hamilton fields. For the H_{q_0} flow we then have:

$$\partial_t x_\pm(t)=\pm\mathrm{V}''(0)^{1/2}x_\pm(t), \qquad \partial_t\xi_\pm(t)=\pm\mathrm{V}''(0)^{1/2}\xi_\pm(t). \tag{2.2}$$

We want to make estimates for the H_q flow. As in [S], we notice the following equivalences between norms: $|(x,\xi)|\sim|x_+|+|x_-|$, $|x_\pm|\sim|\xi_\pm|$. Here and in the following $|\,.\,|$ will denote the l^∞-norm as long as nothing else is indicated. We make the following general remark: Let $\nu(x,\partial_x)=\nu_0(x,\partial_x)+\nu_1(x,\partial_x)$ be a vector field and assume that at some point x_0: $|\nu_1(x_0,\partial_x)|\leqq\varepsilon$. Let $x(t)$ and $x_0(t)$ be the integral curves of ν and ν_0 with $x(0)=x_0(0)=x_0$. Then $|x(t)-x_0(t)|\leqq\varepsilon|t|+\mathcal{O}_\mathrm{N}(|t|^2)$. Hence:

$$|(|x(t)|-|x(0)|)-(|x_0(t)|-|x_0(0)|)|\leqq\varepsilon|t|+\mathcal{O}_\mathrm{N}(|t|^2), \tag{2.3}$$

so if $\partial_t |x(t)|$, $\partial_t |x_0(t)|$ exist at $t=0$, then

$$|(\partial_t |x(t)|)_{t=0} - (\partial_t |x_0(t)|)_{t=0}| \leqq \varepsilon. \tag{2.4}$$

We notice that by Proposition 1.1 of [S]:

$$|H_q - H_{q0}| = \mathcal{O}(|x|^2). \tag{2.5}$$

From (2.2) and the estimates on $\exp(-t V''(0)^{1/2})$ of the preceding section, we notice that for the H_{q0} flow:

$$\partial_t |\xi_\pm| = \pm (\sim 1) |\xi_\pm| \quad \text{almost everywhere}, \tag{2.6}$$

and hence in view of (2.5) and the general estimate (2.3) [*cf.* (2.4)] we get for the H_q flow:

$$\partial_t |\xi_\pm| = \pm (\sim 1) |\xi_\pm| + \mathcal{O}(|x|^2). \tag{2.7}$$

We now restrict the attention to the domain:

$$|\xi_-| \leqq \varepsilon_1 |x|, \quad |x| \leqq \varepsilon_2, \tag{2.8, $\varepsilon_1, \varepsilon_2$}$$

where ε_1 and ε_2 will be small and conveniently chosen. We first look at $\partial_t |x|$ in this domain, where we now differentiate in the H_q direction. We have:

$$\left.\begin{aligned} \partial_t x(t) = \xi(t) = V''(0)^{1/2} x(t) + y(t), \\ |y(t)| \leqq \varepsilon_1 |x(t)| + C |x(t)|^2, \end{aligned}\right\} \tag{2.9}$$

and hence,

$$\partial_t |x(t)| = (\sim 1) |x(t)| + \mathcal{O}(1)(\varepsilon_1 + C|x(t)|) |x(t)|, \tag{2.10}$$

so if ε_1 and ε_2 are small enough:

$$\partial_t |x(t)| = (\sim 1) |x(t)| \text{ (a.e.)}, \tag{2.11}$$

for H_q-integral curves in the domain (2.8, ε_1, ε_2).

We next look at the evolution of $|\xi_-|$. On the part of the boundary of (2.8), where $|\xi_-| = \varepsilon_1 |x|$, we get from (2.7):

$$\begin{aligned} \partial_t |\xi_-| &= -(\sim 1) |\xi_-| + \mathcal{O}(|\xi_-|^2/\varepsilon_1^2) \\ &= -(\sim 1)(1 + \mathcal{O}(|\xi_-|/\varepsilon_1^2)) |\xi_-| \\ &= -(\sim 1)(1 + \mathcal{O}(\varepsilon_2/\varepsilon_1)) |\xi_-| = -(\sim 1) |\xi_-|, \end{aligned} \tag{2.12}$$

where the last equality holds provided that $\varepsilon_2/\varepsilon_1$ is small enough. We have proved:

PROPOSITION 2.1. — *If ε_1 and $\varepsilon_2/\varepsilon_1$ are positive and small enough, then the H_q flow enters the domain (2.8, ε_1, ε_2) through the part of the boundary where $|\xi_-| = \varepsilon_1 |x|$, $|x| < \varepsilon_2$ and leaves through the part where $|x| = \varepsilon_2$.*

Since we are in the holomorphic case (as in [S]), we conclude that there exists a holomorphic function $\varphi(x)$, defined for $|x| < \varepsilon_2$, with

$$|\nabla \varphi(x) - V''(0)^{1/2} x| \leqq \varepsilon_1 |x|, \tag{2.13}$$

such that

$$q(x, \partial_x \varphi(x))=0. \tag{2.14}$$

Moreover,

$$\varphi''(0)=V''(0)^{1/2}, \tag{2.15}$$

and (2.11) implies that

$$|\exp(-t\nabla\varphi\cdot\partial_x)(x)| \leqq e^{-t/C}|x|, \qquad t \geqq 0. \tag{2.16}$$

Combining the last estimate and Proposition 1.2, it is easy to prove that there is a constant $C>0$ independent of the dimension, such that for all $p\in[1, \infty]$:

$$\left.\begin{array}{l} \| d_x \exp(-t\nabla\varphi\cdot\partial_x)(x)\|_{\mathscr{L}(l^p, l^p)} \leqq C e^{-t/C}, \\ |x| < \varepsilon_2, \qquad t\geqq 0. \end{array}\right\} \tag{2.17}$$

In the case $p=\infty$, we could get (2.17) immediately from (2.16) by using the Cauchy inequalities as in [S].

3. CONSEQUENCES FOR THE SPECTRUM FOR THE DIRICHLET PROBLEM IN A BOX

We now make a slight change of notations and denote by φ_0, the solution of the eiconal equation, constructed in the preceding section. We can then proceed exactly as in section 3 of [S] and construct

$$\varphi(x; h) \sim \varphi_0(x)+h\varphi_1(x)+h^2\varphi_2(x)+\dots,$$

with φ_j defined in a complex l^∞ ball $B(0, r)$ for some fixed sufficiently small $r>0$, and $E(h)\sim E_0+E_1 h+E_2 h^2+\dots$ such that in the sense of formal power series expansions with respect to h:

$$e^{\varphi/h}\left(-\frac{1}{2}\Delta+V-hE\right)(e^{-\varphi/h})=V(x)-\frac{1}{2}(\nabla\varphi)^2+h\left(\frac{1}{2}\Delta\varphi-E\right)\sim 0. \tag{3.1}$$

In other words, we get a solution of the sequence of equations:

(E) $$V(x)-\frac{1}{2}(\nabla\varphi_0)^2=0,$$

(T1) $$\nabla\varphi_0(x)\cdot\partial_x\varphi_1(x)=\frac{1}{2}\Delta\varphi_0(x)-E_0$$

(T2) $$\nabla\varphi_0(x)\cdot\partial_x\varphi_2(x)=\frac{1}{2}\Delta\varphi_1(x)-\frac{1}{2}(\nabla\varphi_1)^2-E_1,$$

⋮

(Tm) $\nabla \varphi_0 \cdot \partial_x \varphi_m = \frac{1}{2} \Delta \varphi_{m-1}$

$$-\frac{1}{2}(\nabla \varphi_1 \cdot \nabla \varphi_{m-1} + \nabla \varphi_2 \cdot \nabla \varphi_{m-2} + \dots + \nabla \varphi_{m-1} \cdot \nabla \varphi_1) - \mathrm{E}_{m-1}$$

⋮

We also have the properties: $\varphi_j(0)=0$, $|\nabla \varphi_j(x)| \leqq \mathrm{C}_j$ for $|x|<r$, $|\mathrm{E}_j| \leqq \mathrm{C}_j \mathrm{N}$. Moreover the numbers E_j are real.

Using this WKB-construction, we can first prove that if we let $\mathrm{E}(h)$ denote an asymptotic sum, well defined modulo $\mathcal{O}(\mathrm{N} h^k)$ for every k, then the distance from $h\mathrm{E}(h)$ to the spectrum of P, the Dirichlet realization of $-\frac{1}{2} h^2 \Delta + \mathrm{V}$ in the real l^∞-ball of radius $r>0$ (sufficiently small but independent of N), is $\mathcal{O}(\mathrm{N} h^\infty)$. To do so we just have to check that the argument of section 4 of [S] goes through. Let $\varphi(x; h)$ be holomorphic in $\mathrm{B}(0, r)$ with the property that $\varphi(0; h)=0$,

$$\left| \nabla \left(\varphi(x; h) - \sum_0^k \varphi_j(x) h^j\right) \right| \leqq \mathrm{C}_k h^{k+1}$$

for every k. Then $(\mathrm{P}-h\mathrm{E})(e^{-\varphi/h}) = r e^{-\varphi/h}$, where $r=\mathcal{O}(\mathrm{N} h^\infty)$, so the required conclusion would follow if we ignore the fact that $e^{-\varphi/h}$ does not satisfy the Dirichlet boundary condition. For this we need to make cut-offs in each of the variables x_j, and as in [S], section 4, this will work if we verify that the function $x_j \mapsto \varphi(x; h)$ has a non-degenerate minimum $x_j(x'; h)$ with $|x_j| \leqq \theta r$ for some fixed $\theta \in [0, 1[$, when $|x'|<r$. Here we write $x'=(x_1, \dots, x_{j-1}, \dots, x_\mathrm{N})$. Let us verify this:

Let $\varphi_{0,0}(x) = \frac{1}{2} \mathrm{V}''(0)^{1/2} x \cdot x$ be the quadratic part of φ_0, so that $\nabla \varphi_{0,0}(x) = (\tilde{\mathrm{D}} + \tilde{\mathrm{A}}) x$ (*cf.* Lemma 1.1). We clearly have $\partial_{x_j}^2 \varphi_{0,0}(x) \geqq \tilde{r}_0 - \tilde{r}_1$, with $\tilde{r}_0 = r_0^{1/2}$, $\tilde{r}_1 = r_0^{1/2} - (r_0 - r_1)^{1/2}$, and we are interested in the point of minimum of the function $x_j \mapsto \varphi_{0,0}(x)$, if x' is fixed. Let x be the corresponding point of minimum. This point satisfies $\partial_{x_j} \varphi_{0,0}(x)=0$, which we can write as $[(\tilde{\mathrm{D}}+\tilde{\mathrm{A}})x]_j=0$, or in other words as $\tilde{d}_j x_j + (\tilde{\mathrm{A}} x)_j = 0$, where $\tilde{d}_j$ denotes the jth diagonal element of $\tilde{\mathrm{D}}$. Since $\tilde{d}_j \geqq \tilde{r}_0$, $\|\tilde{\mathrm{A}}\|_{\mathscr{L}(l_\infty, l_\infty)} \leqq \tilde{r}_1$, we get: $|x_j| \leqq (\tilde{r}_1/\tilde{r}_0)|x|$. In particular, $|x| = \max_{k \neq j} |x_k| = |x'|$, and we get:

$$|x_j| \leqq (\tilde{r}_1/\tilde{r}_0)|x'|. \tag{3.2}$$

We next look at the function $\varphi_0(x)$, which satisfies

$$\left.\begin{aligned} \partial_{x_j}^2 \varphi_0(x) &\geqq \tilde{r}_0 - \tilde{r}_1 - \mathcal{O}(|x|), \\ \partial_{x_j} \varphi_0(x) &= \partial_{x_j} \varphi_{0,0}(x) + \mathcal{O}(|x|^2). \end{aligned}\right\} \tag{3.3}$$

Assuming that $r>0$ is sufficiently small, we obtain that for every x' with $|x'|<r$, the function $x \mapsto \varphi_0(x)$ has a unique point of minimum in $]-r, r[$, and if x denotes the corresponding point in $B_{\mathbb{R}}(0, r)$, then

$$|x_j| \leqq ((\tilde{r}_1/\tilde{r}_0) + \mathcal{O}(|x'|))|x'|. \tag{3.4}$$

The last step is to extend this estimate to the function $x_j \mapsto \varphi(x; h)$, and this is straight forward, if we take into account that φ has a point of local minimum x_0 which satisfies: $|x_0| = \mathcal{O}(h)$. We then get the same estimate for $x_j \mapsto \varphi(x; h)$ after translation in the coordinates: If $|x'|<r$, then the function $x_j \mapsto \varphi(x; h)$ has a unique minimum $x_j(x'; h)$ in the interval $]-r, r[$ which satisfies

$$|x_j - (x_0)_j| \leqq \theta |x' - x'_0|, \tag{3.5}$$

with some fixed $\theta < 1$. Moreover, we have

$$\varphi(x; h) - \varphi((x', x_j(x'; h)); h) \geqq C^{-1} |x_j - x_j(x'; h)|^2,$$

[here we write $x=(x', x_j)$]. From this point, the argument of [S], section 4 goes through without any change, and we get

$$\inf \sigma(P) \leqq h E(h) + \mathcal{O}(N h^\infty). \tag{3.6}$$

Also the estimate from below of the spectrum goes through without any changes and we get:

$$\inf \sigma(P) \geqq h E(h) - \mathcal{O}(N h^\infty). \tag{3.7}$$

In fact this follows immediately from the fact that

$$P - hE = \sum_1^N Z_k^* Z_k + R, \tag{3.8}$$

where R is holomorphic in $B(0, r)$ and satisfies $|\nabla R| = \mathcal{O}(h^\infty)$, $R = \mathcal{O}(N h^\infty)$. Here, we have put

$$Z_k = h \partial_{x_k} + \partial_{x_k}(\varphi) = e^{-\varphi/h} \circ h \partial_{x_k} \circ e^{\varphi/h}. \tag{3.9}$$

Summing up our results so far, we get:

THEOREM 3.1. — *Assume that* V *satisfies the assumptions* (1.1), (1.2), *and let* P *denote the Dirichlet realization of* $-h^2 \Delta + V$ *in the ball* $B(0, r)$ *with respect to the* l^∞ *norm. Let* $E(h)$ [*welldefined modulo* $\mathcal{O}(N h^\infty)$] *be the number defined above by means of the WKB-construction of section* 3 *in* [S]. *If* $r>0$ *is sufficiently small independently of the dimension, then for* $h>0$ *sufficiently small:*

$$\inf \sigma(P) = h E(h) + \mathcal{O}(N h^k) \quad \textit{for every } k \in \mathbb{N}. \tag{3.10}$$

We shall end this paper by giving an estimate from below on the gap between the first and the second eigenvalue of P. In [S] this is done under the assumption that

$$N=\mathcal{O}(h^{-N_0}), \tag{3.11}$$

for some fixed N_0, by a change of variables, which reduces the system of operators Z_k to the corresponding system with φ replaced by $x^2/2$. It turned out however that a somewhat simpler proof can be obtained by a more direct use of the operators Z_k. Then we found that the appendix B of the paper by Singer-Wong-Yau-Yau [SiWYY] contains an estimate on the hessian of the logarithm of the first eigenfunction which can be used to make our proof non-asymptotic, and we got a more general result valid without the assumption (3.11). This estimate is due to Brascamp-Lieb [BL] when the domain Ω (below) is $\mathbb{R}^N$.

Let Ω be a convex bounded open set in $\mathbb{R}^N$ and let $V\in C^\infty(\bar{\Omega})$ be a real valued convex potential. Let P denote the Dirichlet realization of $-\frac{1}{2}\Delta+V$ in Ω, and let $\mu_0<\mu_1$ be the two lowest eigenvalues of P. Let $\lambda_0>0$ denote the infimum over Ω of the lowest eigenvalue of $V''(x)$. Then we have:

THEOREM 3.2. – *We have* $\mu_1-\mu_0\geqq\lambda_0^{1/2}h$.

Proof. – Let $P_{\Omega,V}$ denote the Dirichlet realization of $-\frac{1}{2}h^2\Delta+V$ in Ω. If $\hat{\Omega}\subset\Omega$ and if $\mu_j(\Omega, V)$ denote the jth eigenvalue of $P_{\Omega,V}$ (starting the counting at $j=0$) then $\mu_j(\hat{\Omega}, V)\geqq\mu_j(\Omega, V)$ and $\mu_j(\hat{\Omega}, V)\to\mu_j(\Omega, V)$ when $\hat{\Omega}\to\Omega$. It is therefore enough to prove the theorem in the case when Ω is strictly convex with smooth boundary, which we shall assume from now on.

Let e_0 be the positive normalized eigenfunction of P associated to μ_0. Then we can write:

$$e_0=e^{-\varphi/h}, \tag{3.12}$$

where $\varphi\in C^\infty(\Omega)$ is real-valued. In appendix B of [SiWYY], the authors use the maximum principle to show that

$$\varphi''(x)\geqq\lambda_0^{1/2}. \tag{3.13}$$

With this new function φ, we define the operators Z_k as above. We then get the formal identity (3.8) with $R=0$ and with hE replaced by μ_0. For $u\in C_0^\infty(\Omega)$ we get:

$$((P-\mu_0)u\,|\,u)=\|u\|_1^2, \tag{3.14}$$

where we have put

$$\|u\|_1^2=(u\,|\,u)_1=\sum_1^N(Z_j u\,|\,Z_j u)\geqq 0. \tag{3.15}$$

We also put

$$\| u \|_2^2 = (u \mid u)_2 = \sum_1^N (Z_j u \mid Z_j u)_1 \geqq 0. \tag{3.16}$$

We then have

$$\begin{aligned} ((P - \mu_0) u \mid u)_1 &= \sum\sum (Z_k Z_j^* Z_j u \mid Z_k u) \\ &= \| u \|_2^2 + \sum\sum ([Z_k, Z_j^*] Z_j u \mid Z_k u), \end{aligned} \tag{3.17}$$

making use also of the fact that $[Z_j, Z_k] = 0$. Now: $[Z_k, Z_j^*] = h\, \partial_{x_j} \partial_{x_k} \varphi$, so the last term in (3.17) can be bounded from below by

$$h \lambda_0^{1/2} \| u \|_1^2 = h \lambda_0^{1/2} ((P - \mu_0) u \mid u).$$

On the other hand, $((P - \mu_0) u \mid u)_1 = \| (P - \mu_0) u \|^2$, so

$$\| (P - \mu_0) u \|^2 \geqq h \lambda_0^{1/2} ((P - \mu_0) u \mid u) \tag{3.18}$$

for all $u \in C_0^\infty (\Omega)$. Let e_1 be a normalized eigenfunction of P associated to the second eigenvalue μ_1. If we could find a sequence $u_j \in C_0^\infty (\Omega)$, $j = 1, 2, \ldots$ such that u_j converges to e_1 in the norm of the domain of P, then replacing u by u_j in (3.18) and passing to the limit, we would get:

$$(\mu_1 - \mu_0)^2 \geqq h \lambda_0^{1/2} (\mu_1 - \mu_0), \tag{3.19}$$

which would imply the theorem, since $\mu_1 > \mu_0$.

Now the domain of P is $H_0^1 (\Omega) \cap H^2 (\Omega)$ and $C_0^\infty (\Omega)$ is not dense in this space, but we notice that all the identities above extend to the case when $u \in C^\infty (\bar{\Omega})$ vanishes on the boundary. In fact, we have $Z_k = e_0 \circ (h\, \partial_{x_k}) \circ e_0^{-1}$, $Z_k^* = e_0^{-1} \circ (-\partial_{x_k}) \circ e_0$, so if u is smooth and vanishes on the boundary then the same holds for $Z_k u$ while $Z_k^* u$ is smooth up to the boundary. We check that if u, v are smooth up to the boundary and if v vanishes on the boundary, then $(Z_k u \mid v) = (u \mid Z_k^* v)$ (by inserting a sequence of cutoff functions converging to 1). However this is all we need to justify all integrations by parts in the above argument for u smooth up to the boundary and vanishing there. Now we get (3.19) directly by inserting $u = e_1$ in (3.18). □

COROLLARY 3.3. — *Under the assumptions of Theorem* 3.1, *there is a constant* $C > 0$ *independent of* N, *such that the gap between the first and the second eigenvalue is* $\geqq h/C$. *Under the additional assumption* (3.11), *we can take* C *to be any number with* $1/C < \inf_{x \in \Omega} \varphi_0'' (x)$, *provided that h is small enough depending on* C.

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