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C. BOLLEY

B. HELFFER

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An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material

by

C. BOLLEY

École Centrale de Nantes,
1, rue de la Noé,
F44072 Nantes Cedex, France

and

B. HELFFER *

Wissenschaftskolleg zu Berlin,
Wallotstrasse 19, D1000 Berlin 33, Germany

ABSTRACT. – This paper is devoted to a precise description of the properties of the supercooling field of a superconducting film submitted to an external magnetic field. Semi-classical analysis is introduced in order to give very accurate asymptotic behavior of the supercooling field as the thickness of the film tends to ∞.

RÉSUMÉ. – Dans cet article nous décrivons précisément les propriétés du champ de retard à la condensation d'un film supraconducteur soumis à un champ magnétique extérieur. Nous utilisons une analyse semi-classique pour démontrer des comportements asymptotiques fins du champ de retard à la condensation quand l'épaisseur du film tend vers l'infini.

(*) On leave of the university of Paris-XI.
Current address: DMI-ENS, 45, rue d'Ulm, 75230 Paris Cedex, France.

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0. INTRODUCTION: GINZBURG-LANDAU EQUATIONS FOR A SUPERCONDUCTING MATERIAL

Let us consider a superconducting film whose thickness is d (*i.e.* $]-d/2, d/2[\times\mathbb{R}^2$) and which is submitted to an external field $\mathbf{H}_e$ parallel to the surface of the material.

In the study of the Ginzburg-Landau theory concerning the different states of superconducting materials whose temperature is sufficiently small, we meet the problem to minimize the functional ΔG which corresponds to the difference of free Gibbs energy and is defined for

$$(f, A)\in H^1(]-d/2, d/2[;\mathbb{R})\times H^1(]-d/2, d/2[;\mathbb{R})$$

by

$$\Delta G(f, A)=\int_{-d/2}^{d/2}[-f(x)^2+(1/2)f(x)^4+(1/\kappa^2)|f'(x)|^2 + (A(x))^2|f(x)|^2+|A'(x)-h|^2]\,dx \quad (0.1)$$

(*cf.* $[\text{Bo}]_{1,2}$, [Gi] and [Du]).

The functional ΔG depend on three parameters d, h and κ, h is proportional to the intensity of the exterior field $\mathbf{H}_e$, and κ is a characteristic of the material which remain fixed in all our discussion.

The pairs (f, A) characterizing the different states of the superconducting material are given by the extrema in (f, A) of the functional ΔG. In particular they satisfy the following so called Ginzburg-Landau equations:

$$-\kappa^{-2}f''-f+f^3+A^2f=0 \text{ in }]-d/2, d/2[\quad (0.2)_1$$
$$f'(\pm d/2)=0 \quad (0.2)_2$$
$$-A''+f^2.A=0 \quad \text{in }]-d/2, d/2[\quad (0.3)_1$$
$$A'(\pm d/2)=h \quad (0.3)_2$$

As a consequence of the equations, f and A belong necessarily to $H^2(]-d/2, d/2[)$. The pairs $(f, A)=(0, h(x+e))$ $(e\in\mathbb{R})$ are solutions of the equations (0.2) and (0.3). They characterize the normal state of the material and will be called "normal solutions".

A superconducting state corresponds to a solution such that $f\neq 0$.

The first problem we shall consider is to determine if a solution $(0, h(x+e))$ is a minimum of ΔG. A global minimum will define a stable solution and a local minimum a metastable solution.

The following proposition reduces this study to a spectral problem:

PROPOSITION 0.1. – *Let $d>0$, $h>0$ and $e\in\mathbb{R}$. Let $\tau=\tau(d, e, h)$ the principal eigenvalue of the Neumann problem:*

$$-\kappa^{-2}\phi''+h^2(x+e)^2\phi=\tau\phi \quad \textit{in }]-d/2, d/2[\quad (0.4)$$
$$\phi'(\pm d/2)=0$$

(i) *If* $-1+\tau(d, e, h)>0$, *then the solution* $(0, h(x+e))$ *is a local minimum of* ΔG.

(ii) *If* $-1+\tau(d, e, h)<0$, *then the solution* $(0, h(x+e))$ *is not a local minimum of* ΔG.

According to this proposition it is natural to ask if there exists a critical value for h corresponding to the property that all the normal solutions are metastable. For this we introduce the following definition:

DEFINITION 0.2. – For any $d>0$, let $\mathscr{H}_{sc}(d)$ the subset of the h in $\mathbb{R}^{+*}$ s.t. every normal solution $(0, h(x+e))(e\in\mathbb{R})$ is locally stable. Then we shall call supercooling field the lower bound $H_{sc}(d)$ of $\mathscr{H}_{sc}(d)$.

We shall study in section 1 the properties of the set $\mathscr{H}_{sc}(d)$.

To relate more deeply the problem (0.4) and the supercooling field we will prove in Section 2 the following

THEOREM 0.3. – (*i*) *For each* $(d, e)\in\mathbb{R}^{+*}\times\mathbb{R}$, *there exists a unique* $h(d, e)$ *s.t.* $\tau(d, e, h(d, e))=1$.

(ii) $H_{sc}(d)=\text{Sup}_e h(d, e)$.

The study of this supercooling field leads us to the study of the critical values of the function: $e\to\tau(d, e, h)$. Let us introduce:

$$\begin{aligned}&\tilde{s}\text{ the subset of the }(d, e)\text{ in }\mathbb{R}^{+*}\times\mathbb{R}\text{ s.t.}\\&\tau(d, e, h)=1, (\partial\tau/\partial e)(d, e, h)=0.\end{aligned}\tag{0.5}$$

Let us recall one result obtained by one of us (C.B.) in $[\text{Bo}]_2$.

THEOREM 0.4.

$$(d, 0)\in\tilde{s}\tag{0.6}$$

Moreover,

$$h(d, 0)\textit{ is monotone decreasing and tends to }\kappa\textit{ as }d\textit{ tends to }\infty.\tag{0.7}$$

As a natural continuation of $[\text{Bo}]_2$, we want to study by semiclassical analysis the structure of s above a point d of $\mathbb{R}^{+*}$ in the limit $d\to+\infty$ *or* $d\to 0$. This method was successfully applied in $[\text{Bo}]_2$ to prove the existence of solutions s.t. $e\neq 0$ when d is large enough and that a nontrivial necessary condition to get bifurcation was satisfied. This permits, modulo some additional conjectured property of transversality (verified only numerically), to predict existence of bifurcations and to explain the nature of the numerically computed results. We shall return more in detail to some of the results obtained in $[\text{Bo}]_2$ in the study of an equivalent model in Part II.

The natural questions for the study of $\tilde{s}$ is the existence of regular curves $d\to e(d)$ defined on some interval s.t. $(d, e(d))\in\tilde{s}$.

The numerical computations of $[\text{Bo}]_2$ (*see* Appendix 1) give the following structure for $\tilde{s}$:

There exists d_0 s.t.

$$\{(d, e) \mid (d, e) \in \tilde{s},\ d \leqq d_0\} = \{(d, 0) \mid d \leqq d_0\}$$

$$\{(d, e) \mid (d, e) \in \tilde{s},\ d > d_0\}$$
$$= \{(d, 0) \mid d > d_0\} \cup \{(d, e(d)) \mid d > d_0\} \cup \{d, -e(d)) \mid d > d_0\}$$

where $d \to e(d)$ is a smooth curve with value in $\mathbb{R}^{+*}$ s.t. $e(d)$ tends to 0 as $d \to d_0$.

We are far to get until now the complete proof of this situation but we shall justify mathematically the structure of the picture outside a black box (probably the most interesting, but *see* $[\text{Bo}]_2$ for partial results) corresponding to the interval $[1/\text{D}, \text{D}]$ (with $0 < (1/\text{D}) < \text{D} < \infty$), where D is a sufficiently large positive real number. Our principal theorems will be the following:

THEOREM 0.5. — *There exists* $\text{D}_1 > 0$, *s. t.*

$$\{(d, e) \mid (d, e) \in \tilde{s},\ d > \text{D}_1\}$$
$$= \{(d, 0) \mid d > \text{D}_1\} \cup \{(d, e(d)) \mid d > \text{D}_1\} \cup \{(d, -e(d)) \mid d > \text{D}_1\}$$

where $d \to e(d)$ *is a smooth curve with value in* $\mathbb{R}^{+*}$ *defined on* $]\text{D}_1, +\infty[$.

Moreover (increasing possibly D_1*) we have the following properties in* $]\text{D}_1, +\infty[$:

$$h(d, e(d)) = \text{H}_{sc}(d) \tag{0.8}$$

$\text{H}_{sc}(d)$ *is monotonically decreasing in* $]\text{D}_1, +\infty[$

$$\textit{and tends to } \kappa/\mu_1^0 \textit{ as } d \textit{ tends to } +\infty \tag{0.9}$$

$$(e(d)/d) \to 1/2 \textit{ as } d \textit{ tends to } +\infty \tag{0.10}$$

Here μ_1^0 *is the minimum over* $\alpha \geqq 0$ *of the first eigenvalue of the Neumann problem of the harmonic oscillator in* $]-\alpha, +\infty[$.

Remark 0.6. — It will be proved in section 4 (*cf*. also [Da-He]), that μ_1^0 is attained at a unique point α_0. The numerical computations (*see* Appendix 2) give: $\alpha_0 \approx 0.73$ and $\mu_1^0 \approx 0.59$.

As expected, the limit obtained in (0.9) coincides with the results obtained by D. St James and P. G. de Gennes [Ge-Ja] for the supercooling field H_{sc}, as they consider a sample with a semi-infinite domain.

THEOREM 0.7. — *There exists* $\text{D}_0 > 0$, *s. t.*

$$\{(d, e) \mid (d, e) \in \tilde{s},\ d \leqq \text{D}_0\} = \{(d, 0) \mid d \leqq \text{D}_0\} \tag{0.11}$$

$$h(d, 0) = \text{H}_{sc}(d) \quad \textit{for} \quad 0 < d \leqq \text{D}_0. \tag{0.12}$$

$$H_{sc}(d) \to \infty \quad \textit{as} \quad d \to 0. \tag{0.13}$$

The study of the supercooling field is finally completed by the following:

THEOREM 0.8. — $d \to \text{H}_{sc}(d)$ *is a continuous, strictly positive, piecewise analytic function.*

This article is organized as follows:

PART I. GINZBURG-LANDAU EQUATIONS AND CRITICAL FIELDS

1. Study of the stability of the normal solutions

In this section we shall give a proof of Proposition 0.1 and study preliminary properties of the supercooling field. We shall complete the statements of proposition 0.1 by the study of the global stability of the normal solutions.

Proof of Proposition 0.1. – To study the functional ΔG near $(0, h(x+e))$, let us consider the following change of functions: $A = h(x+e) + t\,B$; $f = tg$.

Then

$$\begin{aligned}\Delta G(tg, h(x+e)+t\,B) = t^2\bigg(\int_{-d/2}^{d/2} & [-g(x)^2+(1/\kappa^2)\,g'(x)^2 \\ & + h^2(x+e)^2 g(x)^2 + B'(x)^2]\,dx \\ & + 2\,ht^3\int_{-d/2}^{d/2} B(x).(x+e)\,g(x)^2\,dx \\ & + t^4\int_{-d/2}^{d/2} [(1/2)\,g(x)^4+g(x)^2\,B(x)^2]\,dx \quad (1.1)\end{aligned}$$

Let us assume (i). $\tau(d, e, h)$ being the principal eigenvalue of (0.4), we get, for every function $g \in H^1(]-d/2, d/2[)$ an every $B \in H^1(]-d/2, d/2[)$:

$$\Delta G(tg, h(x+e)+tB) \geqq t^2[(-1+\tau(d, e, h))(\|g\|_{L^2})^2 + \|B'\|^2] + 2ht^3 \int_{-d/2}^{d/2} B(x).(x+e)g(x)^2\,dx$$

If we normalize by:

$$\|g\|_{H^1}^2 + \|B\|_{H^1}^2 = 1, \tag{1.2}$$

then

$$\begin{aligned} 2h\int_{-d/2}^{d/2} B(x).(x+e)g(x)^2\,dx &\leqq 2h((d/2)+|e|)\|B(x)\|_{L^\infty}.(\|g\|_{L^2})^2 \\ &\leqq C\|B\|_{H^1}.(\|g\|_{L^2})^2 \\ &\leqq \tilde{C}(\|g\|_{L^2})^2 \end{aligned}$$

where we have used (1.2) and the injection of Sobolev.

We get the following inequality

$$\Delta G(tg, h(x+e)+tB) \geqq t^2[(-1+\tau(d, e, h)-\tilde{C}|t|)(\|g\|_{L^2})^2 + (\|B'\|_{L^2})^2],$$

and the r.h.s. is positive if:

$$|t| \leqq t_0 = (-1+\tau(d, e, h))/\tilde{C}.$$

Assumption (i) implies $t_0 > 0$. So we get for all $t \in [-t_0, t_0]$:

$$\Delta G(tg, h(x+e)+tB) \geqq 0,$$

i.e. $(0, h(x+e))$ is a local minimum.

(ii) Let us assume now $\tau(d, e, h)-1<0$. Choose $g=\phi$ where ϕ is the positive L^2 normalized eigenstate of (0.4). Then

$$\Delta G(t\phi, h(x+e)) = t^2.[(\tau(d, e, h)-1)+(1/2)t^2(\|\phi\|_{L^4})^4]$$

which is strictly negative if $0<|t|<t_1$ with t_1 small enough.

Remark 1.1. – Proposition 0.1 is different of the classical statements on linearized stability because of the singularities of equation (0.3). Let us recall that a solution (f_0, A_0) of equations (0.2) (0.3) is called linearly stable if all the eigenvalues μ of the linearized problem in (f_0, A_0):

$$-\kappa^{-2}\phi'' - \phi + 3f_0^2\phi + A_0^2\phi + 2A_0 f_0 A = \mu\phi \text{ in }]-d/2, d/2[\tag{1.3$_1$}$$
$$-A'' + 2f_0 A_0 \phi + f_0^2 A = \mu A \tag{1.3$_2$}$$
$$\phi'(\pm d/2) = 0, A'(\pm d/2) = 0 \tag{1.3$_3$}$$

are strictly positive.

In the case where $(f_0, A_0) = (0, h(x+e))$, (1.3) is:

$$-\kappa^{-2}\phi'' + h^2(x+e)^2\phi - \phi = \mu\phi \text{ in }]-d/2, d/2[\tag{1.4$_1$}$$
$$-A'' = \mu A \tag{1.4$_2$}$$
$$\phi'(\pm d/2) = 0, A'(\pm d/2) = 0. \tag{1.4$_3$}$$

$\mu=0$ is always an eigenvalue of this problem with corresponding eigenspace $\phi=0$, $A=\mathrm{cst}$. Consequently we have never linearized stability at the points $(0, h(x+e))$. This is quite evident if we observe that $e \to (0, h(x+e))$ is a curve of critical points of the functional. The tangent space at a point $(0, h(x+e_0))$ is given by $(0, A)$ with $A=\mathrm{Cste}$ and is automatically a solution of (1.4) with $\mu=0$. So we can hope only transversal linearized stability in the sense that we look at the spectral problem (1.4) in adding an orthogonality condition to this tangent space:

$$\int A(x)\, dx=0 \tag{1.4$_4$}$$

and we shall say that the problem is transversally linearly stable if all the eigenvalues of this new problem are strictly positive.

We then observe that if $\tau(d, e, h)>1$, the problem is transversally linearly stable and that if $\tau(d, e, h)\leqq 1$ the problem is not transversally stable at $(0, h(x+e))$.

Supercooling field. – It is well known from the physicists that a superconducting material which is submitted to an exterior field $\mathbf{H}_e$ can change of state when the intensity h of this field varies.

The definition of the supercooling field we have proposed in the introduction is justified by the following proposition:

PROPOSITION 1.2. – *For any $d>0$, $\mathscr{H}_{sc}(d)$ is a semiinfinite interval* $(\mathrm{H}_{sc}(d), +\infty[$.

More intuitively, this corresponds to the idea that by decreasing h a normal state can stay locally stable until a critical value $\mathrm{H}_{sc}(d)$.

An interesting still open question is to determine if $\mathrm{H}_{sc}(d)$ *belongs to* $\mathscr{H}_{sc}(d)$ *or not.*

Proof of Proposition 1.2. – We first prove that if for some value of h say h_0 all the solutions $(0, h_0(x+e))$ are locally stable, then the same property is true for any h s.t. $h\geqq h_0$.

Let us assume that for a given pair (e, h) we have the property:

There exists $t_0>0$ s.t., for all $|t|\leqq t_0$, and for all normalized pair (g, B) (as in (1.2)), $\Delta G(tg, h(x+e)+tB)\geqq 0$.

Then let us observe that this property is equivalent to:

There exists $\tilde{t}_0>0$ s.t., for all $|t|\leqq \tilde{t}_0$, and for all normalized pair (g, B) (as in (1.2)), $\Delta G(tg, h(x+e+tB))\geqq 0$

But

$$\Delta G(tg, h(x+e)+thB) = t^2 \int_{-d/2}^{d/2} [-g(x)^2+(1/\kappa^2) g'(x)^2+h^2(x+e)^2 g(x)^2+h^2 B'(x)^2]\, dx$$

$$+2h^2 t^3 \int_{-d/2}^{d/2} \mathrm{B}(x).(x+e)\, g(x)^2\, dx$$
$$+t^4 \int_{-d/2}^{d/2} [(1/2)\, g(x)^4 + h^2 g(x)^2\, \mathrm{B}(x)^2]\, dx$$

is an increasing function of $h>0$. We then get the property that $\mathscr{H}_{sc}(d)$ is a semiinfinite interval.

The critical thermodynamic (2) *field*

PROPOSITION 1.3. – *For any* $d>0$, *let* $\mathscr{H}_b(d)$ *the subset of the* h *in* $\mathbb{R}^{+*}$ *s.t. every normal solution* $(0, h(x+e))$ $(e\in\mathbb{R})$ *is stable.*

Then $\mathscr{H}_b(d)$ *is a semiinfinite interval* $(\mathrm{H}_b(d), +\infty[$. *The lower bound of* $\mathscr{H}_b(d)$, $\mathrm{H}_b(d)$, *will be called the critical thermodynamic field.*

Proof of Proposition 1.3. – The proof that it is an interval is the same as for the supercooling field. Let us give a proof that $\mathrm{H}_b(d)<\infty$.

LEMMA 1.4. – *For each* d, *there exists* $h(d)$ *s.t.* $h \geqq h(d)$ *implies that the functional* $\Delta\mathrm{G}$ *is positive.*

Proof of Lemma 1.4. – To normalize we shall rewrite sometimes a general solution (g, A) as:

$$(g, \mathrm{A})=(g, h(x+e+\mathrm{B}(x)))$$

with $e=\mathrm{cste}$ and $\int \mathrm{B}(x)\, dx=0$.

Step 1. – We first remark that

$$\Delta\mathrm{G}(g, \mathrm{A}) \geqq \int_{-d/2}^{d/2} [-g(x)^2+(1/\kappa^2)\, g'(x)^2+(1/2)\, g(x)^4]\, dx \tag{1.5}$$

As a consequence we get that $\Delta\mathrm{G} \geqq 0$ if

$$\int_{-d/2}^{d/2} g(x)^2\, dx \leqq \int_{-d/2}^{d/2} [(1/\kappa^2)\, g'(x)^2+(1/2)\, g(x)^4]\, dx \tag{1.6}$$

Let us now observe that (1.6) is satisfied if

$$\int_{-d/2}^{d/2} g(x)^2\, dx \leqq (1/2) \int_{-d/2}^{d/2} g(x)^4\, dx.$$

But by Hölder, we know that:

$$\int_{-d/2}^{d/2} g(x)^2\, dx \leqq \sqrt{d} \left(\int_{-d/2}^{d/2} g(x)^4\, dx \right)^{1/2}. \tag{1.7}$$

(2) This field is called "champ magnétique critique" in [Du].

This implies that (1.6) is satisfied if

$$\int_{-d/2}^{d/2} g(x)^4 \, dx \geqq 4d. \tag{1.8}$$

(1.8) permits to reduce the study of the positivity of ΔG to the case where:

$$\int_{-d/2}^{d/2} g(x)^4 \, dx \leqq 4d. \tag{1.9}$$

It is however not sufficient for our step 2, and we establish now the partially stronger statement:

There exists $C(d)$ s. t. (1.6) is satisfied if

$$\int_{-d/2}^{d/2} [(1/\kappa^2) g'(x)^2 + (1/2) g(x)^4] \, dx \geqq C(d) \tag{1.10}$$

Indeed we known already from (1.8) that this is the case if

$$\int_{-d/2}^{d/2} g(x)^4 \, dx \geqq 4d.$$

So we need only to prove (1.10) under the additional assumption (1.9).

But (1.7) and (1.9) imply that:

$$\int_{-d/2}^{d/2} g(x)^2 \, dx \leqq 2d$$

and (1.6) is satisfied with $C(d) = 2d$.

The conclusion of step 1 is that

$$\Delta G \geqq 0 \text{ if } \int_{-d/2}^{d/2} [(1/\kappa^2) g'(x)^2 + (1/2) g(x)^4] \, dx \geqq 2d. \tag{1.11}$$

Step 2. – We study now the problem of the positivity under the additional assumption that

$$\int_{-d/2}^{d/2} [(1/\kappa^2) g'(x)^2 + (1/2) g(x)^4] \, dx \leqq 2d. \tag{1.12}$$

In particular, it implies that we can assume that g stays in a ball in H^1, that is that there exists a constant $C(d)$ s. t.

$$\| g \|_{H^1} \leqq C(d). \tag{1.13}$$

Let us consider the following rewriting of ΔG:

$$\Delta G(g, h(x+e+B)) = \int_{-d/2}^{d/2} [-g(x)^2 + (1/\kappa^2) g'(x)^2 + \alpha h^2 (x+e)^2 g(x)^2] \, dx$$

$$+2h^2\int_{-d/2}^{-d/2} B(x).(x+e)g(x)^2\,dx$$
$$+\int_{-d/2}^{d/2} [(1/2)g(x)^4+h^2 g(x)^2 B(x)^2]\,dx$$
$$+\int_{-d/2}^{d/2} (1-\alpha)h^2(x+e)^2 g(x)^2\,dx+h^2\int_{-d/2}^{d/2} B'(x)^2\,dx$$

where α is a parameter independent on h satisfying $0<\alpha\leqq 1$.

Using Cauchy-Schwarz inequalities we get the following minoration for ΔG:

$$\Delta G(g, h(x+e+B))\geqq\int_{-d/2}^{d/2} [-g(x)^2+(1/\kappa^2)g'(x)^2 + (\sqrt{\alpha}\,h)^2(x+e)^2 g(x)^2]\,dx$$
$$+h^2[(-\alpha/(1-\alpha))\int_{-d/2}^{d/2} g(x)^2 B(x)^2\,dx+\int_{-d/2}^{d/2} B'(x)^2\,dx] \quad (1.14)$$

To get the positivity of the r.h.s. we shall first choose α s.t.:

$$(-\alpha/(1-\alpha))\int_{-d/2}^{d/2} g(x)^2 B(x)^2\,dx+\int_{-d/2}^{d/2} B'(x)^2\,dx\geqq 0 \quad (1.15)$$

for any g satisfying (1.13) and any B satisfying $\int B\,dx=0$.

For this we observe the two inequalities:

$$\int_{-d/2}^{d/2} h(x)^2 B(x)^2\,dx\leqq\|g\|_{L^\infty}^2(\|B\|_{L^2})^2\leqq C_1(d)(\|B\|_{L^2})^2$$

under condition (1.13), and

$$\|B\|_{L^2}\leqq C_2(d)\|B'\|_{L^2} \text{ under the condition } \int B\,dx=0.$$

$C_2(d)$ is the inverse of the first nonzero eigenvalue of the Neumann problem for the Laplacian in $[-d/2, d/2]$.

Then we choose $\alpha(d)$ s.t.:

$$(\alpha/(1-\alpha))\,C_1(d)\,C_2(d)^2=1/2 \quad \text{and} \quad 0<\alpha<1. \quad (1.16)$$

We observe now that if

$$\sqrt{\alpha(d)}\,.\,h\geqq H_{sc}(d), \quad (1.17)$$

the first term in the r.h.s of (1.14) is positive and we get Lemma 1.4.

On the comparison between the supercooling field and the critical thermodynamic field. Of course we have always

$$H_{sc}(d)\leqq H_b(d), \quad (1.18)$$

but numerical computations (in [Du]) suggest the equality of the two critical fields for d small enough. Actually we shall prove:

PROPOSITION 1.5. — *There exists* d_1 *s. t.*

$$H_{sc}(d)=H_b(d) \quad \textit{for} \quad 0<d\leqq d_1. \tag{1.19}$$

Proof of Proposition 1.5. — We shall show the existence of d_1 s. t. $\Delta G(g, h(x+e)+B)\geqq 0$ for $h=H_{sc}(d)$, any e, any $d\leqq d_1$ and any B for which we can assume without loss of generality that:

$$B(0)=0. \tag{1.20}$$

We shall use also a result in $[Bo]_2$ (*see* also Lemma 7.6)

$$H_{sc}(d)\approx\sqrt{12}/d \text{ for } d \text{ small enough.} \tag{1.21}$$

Step 1. — As in the proof of Lemma 1.4 step 1, we can by easy estimates reduce the proof of positivity of $\Delta G(g, h(x+e)+B)$ to the case where the additional following assumption on (e, B, g) is satisfied [*cf*. (1.12)]:

$$\int_{-d/2}^{d/2} [B'^2+(1/\kappa^2)g'(x)^2+(1/2)g(x)^4]\,dx\leqq C_1 . d \tag{1.22}$$

where C_1 is a constant independent of d.

From this we deduce in particular that:

$$|B(x)|\leqq\left|\int_0^x |B'(t)|\,dt\right|\leqq\sqrt{d}\,\|B'\|^{1/2}\leqq C_2 . \mathrm{d}. \tag{1.23}$$

Using (1.21), (1.22) and (1.23), we observe the positivity of $\Delta G(g, h(x+e)+B)$ for $|e|\geqq C_3 d$, for C_3 large enough.

The result of step 1 is consequently that there exists $C>0$ and $d_3>0$ s. t. $\Delta G(g, h(x+e)+B)\geqq 0$ if $0<d\leqq d_3$, $h=H_{sc}(d)$ and

$$\int_{-d/2}^{d/2} [B'^2+(1/\kappa^2)g'(x)^2+(1/2)g(x)^4]\,dx+|e|>C d \tag{1.24}$$

Step 2. — We can now assume that

$$\int_{-d/2}^{d/2} [B'^2+(1/\kappa^2)g'(x)^2+(1/2)g(x)^4]\,dx+|e|\leqq C d. \tag{1.25}$$

Let us write the following inequality for $\Delta G(g, h(x+e)+B)$ [with $h=H_{sc}(d)$]:

$$\Delta G(g, h(x+e)+B)\geqq 2h\int_{-d/2}^{d/2} B(x).(x+e).g(x)^2\,dx + \int_{-d/2}^{d/2} B'(x)^2\,dx+(1/2)\int_{-d/2}^{d/2} g(x)^4\,dx.$$

We have just to prove the positivity of the r.h.s. for d small enough.

For this we observe simply using (1.21), (1.23) and (1.25) that:

$$\left|2h\int_{-d/2}^{d/2} \mathrm{B}(x).(x+e).g(x)^2\,dx\right| \leqq \mathrm{C}_4.\int_{-d/2}^{d/2} |\mathrm{B}(x)|.g(x)^2\,dx$$
$$\leqq \mathrm{C}_4.\|\mathrm{B}\|_{\mathrm{L}^2}\|g\|_{\mathrm{L}^4}^2 \leqq \mathrm{C}_5.d.\|\mathrm{B}'\|_{\mathrm{L}^2}\|g\|_{\mathrm{L}^4}^2$$
$$\leqq (\mathrm{C}_5 d)^2\int_{-d/2}^{d/2} \mathrm{B}'(x)^2\,dx + \int_{-d/2}^{d/2} g(x)^4\,dx.$$

We then get this positivity easily by choosing d small enough.

2. Qualitative properties of the supercooling field

LEMMA 2.1. – *For fixed $d>0$, the map*

$$\mathbb{R}^{+*} \ni h \to \theta(d, h) = \operatorname{Inf}_e \tau(d, e, h)$$

is a continuous, strictly positive, strictly monotone, piecewise analytic function.

Proof. – Let us first observe that the map:

$$(\mathbb{R}^{+*}\times\mathbb{R}\times\mathbb{R}^{+*}) \ni (d, e, h) \to \tau(d, e, h) \in \mathbb{R} \tag{2.1}$$

is an analytic function with respect to the three variables (*see* for example [Ka]).

As a consequence of the minimax principle in (0.4), we get:

$$h \to \tau(d, e, h) \text{ is increasing on } \mathbb{R}^{+*}. \tag{2.2}$$

More precisely we have at a point (d, e, h):

$$(\partial\tau/\partial h)(d, e, h) = 2h\int (x+e)^2\phi^2\,dx \Big/ \int \phi^2\,dx > 0 \tag{2.3}$$

(where ϕ is the L^2-normalized positive first eigenfunction of the Neumann-problem) which implies the strict monotonicity.

It is clear from the fact that $\tau(d, e, h)$ is always $\geqq 0$, that

$$\theta(d, h) \geqq 0. \tag{2.4}$$

To see if this infimum is in fact a minimum we have to analyze the behavior of $e \to \tau(d, e, h)$ as $|e| \to \infty$.

Let us just observe for the moment that we have the following simple inequality for $|e| \geqq d/2$:

$$\tau(d, e, h) \geqq h^2\int (x+e)^2\phi^2\,dx \Big/ \int \phi^2\,dx \geqq h^2(|e|-(d/2))^2 \tag{2.5}$$

which gives the behavior of $e \to \tau(d, e, h)$ at ∞.

If we observe that this last function is continuous, we obtain immediately the existence of a non empty compact set $\mathscr{E}(d, h)$ s. t.

$$\tau(d, e, h)=\theta(d, h) \text{ for } e\in\mathscr{E}(d, h). \tag{2.6}$$

Let us just localize this set more precisely in proving

$$\mathscr{E}(d, h)\subset\{e\in\mathbb{R}\,|\,|e|\leqq d/2\}. \tag{2.7}$$

We just observe that at a point e in $\mathscr{E}(d, h)$, we have

$$(\partial\tau/\partial e)(d, e, h)=0. \tag{2.8}$$

So we get, observing that

$$(\partial\tau/\partial e)(d, e, h)=2h^2\int(x+e)\phi^2\,dx\Big/\int\phi^2\,dx,$$

$$\int(x+e)\phi^2\,dx=0 \tag{2.9}$$

and (2.7) follows immediately (*see* also Lemma 3.1).

Let us be more precise for the future in observing that, because

$$\mathbb{R}\ni e\to(\partial\tau/\partial e)(d, e, h)$$

is a non constant analytic function due to (2.1) and (2.5), the set $\mathscr{E}(d, h)$ which is contained in the zero set of $e\to(\partial\tau/\partial e)(d, e, h)$ has the following property

$$\mathscr{E}(d, h) \text{ is a finite set} \tag{2.10}$$

Let us show now the continuity of the map as a function of $h\in\mathbb{R}^{+*}$.

Let $h_0>0$ and let us consider for $h_0>\rho_0>0$,

$$\bar{\mathrm{B}}(h_0, \rho_0)=[h_0-\rho_0, h_0+\rho_0].$$

The continuity is an immediate consequence by standard arguments of (2.6), (2.7) and of the uniform continuity of $(e, h)\to\tau(d, e, h)$ on the compact $[-d/2, d/2]\times\bar{\mathrm{B}}(h_0, \rho_0)$.

Let us give the argument for the strict positivity. Because the infimum is a minimum, this is an immediate consequence of

$$\tau(d, e, h)\geqq h^2\int(x+e)^2\phi(x)^2\,dx>0\ (\phi \text{ is strictly positive}). \tag{2.11}$$

We have finally to consider the problem of the piecewise analyticity, that means that:

Outside a locally finite set $\mathscr{H}(d)$ of h,

$$\text{the function } h\to\theta(d, h) \text{ is analytic.} \tag{2.12}$$

Let us take some point $h_0>0$. We shall prove that there exists $\rho_0>0$ s. t. in $\mathrm{B}(h_0, \rho_0)\setminus\{h_0\}$, $h\to\theta(d, h)$ is analytic.

Let us consider the finite set $\mathscr{E}(d, h_0)$ and let $e_i(h_0)$ a point in $\mathscr{E}(d, h_0)$. We have already used that $e \to (\partial\tau/\partial e)(d, e, h)$ is a non identically 0, analytic function. In particular, there exists k_i s.t. $(\partial^{k_i+1}\tau/\partial e^{k_i+1})(d, e_i, h_0) \neq 0$. Using the Weierstrass preparation Theorem, we get that:

$$(\partial\tau/\partial e)(d, e, h) = a^i(e, h)((e-e_i)^{k_i} + b_{1,i}(h)(e-e_i)^{k_i-1} + \ldots + b_{k_i,i}(h))$$

where $a^i(e, h) > 0$ in a neighborhood of (e_i, h_0), and the $b_{l,i}(h)$ are analytic functions vanishing at h_0. Then we use the classical result on the zeros of a polynomial with analytic coefficients (*see* for example Theorem X.12 in [Re-Si]), to get branches of analytic curves (with a possible puiseux singularity at h_0). Then selecting only the curves which stay in the real, we get that in some $B(h_0, \rho_0) \setminus \{h_0\}$, $h \to \mathrm{Inf}_e \tau(d, e, h)$ is obtained as the minimum over a finite, non empty set (3) of analytic curves. By changing possibly ρ_0, we then get (d is fixed in the discussion) that for any h_0 there exists ρ_0 s.t. $h \to \mathrm{Inf}_e \tau(d, e, h)$ is analytic in some $B(h_0, \rho_0) \setminus \{h_0\}$.

For any compact K in $\mathbb{R}^{+*}$, we can consider a finite covering of K by a finite family of $B(h_j, \rho_j)$ and we have proved the statement (2.12). To prove the strict monotonicity we observe that outside a locally finite set we can compute the derivative of θ using (2.3) and we observe immediately that this derivative is strictly positive.

COROLLARY 2.2. – *For all $d>0$, there exists a unique $h^c(d)$ s.t.:*

$$\theta(d, h^c(d)) = 1.$$

Moreover:
For $h > h^c(d)$, $\theta(d, h) > 1$.
For $h < h^c(d)$, $\theta(d, h) < 1$.

Proof of Corollary 2.2. – As h tends to 0, $\tau(d, e, h)$ tends (by regular perturbation theory) to the first eigenvalue of the free (without potential) Neumann problem, in particular we have

$$\lim_{h \to 0} \tau(d, e, h) = 0. \tag{2.13}$$

It is also an immediate consequence of the following inequality (simple application of the minimax principle with the constant function 1):

$$0 \leqq \tau(d, e, h) \leqq (h^2/d) \int (x+e)^2\, dx \tag{2.14}$$

(3) Let us assume indeed that for all i all these analytic branches are not real for h sufficiently near h_0. Then we get the contradiction with the fact that for any h there exists a non empty set in the zero set of $e \to (\partial\tau/\partial e)(d, e, h)$. Here we use the fact that $\varepsilon(d, h)$ stays in a fixed compact (*cf*. (2.7)).

In particular

$$0 \leqq \theta(d, h) \leqq h^2 . d^2/12. \tag{2.15}$$

The study as h tends to infinity is more delicate. (2.7) gives that it is sufficient to study $\tau(d, e, h)$ for e satisfying $|e| \leqq d/2$.

We shall prove later (*see* section 7, lemma 7.2) the following

$$\left.\begin{array}{c}\text{For all } \delta>0, \text{ there exists } \eta>0 \text{ and } A>0 \text{ s.t.} \\ (\tau(d, e, h)/h)>\eta, \text{ for all } (e, d) \text{ s.t. } d \geqq \delta,\ |e| \leqq d/2 \text{ and } |h| \geqq A.\end{array}\right\} \tag{2.16}$$

As a simple consequence, for fixed d, $\tau(d, e, h)$ tends to infinity as h tends to infinity.

Remark 2.3. – The proof gives in particular the existence for each (d, e) of a unique $h(d, e)$ s.t. $\tau(d, e, h)=1$.

This gives us immediately the proof of the first part of Theorem 0.3 (using also the analytic implicit function Theorem and (2.3)).

COROLLARY 2.4. – *For all $d>0$, $h^c(d)$ is the supercooling field.*

Proof. – We just remember Proposition 0.1 and Definition 0.2 of the supercooling field.

We pursue this section with the following lemma which gives some regularity properties of the supercooling field as a function of the thickness d.

PROPOSITION 2.5. – *$d \to h^c(d)$ is a continuous, strictly positive, piecewise analytic function.*

Proof of Proposition 2.5. – *Let us first study the continuity property near d_0.*

Let d_n a sequence s.t. $d_n \to d_0$.

As a consequence of (2.15) and (2.16), we have the following property:
There exists a compact K in $\mathbb{R}^{+*}$ s.t. $h^c(d_n) \in K$. (2.17)

It is sufficient to prove that one can extract a subsequence d'_n s.t. $h^c(d'_n)$ tends to $h^c(d_0)$.

According to (2.17), there exists a subsequence $\tilde{d}_n$ s.t.:

$$h^c(\tilde{d}_n) \to \tilde{h}.$$

Using (2.7), we can find (by new extraction) a sequence d'_n and a sequence e_n s.t. e_n tends to e_0, e_n is a minimum of

$$e \to \tau(d'_n, e, h^c(d'_n)),$$
$$1=\tau(d_n, e_n, h^c(d_n)).$$

By continuity we get $\tau(d_0, e_0, \tilde{h})=1$.

On the other hand, e_0 corresponds to a minimum of $e \to \tau(d_0, e, \tilde{h})$. If it was not true we would find $\tilde{e}_0$ s.t. $\tau(d_0, \tilde{e}_0, \tilde{h}) < \tau(d_0, e_0, \tilde{h})=1$ and by continuity we would get $\tau(d'_n, \tilde{e}_0, h^c(d'_n))<1$ for n large enough which contradicts the assumption that e_n is a minimum.

So we get $\operatorname{Inf}_e \tau(d_0, e, \tilde{h})=1$ and by unicity $\tilde{h}=h^c(d_0)$.

Let us prove now:

$$\text{Outside a locally finite set in } \mathbb{R}^{+*},\ d \to h^c(d) \text{ est analytic.} \tag{2.18}$$

The proof is quite similar to the proof of the analyticity in Lemma 2.1 but in this lemma we did not analyze the dependence with respect to d. We have seen before that for all d, there exists $e(d)$ s. t.:

$$(\partial\tau/\partial e)(d, e(d), h^c(d))=0, \qquad \operatorname{Inf}_e \tau(d, e, h^c(d))=\tau(d, e(d), h^c(d))=1$$

The natural idea is to consider the system:

$$\tau(d, e, h)=1;\ (\partial\tau/\partial e)(d, e, h)=0 \tag{2.19}$$

near the point $(d_0, e(d_0), h^c(d_0))$.

To arrive to a situation similar to the situation in the proof of Lemma 2.1, we observe using (2.3) that we can parametrize $\tau=1$ by an equation $h=h(d, e)$ depending analytically on (d, e). Then we are essentially reduced to the study of

$$\left.\begin{array}{c}\psi(d, e)=0\\ \text{with } \psi(d, e)=(\partial\tau/\partial e)(d, e, h(d, e))\end{array}\right\} \tag{2.20}$$

and the basic remark is then that:

Let (d_0, e_0) a point s. t. $\psi(d_0, e_0)=0$. Then there exists an open set $]d_0-\delta_0, d_0+\delta_0[\times]e_0-\varepsilon_0, e_0+\varepsilon_0[$ s. t. the solution of (2.20) are described by a finite set of continuous curves $d \to e^{(l)}(d)$, analytic outside of d_0, s. t. $e^{(l)}(d_0)=e_0$. (2.21)

To prove (2.21) as in Lemma 2.1. (according to the Weierstrass preparation Theorem) we have just to verify that:

$$e \to (\partial\psi/\partial e)(d_0, e) \text{ is a non identically vanishing, analytic function.} \tag{2.22}$$

We recall first from (2.6)-(2.7) that $e \to (\partial\tau/\partial e)\ (d_0, e, h_0)$ is not identically 0. Therefore there exists $k \geqq 2$ s. t.:

$$(\partial^l \tau/\partial e^l)(d_0, e_0, h_0)=0 \text{ for } l \leqq k-1 \tag{2.22}$$

and

$$(\partial^k \tau/\partial e^k)(d_0, e_0, h_0) \neq 0. \tag{2.23}$$

We recall also that:

$$\tau(d, e, h(d, e))=1. \tag{2.24}$$

From these two properties we deduce:

$$((d/de)^l (\partial\tau/\partial e)(d_0, e, h(d_0, e)))_{e=e_0} = 0 \qquad \text{for} \quad l \geqq k-2 \tag{2.25}$$

and

$$((d/de)^{k-1} (\partial\tau/\partial e)(d_0, e, h(d_0, e)))_{e=e_0} \neq 0. \tag{2.26}$$

Indeed we can differentiate (2.24) with respect to e and, using $\partial\tau/\partial h \neq 0$, we get:

$$\partial^l h(d_0, e_0)/\partial e^l = 0 \qquad \text{for} \quad l \leqq k-1. \tag{2.27}$$

Then (2.25)-(2.26) are immediate consequences of (2.22)-(2.23).

This gives modulo some details left to the reader the proof of (2.18) and of Proposition 2.5.

Remark 2.6. – Corollary 2.4 and Proposition 2.5 give the proof of Theorem 0.8.

PROPOSITION 2.7. – $h^c(d) = \operatorname{Sup}_e h(d, e)$.

Proof.

Step 1. – We first remark that:

$$\operatorname{Sup}_e h(d, e) = \operatorname{Sup}_{\{e,\, (\partial\tau/\partial e)(d,\, e,\, h(d,\, e)) = 0\}} h(d, e).$$

Step 2. – Let us prove now that

$$\operatorname{Sup}_{\{e,\, (\partial\tau/\partial e)(d,\, e,\, h(d,\, e)) = 0\}} h(d, e) \geqq h^c(d) \tag{2.28}$$

If $h > \operatorname{Sup}_{\{e,\, (\partial\tau/\partial e)(d,\, e,\, h(d,\, e)) = 0\}} h(d, e)$, we get by the monotonicity of τ with respect to $h : \tau(d, e, h) > \tau(d, e, h(d, e)) = 1$ for each e.

This implies (2.28) by Corollary 2.2.

Step 3. – Let us conversely prove that

$$\operatorname{Sup}_{\{e,\, (\partial\tau/\partial e)(d,\, e,\, h(d,\, e)) = 0\}} h(d, e) \leqq h^c(d) \tag{2.29}$$

Let e s.t. $(\partial\tau/\partial e)(d, e, h(d, e)) = 0$, $\tau(d, e, h(d, e)) = 1$.

Then by definition

$$\theta(d, h(d, e)) \leqq \tau(d, e, h(d, e)) = 1$$

and this implies $h(d, e) \leqq h^c(d)$ again by Corollary 2.2.

Remark 2.8. – Proposition 2.7 gives in particular the second assertion of Theorem 0.3.

LEMMA 2.9.. – $d \to H_{sc}(d)$ *is monotone decreasing.*

Proof of Lemma 2.9. – According to Proposition 2.5, we have just to prove that the derivative of $H_{sc}(d)$ is negative outside a locally finite set of points $\mathscr{D}$. It follows from the proof of this proposition that outside $\mathscr{D}$, one can always find locally an analytic positive function $d \to e(d)$, s. t.

$$H_{sc}(d) = h(d, e(d)). \tag{2.30}$$

Let us now differentiate with respect to d, the identity: $\tau(d, e(d), H_{sc}(d)) = 1$.

If we remember (2.19), we get:

$$(\partial\tau/\partial d)(d, e(d), H_{sc}(d)) + (\partial\tau/\partial h)(d, e(d), H_{sc}(d)).H'_{sc}(d) = 0. \tag{2.31}$$

According to (2.3), we are reduced to prove that

$$(\partial\tau/\partial d)(d, e(d), H_{sc}(d)) \geqq 0. \tag{2.32}$$

Let us compute $(\partial\tau/\partial d)(d, e, h)$ at a point where $(\partial\tau/\partial d)(d, e, h) = 0$. We observe as in [Da-He] (*see* also section 5), that we can write the following relations for the derivatives (it is convenient to translate by e, to arrive to a Neumann problem for the harmonic oscillator operator in $]-(d/2)+e, (d/2)+e[$:

$$\begin{aligned}(\partial\tau/\partial e)(d, e, h) = (-\tau(d, e, h) + h^2((d/2)+e)^2).\nu_+(d, e, h)\\ -(-\tau(d, e, h) + h^2((d/2)-e)^2).\nu_-(d, e, h)\\ (\partial\tau/\partial d)(d, e, h) = (1/2)[-\tau(d, e, h) + h^2((d/2)+e)^2).\nu_+(d, e, h)\\ +(-\tau(d, e, h) + h^2((d/2)-e)^2).\nu_-(d, e, h)]\end{aligned}$$

with $\nu_\pm(d, e, h) > 0$.

Using now the relation $(\partial\tau/\partial e)(d, e, h) = 0$, we get

$$(\partial\tau/\partial d)(d, e, h) = (-\tau(d, e, h) + h^2((d/2)+e)^2).\nu_+(d, e, h)$$

So we are just reduced to the proof that:

$$1 = \tau(d, e, h) \leqq h^2((d/2)+e)^2 \quad \text{for} \quad e = e(d),\ h = H_{sc}(d).$$

Since $e(d) \geqq 0$ and using (2.15) under the form $1 \leqq H_{sc}(d)^2 d^2/12$, we get the result.

Remark 2.10. – The monotonicity of $d \to h(d, 0)$ was already proved in $[\text{Bo}]_2$ (Proposition 2.5).

PART II. NEUMANN PROBLEMS WITH VARIABLE BOUNDARY

It is easier for the study of some properties to look to a scaled problem obtained by using the following map:

$$x = (\kappa h)^{-1/2}(y - c)$$

Then we get

$$\lambda(a, c) = \kappa\tau(d, e, h)/h$$
$$a = (\kappa h)^{1/2}(d/2), \qquad c = (\kappa h)^{1/2} e$$

where $\lambda(a, c)$ is the first eigenvalue of the Neumann problem

$$-g''(y) + y^2 g(y) = \lambda g(y) \quad \text{in }]-a+c, a+c[.$$

This part (Sections 3, 4, 5 and 6) will be devoted to the study of this problem. We shall describe more precisely in section 7 how to use the results of this part to prove the statements of the theorems announced in section 0.

3. Presentation of the problem and of the statements

Let us first introduce the following definition.

$\mathscr{S}$ is the set of the (a, c, λ, u) in $\mathbb{R}^{+*} \times \mathbb{R} \times \mathbb{R}^{+*} \times \mathrm{H}^2(]-a+c, a+c[, \mathbb{R})$ (3.1)

s. t. the following conditions (3.2)-(3.6) are satisfied:

$$\mathrm{P}u = \lambda u \quad \text{in }]-a+c, a+c[\tag{3.2}$$

where P is the harmonic oscillator:

$$\mathrm{P} = -d^2/dx^2 + x^2$$
$$u'(\pm a + c) = 0 \quad \text{(Neumann conditions)} \tag{3.3}$$
$$u > 0 \quad \text{in }]-a+c, a+c[\tag{3.4}$$
$$\int_{-a+c}^{a+c} u(x)^2\, dx = 1. \tag{3.5}$$
$$\int_{-a+c}^{a+c} x u(x)^2\, dx = 0 \tag{3.6}$$

One can always solve uniquely (3.2)-(3.5). The condition (3.4) implies that necessarily $\lambda = \lambda(a, c)$ is the first (simple) eigenvalue of the Neumann realization of P in $]-a+c, a+c[$ and that u is a corresponding eigenfunction. (3.5) permits to get uniqueness. Then the question is of course to satisfy (3.6). Let us observe that:

$\mathscr{S}$ is in bijection with the set s

obtained by projecting $\mathscr{S}$ on the two first variables. (3.7)

That means that if (a, c) belongs to s there exists a unique pair $(\lambda(a, c), u(a, c))$ such that $(a, c, \lambda(a, c), u(a, c))$ belongs to $\mathscr{S}$.

Analogously to Theorem 0.4 (*cf.* $[\mathrm{Bo}]_2$), we have the property:

$$\text{For each } a \text{ in } \mathbb{R}^{+*}, \qquad (a, 0) \in s \tag{3.8}$$

which is easy to prove according to the symmetry property of the potential and of the domain in the case $c=0$.

The same symmetry property implies to following property for s

$$(a, c)\in s \quad \text{iff } (a, -c)\in s. \tag{3.9}$$

Let us observe now that:

$$\lambda(a, 0) \text{ tends as } a \text{ tends to } \infty \text{ to } 1, \tag{3.10}$$

which is a relatively standard result because 1 is the eigenvalue of P in $\mathbb{R}$ and that the domain $]-a, a[$ tends to $\mathbb{R}$ as a tends to ∞.

We get a first localization (*see* (2.7)) of s by the following:

LEMMA 3.1. – $s\subseteq\{(a, c) \| c| < a\}$.

Proof. – Suppose for example that $c \geqq a > 0$. Then in (3.6) we observe that we integrate over an interval in $\mathbb{R}^+$. In particular $x.u(x)$ is $\geqq 0$ on this interval. Then (3.6) implies that $xu(x)=0$ and we get the contradiction with (3.5). The case when $c \leqq -a < 0$ is treated on the same way.

In a preceding paper ([Bo]$_2$), the existence of solutions for (3.2)-(3.6) with $c \neq 0$ was studied in connection with the existence of bifurcation (*see* also [Bo]$_2$). Let us just recall some facts of [Bo]$_2$ which will be useful in our discussion.

First we remark that condition (3.6) is equivalent to:

$$(\partial\lambda/\partial c)(a, c)=0. \tag{3.11}$$

Then

$$s=(\partial\lambda/\partial c)^{-1}(0) \tag{3.12}$$

We observe also that according to the analyticity of $\lambda(a, c)$, if $(a_0, c_0)\in s$ and $\nabla(\partial\lambda/\partial c)(a_0, c_0)\neq 0$, s is locally a submanifold in $(\mathbb{R}^{+*})^2$.

As we remark before: $(a, 0)\in s$. Now if $\nabla(\partial\lambda/\partial c)(a, 0)\neq 0$ for each a in $\mathbb{R}^{+*}$ then the line $c=0$ is isolated and one can not hope any bifurcation. For this reason, it was interesting to prove the existence of solutions of:

$$(\partial^2\lambda/\partial c^2)(a, 0)=0. \tag{3.13}$$

(we observe that $(\partial^2\lambda/\partial c\,\partial a)(a, 0)$ always vanishes).

This was proved by a careful analysis of this quantity as a tends to 0 and as a tends to ∞ exhibiting a change of sign of the function:

$$a \to (\partial^2\lambda/\partial c^2)(a, 0).$$

More precisely it is proved that:

For A large enough, 0 is, for each $a \geqq$ A, a local maximum of $c \to \lambda(a, c)$ and, for each $a \leqq 1/\text{A}$, a local minimum of $c \to \lambda(a, c)$. (3.14)

Now let us compare $\lambda(a, 0)$ and $\lambda(a, a)$. An easy computation shows that:

$$\lim_{a \to +\infty} \lambda(a, a) = 1. \tag{3.15}$$

But this is not sufficiently precise for our purpose. Let us recall that it was proved in [Da-He] (Theorem 4.2) by elementary Sturm-Liouville techniques that:

The first eigenvalue of the Neumann problem in $]0, +a[$ increases monotonically from 0 to 1 as a goes from 0 to $+\infty$. (3.16)

We observe also that this eigenvalue is the same that the first eigenvalue of the Neumann problem in $]-a, a[$. This implies (observing that $\lambda(a, a)$ is the first eigenvalue of the Neumann problem in $]0, 2a[$) the inequality:

$$\lambda(a, a) > \lambda(a, 0).$$

As a consequence, this will give us the existence for a large enough of two symmetric minima and the following lemma:

LEMMA 3.2 (*cf.* Proposition 2.25 in $[\mathrm{Bo}]_2$). — *There exists* $A>0$, *s.t., for each* $a>A$, $\#\{c \mid (a, c) \in s\} \geqq 3$.

We now formulate the two results which are the analogs of Theorem 0.5 and Theorem 0.7 in the introduction.

THEOREM 3.3. — *There exists* A *s.t. for* $a>A$ *there is a unique* $c=c(a)>0$ *s.t.* $(a, c) \in s$. *Moreover (taking possibly a larger* A) $c(a)$ *has the following properties:*

$$c(a)/a \to 1 \text{ as } a \text{ tends to} +\infty, \tag{3.16}$$

$$a - c(a) \to \alpha_0 \text{ exponentially rapidly as tends to } +\infty, \tag{3.17}$$

$\lambda(a, c(a))$ *is monotonically increasing and tends to* μ_1^0 *exponentially rapidly as a tends to* $+\infty$ *(with* $0<\mu_1^0<1$), (3.18)

$$\lambda(a, c(a)) = \operatorname{Min}_{c \in \mathbb{R}} \lambda(a, c) \text{ for } a>A. \tag{3.19}$$

THEOREM 3.4. — *These exists* $A>0$, *s.t.*

$$\{(a, c) \mid (a, c) \in s,\ a<(1/A)\} = \{(a, 0) \mid a<(1/A)\} \tag{3.20}$$

$$\lambda(a, 0) = \operatorname{Min}_{c \in \mathbb{R}} \lambda(a, c) \quad \text{for} \quad a<(1/A). \tag{3.21}$$

4. The Neumann problem on a semi-axis

In this section we recall useful results of $[\mathrm{Bo}]_2$, [Da-He], [Gu] on the Neumann model problem $(\mathscr{Q})_\alpha$:

Find ϕ in $H^2(]-\alpha, +\infty[)$ s.t.

$$P\phi = \mu\phi \text{ in }]-\alpha, +\infty[\tag{4.1$_a$}$$

where P is again the harmonic oscillator:

$$P = -d^2/dy^2 + y^2$$

and

$$\phi'(-\alpha) = 0 \qquad (4.1)_b$$

(Neumann condition at $-\alpha$)

$$\phi > 0 \text{ in }]-\alpha, +\infty[\qquad (4.1)_c$$

Let us first recall that it has been proved in [Da-He] or [Gu] by elementary methods that, if we denote by $\mu_1(\alpha)$ the first eigenvalue of the Neumann problem in $]-\alpha, +\infty]$,

the function $\alpha \to \mu_1(\alpha)$ has a unique minimum μ_1^0 on $]0, \infty[$
at $\alpha = \alpha_0$ which satistifes: $\mu_1(\alpha_0) = \alpha_0^2$. (4.2)

Let us also recall that this minimum is < 1 because:

$$\mu_1(0) = \mu_1(\infty) = 1. \qquad (4.3)$$

In particular:

$$\mu_1(\alpha) \in M = [\alpha_0^2, 1] \qquad (4.4)$$

which is compact in $\mathbb{R}^{+*}$, and this will be very important to have uniform control of the results. By standard o.d.e. techniques (*see* Sibuya [Si], Theorems 6.1 and 7.1) we know that we can for any parameter μ choose a basis $\phi_1(., \mu)$, $\phi_2(., \mu)$ of the solutions of the operator $(P-\mu)$ in $\mathbb{R}$ s.t.:

$$\phi_1(y, \mu) = \exp(-y^2/2) \,.\, y^{(\mu-1)/2}(1 - [(\mu-1)(\mu-3)/16y^2] + O(1/y^4)) \qquad (4.5)$$

$$\phi_2(y, \mu) = \exp(y^2/2) \,.\, y^{-(\mu+1)/2} \,.\, (1 + [(\mu+1)(\mu+3)/16y^2] + O(1/y^4)) \qquad (4.6)$$

where all these expansions are valid as $y \to +\infty$ and the O are locally uniform with respect to μ.

Moreover, we have similar expansions for the derivatives, and in particular:

$$\phi_1'(y, \mu) = -\exp(-y^2/2) \,.\, y^{(\mu+1)/2}(1 - [(\mu-1)(\mu+5)/16y^2] + O(1/y^4)) \qquad (4.7)$$

when $y \to +\infty$.

$$\phi_2'(y, \mu) = \exp(y^2/2) \,.\, y^{-(\mu-1)/2}(1 + [(\mu+1)(\mu-5)/16y^2] + O(1/y^4)) \qquad (4.8)$$

when $y \to +\infty$.

Let us observe now that if $\mu = \mu_1(\alpha)$, then a first eigenfunction $\tilde{\phi}_1(y, \alpha)$ of the Neumann problem is necessary of the form:

$$\tilde{\phi}_1(y, \alpha) = C_\alpha \phi_1(y, \mu_1(\alpha)) \quad \text{with} \quad C_\alpha \neq 0 \qquad (4.9)$$

We have more presisely proved the following:

LEMMA 4.1. — *The normalized first eigenfunction* $\tilde{\phi}_1(., \alpha)$ *of the Neumann problem in* $]-\alpha, +\infty]$ *has the following behavior:*

$$\tilde{\phi}_1(y, \alpha) = g(\alpha) \exp(-y^2/2) \,.\, y^{(\mu_1(\alpha)-1)/2} \times (1 - [(\mu_1(\alpha)-1)(\mu_1(\alpha)-3)/16y^2] + O(1/y^4)) \quad (4.10)$$

as $y \to +\infty$, *where* O *is uniform wich respect to* α
and

$$g(\alpha) \text{ is a continuous strictly positive function on } [0, \infty[\quad (4.11)$$

and

$$\left.\begin{aligned} & g(0) = 2^{1/2} \,.\, \pi^{-1/4} \\ \text{and} \\ & g(\alpha) = \pi^{-1/4}(1 + O(\exp(-\alpha^2/2))) \text{ as } \alpha \to +\infty. \end{aligned}\right\} \quad (4.12)$$

Proof. — We just have to compute: $\| \phi_1(., \mu) \|_{L^2(]-\alpha, +\infty[)}$ and observe that the construction of ϕ_1 is regular with respect to μ.

We observe also that $\mu_1(\alpha)$ is a C^∞ function of α on $[0, \infty[$ and that as α tends to $+\infty$, $\mu_1(\alpha)$ tends to $+1$ the first eigenvalue of the harmonic oscillator. Observe also that for $\mu = 1$, $\phi_1(y, \mu)$ is exactly the function $\exp(-y^2/2)$. From all these remarks it is clear that:

$$\alpha \to g(\alpha) = \| \phi_1(y, \mu)) \|^{-1}_{L^2(]-\alpha, +\infty[)} \qquad (\text{for } \mu = \mu_1(\alpha))$$

is a continuous strictly positive function on $[0, +\infty[$ and that:

$$g(0) = 2^{1/2} \,.\, \pi^{-1/4}.$$

It remains to prove that $g(\alpha)$ tends to $\pi^{-1/4}$ as α tends to $+\infty$ to get a complete answer. This is proved in $[\text{Bo}]_2$ by comparing again the Neumann problem in $]-\alpha, +\infty]$ and the global problem in $]-\infty, +\infty[$.

The last observation in this section is the following:

LEMMA 4.2. — *If we denote by* $\mu_2(\alpha)$ *the second eigenvalue of the Neumann problem in* $]-\alpha, +\infty]$, *there exists a constant* $\nu_0 > 0$ *s. t. for each* $\alpha \in [0, \infty[$ *we have:*

$$\mu_2(\alpha) - \mu_1(\alpha) \geqq \nu_0 > 0. \quad (4.13)$$

Proof. — This is clearly true on every compact of $[0, \infty[$. At $+\infty$, this is a consequence of the semi-classical study:

$$\mu_2(\alpha) - \mu_1(\alpha) \text{ tends to } 2 \text{ as } \alpha \text{ tends to } +\infty. \quad (4.14)$$

2 is indeed the splitting between the two first eigenvalues of the harmonic oscillator. Note also as a remark that:

$$\mu_2(0) - \mu_1(0) = 4. \quad (4.15)$$

5. Construction of quasimodes for the Neumann problem in $]-a+c, a+c[$

In this section, we shall try to have a very precise estimate for the first eigenvalue and the corresponding eigenfunction of the Neumann-problem in $]-a+c, a+c[$ for each (a, c) in the limit where a is large and for c satisfying

$$-a<c<a. \tag{5.1}$$

For this analysis, we shall use the construction of quasimodes in the semi-classical spirit (*see* [He-Sj]). This will be partially a substitute for ordinary differential techniques. Let us define:

$$\alpha=a-c, \qquad \beta=a+c$$

Let us now construct a good candidate to approximate the Neumann problem under the condition that β is a large positive number. Typically we shall need these results under the condition that:

$$c>-\rho_2 . a \text{ for some fixed } \rho_2 \text{ satisfying } 0<\rho_2<1. \tag{5.2}$$

Under assumption (5.2) β large will be a consequence of:

$$\beta \geqq (1-\rho_2) a \tag{5.3}$$

and we have always:

$$\alpha>0. \tag{5.4}$$

We denote by $\mathscr{A}(\rho_2, A)$ the set of (a, c) satisfying (5.1), (5.2) and $a \geqq A$. We start from the solutions given in section 4 with $\mu=\mu_1(\alpha)$ and we look *a priori* for linear combination of the two functions:

$$\psi(y, a, c)=\gamma(a, c) \phi_1(y, \mu_1(\alpha))+\delta(a, c) \phi_2(y, \mu_1(\alpha))$$

This is bad because the boundary conditions are not satisfied. Recall that this problem was studied in $[\mathrm{Bo}]_2$ in the case where $c=0$ and we shall follow the same strategy. The only thing we know is that ϕ_1 satisfies the condition at $-\alpha$. On the other side we must think to the term $\delta(a, c) \phi_2(y, \mu_1(\alpha))$ as a correction to take account of boundary effects at β (ϕ_1 does not satisfy the condition at β but is very small there). So it is natural to multiply this function by a cutoff function χ s. t.:

$$\chi(x)=0 \quad \text{if } x \leqq 1/2; \qquad \chi(x)=1 \quad \text{if } x \geqq 3/4,$$

and to consider as a candidate to be a good quasimode the function:

$$\psi(y, a, c)=\gamma(a, c) \phi_1(y, \mu_1(\alpha))+\delta(a, c) \chi(y/\beta) \phi_2(y, \mu_1(\alpha)) \tag{5.5}$$

We have now to choose the coefficients γ and δ s. t.:
- the Neumann-boundary conditions are satisfied
- the function ψ is normalized
- the error is sufficiently small.

Let us start with the Neumann condition. By construction, the boundary condition is automatically satisfied at $-\alpha$. So we have to look only to a condition at β, which is: $\gamma(a, c)\phi_1'(\beta, \mu)+\delta(a, c)\phi_2'(\beta, \mu)=0$, or

$$\delta(a, c)=-\gamma(a, c)\phi_1'(\beta, \mu)/\phi_2'(\beta, \mu) \quad \mu=\mu_1(\alpha). \tag{5.6}$$

From (4.7) and (4.8) and observing that β tends to $+\infty$ as a tends to $+\infty$ and that $\mu_1(\alpha)$ belongs to a compact we obtain:

$$\delta(a, c)=\gamma(a, c)\exp(-\beta^2)(\beta)^{\mu}.(1-[(\mu^2-5)/8\beta^2]+O(1/a^4)) \tag{5.7}$$

as a tends to $+\infty$ where O is uniform with respect to (a, c) satisfying the conditions (5.3)-(5.4).

Let us now determine $\gamma(a, c)$ by the normalization condition:

$$\|\psi\|_{L^2(]-\alpha, \beta[)}=1. \tag{5.8}$$

We shall prove the following:

LEMMA 5.1. – *There exists* A *and two constants* γ_1 *and* γ_2 *s.t., for all* $(a, c)\in\mathscr{A}(\rho_2, A)$, *we have:*

$$0<\gamma_1\leqq\gamma(a, c)\leqq\gamma_2. \tag{5.9}$$

Proof. – The general idea is again that this is the first term in (5.5) which will be dominant. This first term was indeed computed in section 4 (Lemma 4.1). Then all the other contributions are exponentially small due to (5.7) and to the estimate:

$$\|\chi(y/\beta)\phi_2(y, \mu_1(\alpha))\|_{L^2(]-\alpha, \beta[)}\leqq 2\exp((\beta)^2/2) \tag{5.10}$$

for all $(a, c)\in\mathscr{A}(\rho_2, A)$ and A large enough.

So we get modulo an exponentially small term that:

$$\gamma(a, c)=g(\alpha)+O(\exp-\beta^2/4) \quad (^4) \tag{5.11}$$

We arrive to the last but most important point in the proof to be sure that we have a good approximation. We have now to estimate precisely the error we have made. We now compute $(P-\mu_1(\alpha))\psi$:

$$(P-\mu_1(\alpha))\psi=\delta(a, c)[P, \chi]\phi_2(., \mu_1(\alpha)) \tag{5.12}$$

Using for the first time the precise choice of the cutoff-function, we get immediately from (5.7) and (5.9) that:

$$\|\delta(a, c)[P, \chi]\phi_2(y, \mu_1(\alpha))\|_{L^2(]-\alpha, \beta[)}\leqq C(A).\exp-(11\beta^2/16)). \tag{5.13}$$

As in [He – Sj] (*cf.* also $[Bo]_2$), we deduce from (5.12)-(5.13) the existence of an eigenvalue of our problem which is near of $\mu_1(\alpha)$ with an error which is of the same order as the r.h.s. of (5.13). This is the immediate

(4) Actually we do not need always an optimal estimate according to stronger errors elsewhere.

consequence of the spectral theorem. Using now Lemma 4.2 (*see* Lemma 5.2), it is not too difficult to see that this eigenvalue is unique. So there exists $A>0$ and $C(A)$ s.t. for each $(a, c) \in \mathscr{A}(\rho_2, A)$ there exists a unique eigenvalue $\lambda(a, c)$ of the Neumann-problem with

$$|\lambda(a, c)-\mu_1(a-c)| \leqq C(A) \exp-(11\,\beta^2/16) \tag{5.14}$$

We now have to analyze more precisely the link between the eigenmode and the real eigenfunction. This is possible because of the uniform control of the splitting between the two first eigenvalues (that we have already used to obtain (5.14)):

LEMMA 5.2. – *There exists a constant* ν_1 *s.t. for all* $(a, c) \in \mathscr{A}(\rho_2, A)$, *we have:*

$$|\lambda_2(a, c)-\lambda_1(a, c)| \geqq \nu_1 \tag{5.15}$$

(*we have written* $\lambda(a, c)=\lambda_1(a, c)$).

Proof. – This is just the consequence of simple semi-classical analysis and Lemma 4.2.

We denote by $\Pi_{a,c}$ the spectral projector attached to $\lambda(a, c)$. Then we have, as a consequence of Lemma 5.2 (*see* for example Proposition 2.5 in [He-Sj]) that:

$$\|(I-\Pi_{a,c})\psi\| \leqq C(A) . \exp-(11\,\beta^2/16). \tag{5.16}$$

Consequently, the corresponding normalized eigenfunction has the following form:

$$u(., a, c)=\Pi_{a,c}\psi/\|\Pi_{a,c}\psi\|=\psi(., a, c)+O(\exp-(11\,\beta^2/16))$$

i.e.

$$\|u-\psi(., a, c)\|_{L^2(]-\alpha, \beta[)} \leqq C(A)(\exp-(11\,\beta^2/16) \tag{5.17}$$

uniformly for $(a, c) \in \mathscr{A}(\rho_2, A)$.

Let us now introduce the spaces:

$$F_{a,c}=\{v \in L^2(]-\alpha, \beta[), \langle v \mid u \rangle_{L^2(]-\alpha, \beta[)}=0\}$$
$$G_{a,c}=\{v \in H^2(]-\alpha, \beta[), v'(-\alpha)=v'(\beta)=0, \langle v \mid u \rangle_{L^2(]-\alpha, \beta[)}=0\}.$$

Then it is clear that $(P-\lambda(a, c))$ is an isomorphism from $G_{a,c}$ onto $F_{a,c}$ and Lemma 5.2 gives us:

$$\|(P-\lambda(a, c))^{-1}\|_{\mathscr{L}(F, F)} \leqq \nu_1^{-1} \tag{5.18}$$

and using that $-d^2/dy^2=P-y^2$ and using a trivial estimate on the potential we get:

$$\|(P-\lambda(a, c))^{-1}\|_{\mathscr{L}(F, G)} \leqq C(A)\,a^2, \qquad \text{for} \quad (a, c) \in \mathscr{A}(\rho_2 A). \tag{5.19}$$

We are now able to compute more precisely $\lambda(a, c)$, using the orthogonality of $\Pi_{a,c}\psi$ and of $\psi-\Pi_{a,c}\psi$:

$$\lambda(a, c)=(\mathrm{P}\,\Pi_{a,c}\psi\,|\,\psi)/\|\Pi_{a,c}\psi\|^2$$
$$=\mu_1(\alpha)+((\mathrm{P}-\mu_1(\alpha))\psi/\psi)+O(\exp-(5\beta^2/4)) \quad (5.20)$$

We arrive to the computation of the expression $((\mathrm{P}-\mu_1(\alpha))\psi/\psi)$ for which we use (5.12):

$$\langle\,\delta(a, c)[\mathrm{P}, \chi]\phi_2(.\,, \mu_1(\alpha))\,|\,\gamma(a, c)\phi_1(.\,, \mu_1(\alpha)) + \delta(a, c)\chi(./(\beta))\phi_2(.\,, \mu_1(\alpha))\,\rangle$$

and we get:

$$\lambda(a, c)-\mu_1(\alpha)=\delta(a, c)\gamma(a, c) < [\mathrm{P}, \chi]\phi_2(.\,, \mu_1(\alpha))\,|\,\phi_1(.\,, \mu_1(\alpha))\,\rangle + O(\exp-(5\beta^2/4)) \quad (5.21)$$

Let us look more carefully to:

$$\langle\,[\mathrm{P}, \chi]\phi_2(.\,, \mu_1(\alpha))\,|\,\phi_1(.\,, \mu_1(\alpha))\,\rangle = \langle\,[\mathrm{P}-\mu_1(\alpha), \chi]\phi_2(.\,, \mu_1(\alpha))\,|\,\phi_1(.\,, \mu_1(\alpha))\,\rangle$$

and by integration by part we get that:

$$\langle\,[\mathrm{P}, \chi]\phi_2(.\,, \mu_1(\alpha))\,|\,\phi_1(.\,, \mu_1(\alpha))\,\rangle = -\phi_2'(\beta, \mu_1(\alpha).\phi_1(\beta, \mu_1(\alpha))+\phi_1'(\beta, \mu_1(\alpha).\phi_2(\beta, \mu_1(\alpha)).$$

Let us now use the formulas (4.5)-(4.8) and we get:

$$\langle\,[\mathrm{P}, \chi]\phi_2(.\,, \mu_1(\alpha))\,|\,\phi_1(.\,, \mu_1(\alpha))\,\rangle = -2(1+O(1/a^2)).$$

Plugging this result in (5.16), we get:

$$\lambda(a, c)-\mu_1(\alpha)=-2\delta(a, c)\gamma(a, c)+O(\exp-(5\beta^2/4))$$
$$=-2\gamma(a, c)^2\exp(-\beta^2)(\beta)^{\mu}(1+O(1/a^2))$$

and using (5.11) :

$$\lambda(a, c)-\mu_1(\alpha)=-2g(\alpha)^2\exp(-\beta^2).\beta^{\mu_1(\alpha)}.(1+O(1/a^2)) \quad (5.22)$$

uniformly for $(a, c)\in\mathscr{A}(\rho_2, \mathrm{A})$.

Let us now return to (5.17) to improve it. We deduce from:

$$(\mathrm{P}-\lambda(a, c))(\psi(.\,, a, c))=(\mu_1(\alpha)-\lambda(a, c))(\psi(.\,, a, c)) + (\mathrm{P}-\mu_1(\alpha))\psi(.\,, a, c),$$

(5.13) and (5.22) that:

$$\|(\mathrm{I}-\Pi_{a,c})(u(.\,, a, c)-\psi(.\,, a, c)\|_{\mathrm{H}^2}=O(\exp(-5\beta^2/8))$$

But we know that:

$$\|\Pi_{a,c}(u(.\,, a, c)-\psi(.\,, a, c))\|_{\mathrm{H}^2} \leq \|u(.\,, a, c)\|_{\mathrm{H}^2}\|u(.\,, a, c)-\psi(.\,, a, c)\|_{\mathrm{L}^2} = O(\exp(-5\beta^2/8)).$$

Using the injection of H^2 in C^1 we get finally that:

$$\|(u(.,a,c)-\psi(.,a,c))\|_{C^1([-\alpha,\beta])}\leqq C(A)\exp(-5\beta^2/8) \quad (5.23)$$

uniformly for $(a, c)\in\mathscr{A}(\rho_2, A)$.

Finally we have proved the following proposition:

PROPOSITION 5.3. – *If $(a, c)\in\mathscr{A}(\rho_2, A)$, then the first eigenvalue of the Neumann-problem satisfies:*

$$|\lambda(a,c)-\mu_1(a-c)\|\leqq C(A)\exp-(11(a+c)^2/16)$$

and the corresponding normalized eigenfunction satisfies:

$$\|(u(.,a,c)-\psi(.,a,c))\|_{C^1([-a+c,a+c])}\leqq C(A)\exp(-5(a+c)^2/8)$$

where $\mu_1(\alpha)$ is the first eigenvalue of the Neumann problem in $]-\alpha, +\infty]$ and ψ is defined in (5.5).

COROLLARY 5.4. – *If $(a, c)\in\mathscr{A}(\rho_2, A)$, we have:*

$$u(a+c,a,c)$$
$$=2g(a-c)\exp(-(a+c)^2/2)(a+c)^{\mu_1(a-c)-1}(1+O(1/a)) \quad (5.24)$$

For $\rho_2 a\geqq c\geqq 0$, $(\rho_2<1)$ we have:

$$u(-a+c,a,c)=2\pi^{-1/4}\exp(-(a-c)^2/2)(1+O(1/a)). \quad (5.25)$$

Proof. – Let us first consider (5.24). We first remark that:

$$u(a+c,a,c)=\psi(a+c,a,c)(1+O(\exp(-(a+c)^2/8))$$
$$\psi(a+c,a,c)=\gamma(a,c)[(\phi_2'\phi_1-\phi_1'\phi_2)/\phi_2'](a+c,\mu_1(\alpha))$$
$$=2\gamma(a,c)/\phi_2'(a+c,\mu_1(\alpha))(1+O(1/a))$$
$$=2\gamma(a,c)\exp(-(a+c)^2/2)(a+c)^{\mu_1(\alpha)-1}(1+O(1/a))$$

So finally we get:

$$u(a+c,a,c)=2\gamma(a,c)\exp(-(a+c)^2/2)(a+c)^{\mu_1(\alpha)-1}(1+O(1/a))$$

and we can use also (5.11) to get (5.24). Let us now consider (5.25). Of course one way will be to use that $u(-a+c, a, c)$ is well approximated by:

$$\psi(-a+c,a,c)=\gamma(a,c)\phi_1(-a+c,\mu_1(\alpha))+O(\exp(-5(a+c)^2/8))$$

and then to study $\phi_1(-\alpha, \mu_1(\alpha))$. This is probably possible but we will use an indirect way using the preceding result.

If we observe that $u(-y, a, c)=u(y, a, -c)$, we get that under the condition that $c\geqq 0$ and $a-c>\rho_0 a$ for some $\rho_0>0$ that:

$$u(-a+c,a,c)=2\gamma(a,-c)\exp(-(a-c)^2/2)(a-c)^{\mu_1(a+c)-1}(1+O(1/a))$$
$$=2g(a+c)\exp(-(a-c)^2/2)(1+O(1/a)).$$

Finally we use (4.12) and $\alpha\geqq(1-\rho_2)a$ to get the conclusion.

Remark 5.5. – (5.25) is still true if you replace the condition $(a-c)\geqq(1-\rho_2)\,a$ by the weaker condition $(a-c)\geqq \mathrm{B}$ with B sufficiently large and if you write (5.25) on the form:

$$u(-a+c, a, c)=2\pi^{-1/4}\exp(-(a-c)^2/2)(1+O(1/(a-c)) \quad (5.26)$$

6. Proof of Theorems 3.3 and 3.4

We have already observe that we can assume that:

$$c\geqq 0 \quad (6.1)$$

and that on s we have consequently the condition

$$0\leqq c<a. \quad (6.2)$$

Let us continue to localize the set s by the following lemma:

LEMMA 6.1. – There exists B and $\mathrm{A}>0$ s.t

$$s\cap\{(a, c)\,|\,a\geqq \mathrm{A}, (\mathrm{B}/a^2)<c<a-\mathrm{B}\}=\varnothing. \quad (6.3)$$

Proof. – As it results of [Da-He] (by a variant of the Hellman formula), (c, a) belongs to s if and only if

$$\Phi(c)-\Phi(-c)=0 \quad (6.4)$$

where:

$$\Phi(c)=((a+c)^2-\lambda(a, c)).u^2(a+c, a, c) \quad (6.5)$$

Now we have already used that:

$$\lambda(a, -c)=\lambda(a, c) \quad (6.6)$$

and

$$u(x, a, c)=u(-x, a, -c) \quad (6.7)$$

Then we can rewrite (6.4) under the form:

$$[((a+c)^2-\lambda(a, c)).u^2(a+c, a, c)]/[((a-c)^2-\lambda(a, c)).u^2(c-a, a, c)]=1 \quad (6.8)$$

Let us write $c/a=\rho(a, c)$.

We shall prove that if $(\mathrm{B}/a^3)\leqq\rho\leqq 1-(\mathrm{B}/a)$, then for a large enough:

$$[((a+c)^2-\lambda(a, c)).u^2(a+c, a, c)]/[((a-c)^2-\lambda(a, c)).u^2(c-a, a, c)]<1.$$

Let us start with the problem near 0 and let us assume:

$$(\mathrm{B}/a^3)\leqq\rho\leqq\rho_2<1.$$

Using the different asymptotics of section 4, we get:

$$\Phi(c)/\Phi(-c)=(1+\rho)^2(1-\rho)^{-2}.\exp(-4a^2\rho)(1+O(1/a)) \quad (6.9)$$

and the result follows by a careful study of the function:

$\rho \to (1+\rho)^2 (1-\rho)^{-2} . \exp(-D\rho)$ in the neighborhood of 0.

If we now consider the problem near 1:

$$0<\rho_1 \leqq \rho \leqq 1-(B/a)$$

we observe (*see* remark 5.5) that:

$$\Phi(c)/\Phi(-c)=(1+\rho)^2 (1-\rho)^{-2} . \exp(-4a^2 \rho)(1+O(1/B)) \quad (6.10)$$

and this is easier.

LEMMA 6.2. — *For each* B, *there exists* A>0 *s.t.*

$$s \cap \{(a, c) \mid a \geqq A,\ 0<c \leqq B a^{-2}\} = \varnothing \quad (6.11)$$

We have now to use properties of the functions $\Phi(c)$ and $\Phi(-c)$ near $c=0$. This is essentially what was made in $[Bo]_2$ to prove that $\partial^2 \lambda(a, 0)/\partial c^2$ was strictly positive for a large enough. We have just to verify that the control which was given for $c=0$ is satisfied for $|c| \leqq B a^{-2}$. If we derive (6.5) with respect to c and if you look to the proof in $[Bo]_2$, the only thing you have to prove is that for $|c| \leqq B a^{-2}$:

$$(\partial u/\partial c)(a+c, a, c)/u(a+c, a, c) = -a+O(1) \quad (6.12)$$

This can be proved by small modifications of the argument in $[Bo]_2$. Using a very accurate approximation (*see* Appendix 3 for details) of $(\partial v/\partial c)(x, a, c)$ where $v(x, a, c)=u(x+c, a, c)$, we justify the asymptotics obtained by deriving formally with respect to c the formula (5.24). In particular we obtain:

$$(\partial u/\partial c)(a+c, a, c)/u(a+c, a, c) = -(a+c)+O(1)$$

and consequently (6.12). We now recall that:

$$(\partial \lambda/\partial c)(a, c)=\Phi(c)-\Phi(-c) \quad (6.13)$$

and we get for some c_0 s. t. $|c_0| \leqq Ba^{-2}$:

$$(\partial \lambda/\partial c)(a, c)=c(\Phi'(c_0)+\Phi'(-c_0)). \quad (6.14)$$

But according to (6.5), we have:

$$\Phi'(c)=u^2(a+c, a, c)[2(a+c)-(\partial \lambda/\partial c)(a, c) + 2((a+c)^2-\lambda(a, c))((\partial u/\partial c)(a+c, a, c)/u(a+c, a, c))]. \quad (6.15)$$

We apply (6.15) for $c=\pm c_0$. Then we use the different asymptotics given in $[Bo]_2$ and (6.11) to obtain in the region we are looking that:

$$(\partial \lambda/\partial c)(a, c)=-4 . a^3 c . u^2(a+c, a, c)(1+O_B(1/a)) \quad (6.16)$$

and, using (5.24)-(5.25), we get finally

$$(\partial \lambda/\partial c)(a, c)=-16 . a^3 c . \exp(-a^2)\pi^{-1/2}(1+O_B(1/a)). \quad (6.17)$$

LEMMA 6.3. — *For each* $B>0$, *there exists* $A>B$, *and* $C>0$ *s. t.*

$$s\cap\{(a,c)\,|\,a\geqq A,\ a-B<c<a\}$$
$$=s\cap\{(a,c)\,|\,a\geqq A,\ |a-c-\alpha_0|\leqq C.\exp-(a^2/2)\} \quad (6.18)$$

Proof. — We return to the formula:

$$((a+c)^2-\lambda(a,c)).u^2(a+c,a,c)/u^2(c-a,a,c)$$
$$=((a-c)^2-\lambda(a,c)). \quad (6.19)$$

giving the equation of s.

We observe that $u^2(c-a,a,c)$ stays in a compact set away from 0 in the domain. More precisely we use the following lemma:

LEMMA 6.4. — *For each* $B>0$, *there exists* $A>B$ *and there exists* $\delta>0$ *s. t. for each* $a\geqq A$, *for each* c *s. t.* $a-B<c<a$, *we have:*

$$u(c-a,a,c)\geqq\delta. \quad (6.20)$$

Proof of Lemma 6.4. — We have just to compare with the corresponding first eigenfunction for the Neumann problem in $]-\alpha,+\infty[$ (with $\alpha=a-c$). The assumptions in the lemma imply that α stays in $[0, B]$ and the result is clear in this case. The approximation by this problem is good for a large enough.

Using (6.20) we obtain from (6.19) the following inequality:

$$|\lambda(a,c)-(a-c)^2|\leqq C.\exp(-a^2/2)$$

Using (5.22) we deduce from this with a new constant C:

$$|\mu_1(a-c)-(a-c)^2|\leqq C.\exp(-a^2/2)$$

Now we recall that the equation $\mu_1(\alpha)=\alpha^2$ has a unique solution at the minimum α_0 of $\mu_1(\alpha)$. Because α_0 is >0, we get with a new constant C:

$$|(a-c)-\alpha_0|\leqq C.\exp(-a^2/2), \quad (6.21)$$

and Lemma 6.3 is proved.

We have now a very precise localization result for s. The last useful lemma for the proof of Theorem 3.3 is the following:

LEMMA 6.5. — *For each* C, *there exists* $A>0$ *s. t. for* $a\geqq A$, *there exists a unique* $c(a)>0$ *s. t.* $(a,c(a))\in s$ *and*

$$|a-c(a)-\alpha_0|\leqq C.\exp-(a^2/2). \quad (6.22)$$

We know already the existence (*see* Lemma 3.2) for each a large enough of some $c(a)$. The only problem which remains is the problem of the uniqueness. Let us consider the equation

$$\psi(a,c)=0 \quad (6.23)$$

with

$\psi(a, c)$:

$$=(a-c)^2-\lambda(a, c)-[(a+c)^2-\lambda(a, c)]u^2(a+c, a, c)/u^2(-a+c, a, c)$$

If we have two solutions c' and c'' of (6.23) in the region

$$\{c \,|\, |a-c-\alpha_0| \leqq C.\exp-(a^2/2)\}, \tag{6.24}$$

we get:

$$\psi(a, c')-\psi(a, c'')=(c'-c'')(\partial\psi/\partial c)(a, \tilde{c}) \tag{6.25}$$

with $\tilde{c}$ also in the same region.

We have simply to control that $c \to (\partial\psi/\partial c)(a, c)$ does not vanish in the same region (6.24). The apparently dominant term is $-2(a-c)$ which is not zero because α_0 is not zero. To prove that the other terms are negligeable we observe that $(\partial\lambda/\partial c)(a, c)$ and $u(a+c, a, c)$ are exponentially small, and we use (6.20) together with the very crude estimates:

$|(\partial u/\partial c)(a+c, a, c)|$, $|(\partial u/\partial c)(c-a, a, c)|$ are slowly increasing with a. (6.26)

Let us prove this weak property very shortly. We start from the equation:

$$(-d^2/dx^2+x^2)u(x, a, c)=\lambda(a, c)u(x, a, c) \text{ in }]-a+c, a+c[$$

satisfying to the Neumann conditions. It is not easy to derive with respect to c because the domain depends on c. As in [Bo]$_2$ we introduce:

$$\tilde{u}(y, a, c)=u(y+c, a, c)$$

and we observe that this is a solution of:

$$(-d^2/dy^2+(y+c)^2)\tilde{u}(y, a, c)=\lambda(a, c)\tilde{u}(y, a, c) \text{ in }]-a, a[$$

with Neumann conditions.

We derive with respect to c and we obtain that:

$$(-d^2/dy^2+(y+c)^2-\lambda(a, c))(\partial\tilde{u}/\partial c)(y, a, c) = -[2(y+c)-\partial\lambda/\partial c]\tilde{u}(y, a, c)$$

Then we return to the initial coordinates to obtain:

$$(-d^2/dx^2+x^2-\lambda(a, c))(\partial\tilde{u}/\partial c)(x-c, a, c)=-[2x-\partial\lambda/\partial c]u(x, a, c)$$

But $(\partial\tilde{u}/\partial c)(y, a, c)=(\partial u/\partial x)(y+c, a, c)+(\partial u/\partial c)(y+c, a, c)$ and consequently:

$$(\partial\tilde{u}/\partial c)(x-c, a, c)=(\partial u/\partial x)(x, a, c)+(\partial u/\partial c)(x, a, c).$$

We now observe that $x \to y(x)=(\partial\tilde{u}/\partial c)(x-c, a, c)$ is orthogonal to u, satisfies the Neumann conditions. From (5.19) we deduce that $\|y\|_{H^2}$ increases polynomially with a. We remark now that

$$(\partial\tilde{u}/\partial c)(a, a, c)=(\partial u/\partial c)(a+c, a, c)$$
$$(\partial\tilde{u}/\partial c)(-a, a, c)=(\partial u/\partial c)(-a+c, a, c)$$

and using the Sobolev's injection of H^2 in $C^1([-a+c, a+c])$ as in (5.23), we obtain the polynomial control of $(\partial u/\partial c)(a+c, a, c)$ and $(\partial u/\partial c)(-a+c, a, c)$. The lemma is proved.

Remark 6.6. — It can be useful to have a better control of the sign of: $a-c(a)-\alpha_0$. This is in fact possible by careful analysis of the proof. We start again from (6.19)

$$((a-c)^2-\lambda(a, c))=((a+c)^2-\lambda(a, c)).u^2(a+c, a, c)/u^2(c-a, a, c).$$

Then we write:

$$\mu_1(a-c)-(a-c)^2=(\mu_1(a-c)-\lambda(a, c))+(\lambda(a, c)-(a-c)^2).$$

Using (5.22), (5.24) we get:

$$\begin{aligned}\mu_1(a-c)-(a-c)^2 &\\ &=2g(a-c)^2\exp(-(a+c)^2).(a+c)^{\mu_1(a-c)}.(1+O(1/a^2))\\ &-[4((a+c)^2-\lambda(a, c)).g(a-c)^2\exp(-(a+c)^2)(a+c)^{2\mu_1(a-c)-2}]/\\ &\qquad u^2(c-a, a, c)(1+O(1/a^2))\end{aligned}$$

For $c=c(a)$, this gives:

$$\begin{aligned}\mu_1(a-c)-(a-c)^2=-4g(\alpha_0)^2\exp(-(2a-\alpha_0)^2)&\\ \times(2a)^{2\mu_1(\alpha_0)}.(1+o(1)).(1/\tilde{\phi}_1(-\alpha_0, \alpha_0)^2)&\end{aligned}$$

where $\tilde{\phi}_1$ is defined in (4.9).

We are now able to give a much more precise result on $(a-c(a)-\alpha_0)$:

$$\begin{aligned}(a-c(a)-\alpha_0)=(2/\alpha_0).(g(\alpha_0)/\tilde{\phi}_1(-\alpha_0, \alpha_0))^2&\\ \times\exp(-(2a-\alpha_0)^2).(2a)^{2\mu_1(\alpha_0)}.(1+o(1))&\end{aligned}\tag{6.27}$$

Remark 6.7. — Another problem is to study the monotonicity of $a\to\Lambda(a)=\lambda(a, c(a))$.

As in section 2, we observe that:

$$\Lambda'(a)=(\partial\lambda/\partial a)(a, c(a))=\Phi(c(a))+\Phi(-c(a))=2\Phi(c(a))>0$$

for a large enough.

As a consequence we have:

There exists A_1 s.t. for $a>A_1$,

the function $a\to\Lambda(a)$ is monotonically increasing. (6.28)

Proof of Theorem 3.3. — This is just the union of the different assertions of the Lemmas 6.1-6.4 and of the Remark 6.7. Note also that Remark 6.6 gives a much more precise statement for (3.17).

Proof of Theorem 3.4. — This was partially proved in $[Bo]_2$. It was indeed proved that for a small enough $(\partial^2\lambda/\partial c^2)(a, 0)$ was >0. We give now the general proof.

After some scaling we arrive to the following question. We consider the Neumann problem in $]-1/2, 1/2[$ but for the operator:

$$-d^2/dz^2+\varepsilon(z+\sigma)^2 \tag{6.29}$$

(with $\varepsilon=a^4$, $\sigma=c/a$).

If $\psi(z, \varepsilon, \sigma)$ is the normalized first eigenfunction, the necessary condition for an extrema (*see* (3.6)) for our initial problem is now:

$$\int_{-1/2}^{+1/2} z\,\psi(z, \varepsilon, \sigma)^2\,dz=-\sigma/2. \tag{6.30}$$

This is of course satisfied for $\sigma=0$ by parity. We have now just to prove that

$$\int_{-1/2}^{+1/2} z\,\psi(z, \varepsilon, \sigma)^2\,dz=\sigma . O(\varepsilon) \tag{6.31}$$

But this is clear by standard perturbation theory if we write:

$$-d^2/dz^2+\varepsilon(z+\sigma)^2=[-d^2/dz^2+\varepsilon z^2+\varepsilon\sigma^2]+\varepsilon\sigma[2z]. \tag{6.32}$$

Then $\psi(z, \varepsilon, \sigma)$ has the following decomposition:

$$\psi(z, \varepsilon, \sigma)=\psi_1(z, \varepsilon, \sigma)+(\varepsilon\sigma)\psi_2(z, \varepsilon, \sigma) \tag{6.33}$$

where ψ_1 is even and ψ_2 is odd.

This gives the result.

PART III. THE LINK BETWEEN PART I AND PART II

7. Asymptotic properties of the supercooling field. End of the proofs of the theorems announced in the introduction

Let us first recall the relation between the notions introduced in Part I and II.

$$a=(\kappa h)^{1/2}(d/2) \tag{7.1}$$

$$c=(\kappa h)^{1/2} e \tag{7.2}$$

$$\lambda(a, c)=\kappa\tau(d, e, h)/h. \tag{7.3}$$

It is *a priori* clear how to associate to (d, e, h) the corresponding $(a, c, \lambda(a, c))$.

LEMMA 7.1. – *For fixed $h>0$, the map*

$$(d, e)\to(a, c)=((\kappa h)^{1/2}(d/2), (\kappa h)^{1/2} e)$$

gives a one to one correspondence between

the set of (d, e) s. t. $(\partial\tau/\partial e)(d, e, h)=0$

and the set of (a, c) s. t. $(\partial\lambda/\partial c)(a, c)=0$.

Proof. — This is an immediate consequence of

$$(\partial\tau/\partial e)(d, e, h) = (h^{3/2}/\kappa^{1/2})(\partial\lambda/\partial c)(a, c) \tag{7.4}$$

LEMMA 7.2. — *There exists* $\eta > 0$ *and* $A_0 > 0$ *s. t.*

$$(\tau(d, e, h)/h) > \eta, \text{ for all } (e, d) \text{ s. t. } h^{1/2} d \geqq A_0, \ |e| \leqq d/2.$$

Proof of Lemma 7.2. — Let us write (7.3) under the form:

$$(\tau(d, e, h)/h) = (1/\kappa)\,\lambda((\kappa h)^{1/2} d/2, (\kappa h)^{1/2} e) \tag{7.5}$$

Then Lemma 7.2, is a direct consequence of the following property of $a \to \operatorname{Inf}_c \lambda(a, c)$:

$$\left.\begin{array}{c}\text{There exists } A_1 \text{ and } \eta > 0 \text{ s. t., for } a > A_1,\\ \text{we have } \operatorname{Inf}_c \lambda(a, c) \geqq \eta\end{array}\right\} \tag{7.6}$$

which is an immediate consequence of (3.18) and (3.19).

Remark 7.3. — A weaker version of Lemma 7.2 was needed in the proof of Corollary 2.2.

Asymptotic behavior as $d \to +\infty$. — Until now we have only use very simple argument coming from Part II. We want now to have a precise analysis of $h^c(d)$ as d tends to $+\infty$.

LEMMA 7.4:

$$\lim_{d \to \infty} h^c(d) = \kappa/\mu_0^1. \tag{7.7}$$

Proof of Lemma 7.4. — We first observe the following relation:

$$\theta(d, h) = (h/\kappa) \operatorname{Inf}_c \lambda((\kappa h)^{1/2} d/2, c) \tag{7.8}$$

and recall that $h^c(d)$ was characterized as the unique h s. t. $\theta(d, h) = 1$.

If we produce by some other mean one solution $\tilde{h}(d)$ of this equation we are done. Let D and $h_0 > 0$ to be determine later. We are looking for a solution s. t.

$$\tilde{h}(d) > h_0 > 0 \qquad \text{for} \quad d \geqq D. \tag{7.8}$$

If we assume that

$$D = 2A(\kappa h_0)^{-1/2}$$

$$\text{and} \qquad h \geqq h_0 \text{ (where A is defined in Theorem 3.3)} \tag{7.9}$$

we know from this theorem the asymptotic behavior of $\operatorname{Inf}_c \lambda((\kappa h)^{1/2} d/2, c)$. As d tends to $+\infty$, it is then natural to look for $\tilde{h}(d)$

near $\kappa/\operatorname{Lim}_{a\to\infty}\lambda(a, c(a))=\kappa/\mu_1^0:=\tilde{h}(\infty)$. We now choose

$$h_0<\tilde{h}(\infty) \tag{7.10}$$

so D and h_0 are defined (once A is fixed).

Then it is now quite easy to see that for each $\varepsilon>0$, there exists

$$A_\varepsilon,\ h_\varepsilon^\pm \text{ s.t. } h_0\leqq\tilde{h}(\infty)-\varepsilon\leqq h_\varepsilon^-\leqq\tilde{h}(\infty)\leqq h_\varepsilon^+\leqq\tilde{h}(\infty)+\varepsilon \quad \text{and} \quad A\leqq A_\varepsilon,$$

s.t. for all $d\geqq D_\varepsilon=2A_\varepsilon(\kappa h_0)^{-1/2}$ the inequality $\theta(d, h_\varepsilon^-)\leqq 1\leqq\theta(d, h_\varepsilon^+)$ is satisfied. In particular, using the continuity of θ with respect to h there exists

$$\tilde{h}(d) \text{ s.t. } |\tilde{h}(d)-\tilde{h}(\infty)|\leqq\varepsilon \quad \text{and} \quad \theta(d, \tilde{h}(d))=1.$$

But $\tilde{h}(d)=h^c(d)$ and we have the proof of Lemma 7.3.

Remark 7.5. – Once we have the convergence of $h^c(d)$ as d tends to $+\infty$ to a strictly positive limit the other statements in Theorem 0.5 follow immediately of the corresponding statements in Theorem 3.3 and of Lemma 7.1.

We have indeed the following relation:

$$(2e(d)/d)=c((\kappa h^c(d))^{1/2}d/2)/((\kappa h^c(d))^{1/2}d/2)\to 1$$

as a consequence of (3.16) which gives (0.10).

In fact we can get much more precise information from Theorem 3.3 than written in Theorem 0.5.

Asymptotic behavior as $d\to+0$. – Most of the results concerning $d\to+0$ are already proved in $[\text{Bo}]_2$ (§ 2.2).

The proof uses a new scaling reducing to $[-1/2, 1/2]$ and in particular it was obtained that $d\to h(d, 0)$ was strictly decreasing for $d\leqq d_0$ and that $\operatorname{Lim}_{d\to 0}(d^2 h(d, 0)^2)=12$.

Let us just see briefly how to recover these properties from Part II and recall also that some estimate can be obtained from 2.15. In fact we proceed quite in parallel with the case $d\to+\infty$.

LEMMA 7.6.

$$\operatorname{Lim}_{d\to 0} d^2 h^c(d)^2=12. \tag{7.11}$$

Proof of Lemma 7.6. – We start again from (7.8) and recall that $h^c(d)$ was characterized as the unique h s.t. $\theta(d, h)=1$. Thus we have to produce by some other mean one solution $\tilde{h}(d)$ of this equation. Let $d_0>0$ to be determine later. We are looking for a solution $\tilde{h}(d)$ s.t.

$$d^2\kappa\tilde{h}(d)<4/A^2 \quad \text{for} \quad d\leqq d_0 \tag{7.12}$$

where $1/A$ was given in Theorem 3.4.

Let us first work heuristically. We know (*see* $[Bo]_2$ or the proof of Theorem 3.4) that $\lambda(a, 0) \approx a^2/3$, and this gives, for a solution of $\theta(d, h)=1$, $h \approx \sqrt{12}/d$. It looks nice because: $h^{1/2} d \approx 12^{+1/4} d^{1/2}$ which is effectively small if d is small enough.

To give a rigorous proof let us find a solution $\tilde{h}(d)$ of:

$$(h/\kappa)\lambda((\kappa h)^{1/2} d/2, 0)=1$$

or equivalently

$$(hd^2/4\kappa)\lambda((\kappa h)^{1/2} d/2, 0)=d^2/4 \tag{7.14}$$

The l.h.s. is a realvalued analytic function $u \to \psi(u)$ (positive for $u \geqq 0$) of $u=\kappa hd^2/4$ ($a \to \lambda(a, 0)$ can be considered as an even function of a) and equivalent to $u^2/3\kappa^2$ at 0. We can always consider this function as the square of an analytic function:

$$\psi(u)=\chi(u)^2 \text{ with } \chi(u) \text{ equivalent to } u/\sqrt{3}\kappa \text{ as } u \to 0. \tag{7.15}$$

Equation (7.14) can be written

$$\chi(u)=d/2 \qquad (\text{with } u=\kappa hd^2/4). \tag{7.16}$$

χ is invertible in $B(0, d_0)$ for d_0 small enough and one get:

$$\kappa hd^2/4=\chi^{-1}(d/2) \tag{7.17}$$

and finally

$$\tilde{h}(d)=4\kappa^{-1} d^{-2} \chi^{-1}(d/2) (\approx \sqrt{12} d^{-1})$$

which gives a complete expansion for $\tilde{h}(d)$.

We now observe that:

$$\theta(d, \tilde{h}(d))=(\tilde{h}(d)/\kappa) \operatorname{Inf}_c \lambda((\kappa \tilde{h}(d))^{1/2} d/2, c)=(\tilde{h}(d)/\kappa)\lambda((\kappa \tilde{h}(d))^{1/2} d/2, 0)$$

The second equality follows from the observation that $\tilde{h}(d) d^2$ is small enough for d small enough and from (3.21). We then get the existence of d_0 s.t. for $d \leqq d_0$ we have $\tilde{h}(d)$ s.t.: $\theta(d, \tilde{h}(d))=1$.

By the unicity of $h^c(d)$ we get $h^c(d)=\tilde{h}(d)$ and Lemma 7.6.

Remark 7.7. – Once we have the asymptotic behavior of $h^c(d)$, it is then easy to deduce the statements of Theorem 0.7 from the corresponding statements of Theorem 3.4.

APPENDIX 1

Numerical computations in $[Bo]_2$ give us in other units, three branches of curves $d \to e(d)$, denoted by $\mathscr{E}^+$, $\mathscr{E}^-$ and $\mathscr{E}^0$ such that $(d, e(d))$ belongs to the $\tilde{s}$ defined in (0.5).

The relations between the units (d, e) and those of $[\mathrm{Bo}]_2$ (say (a, c)) are given in (7.1), (7.2).

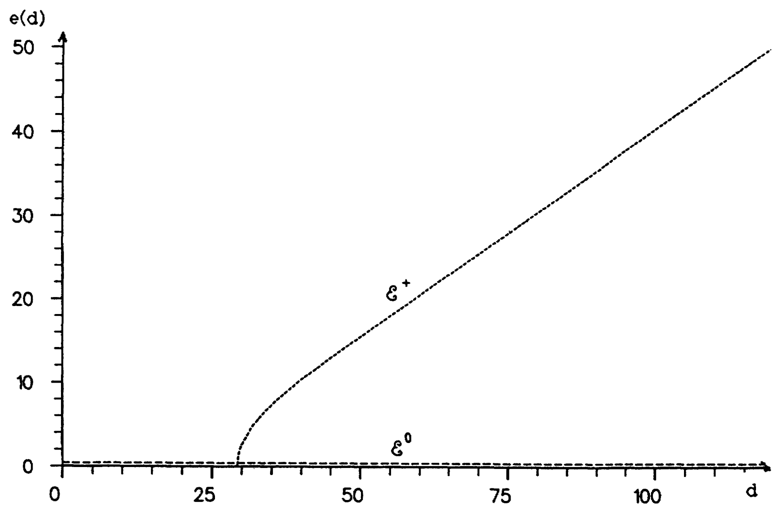

FIG. 1. – Numerically computed branches $\mathscr{E}^+$ and $\mathscr{E}^0$.

Figure 1 shows the numerically computed branches $\mathscr{E}^+$ and $\mathscr{E}^0$ when the parameter κ is equal to 0,062 (characteristic value for Indium). The branch $\mathscr{E}^-$ is obtained from $\mathscr{E}^+$ by symmetry with respect to the d-axis.

APPENDIX 2

A numerical study of the Neumann problem (4.1) gives us approximate values of α_0 and of the minimum $\mu_1^0 = \mu(\alpha_0)$. The following picture shows the numerically computed graph of $\alpha \rightsquigarrow \mu_1(\alpha)$. The eigenvalue $\mu_1(\alpha)$ is given by an inverse iterated power method applied to a discretization of problem (4.1).

The computations give us, in particular, the following approximate values:

$$\alpha_0 \approx 0{,}73, \quad \mu_1^0 = \mu_1(\alpha_0) \approx 0{,}59.$$

Coming back to our initial units, we obtain that the limit of $h(d, e(d))$ as d tends to ∞ is around $1{,}7\ \kappa$.

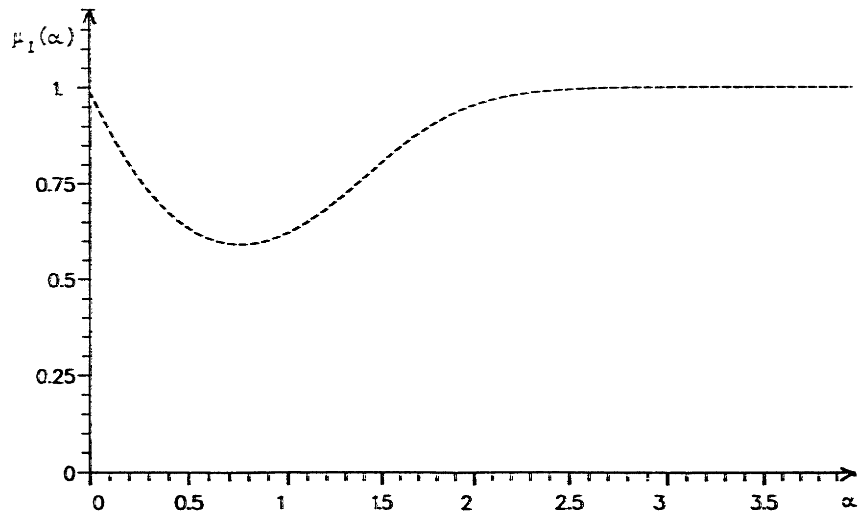

FIG. 2. – The first eigenvalue μ_1 (α) of (4.1).

APPENDIX 3

Construction of an approximation of $(\partial u/\partial c)$ as a tends to $+\infty$ and c lives close to 0

In this appendix we give more details on the proof of lemma 6.2.

We are looking for an approximation as a tends to $+\infty$ of the derivative $\partial u/\partial c$ with respect to c of the solution $u=u(y, a, c)$ of the problem:

$$\left.\begin{aligned} -u''+y^2u=\lambda u \quad \text{in }]-a+c, a+c[\\ u\in H^2(]-a+c, a+c[) \end{aligned}\right\} \qquad (A\,1)_1$$

$$u'(\pm a+c)=0 \qquad (A\,1)_2$$

$$u>0 \text{ in}]-a+c, a+c[\qquad (A\,1)_3$$

This was already made in $[Bo]_2$ in the case $c=0$, but we need now to consider the more general case where c satisfies:

$$0\leqq c\leqq B\,a^{-2} \text{ for some fixed B.} \qquad (A\,2)$$

We denote by b(A, B) the set of (a, c) satisfying (A 2) and $a\geqq A$.

As in section 6, we introduce the translated function v defined by

$$v(x, a, c)=u(x+c, a, c) \qquad \text{for} \quad x\in]-a, a[$$

v is then a solution of:

$$\left.\begin{array}{c} -v''+(x+c)^2 v=\lambda v \quad \text{in }]-a, a[, \\ v\in \mathrm{H}^2(]-a, a[) \end{array}\right\} \qquad (\mathrm{A}\,3)_1$$

$$v'(\pm a)=0 \qquad (\mathrm{A}\,3)_2$$

$$v>0 \quad \text{in }]-a, a[\qquad (\mathrm{A}\,3)_3$$

It is easier to differentiate $(\mathrm{A}\,3)_1$ than $(\mathrm{A}\,1)_1$ with respect to c because the domain of $(\mathrm{A}\,3)_1$ is independent of c.

Let us define

$$\partial_c v \equiv \partial v/\partial c$$

then $\partial_c v$ is the unique solution in $\mathrm{H}^2(]-a, a[)$ of:

$$-w''+(x+c)^2 w-\lambda w=-2(x+c)v+\partial_c\lambda . v \quad \text{in }]-a, a[\qquad (\mathrm{A}\,4)_1$$

$$w'(\pm a)=0 \qquad (\mathrm{A}\,4)_2$$

$$(w, v)_{\mathrm{L}^2(]-a, a[)}=0 \qquad (\mathrm{A}\,4)_3$$

We have:

$$(\partial_c u)(x+c, a, c)=(\partial_c v)(x, a, c)-(\partial_x v)(x, a, c)$$

so that

$$(\partial_c u)(\pm a+c, a, c)=(\partial_c v)(\pm a, a, c)$$

We can rewrite the problem (A 4) equivalently as the problem to find z in $\mathrm{H}^2(]-a, a[)$ solution of:

$$-z''+y^2 z-\lambda z=-2yu+(\delta_c\lambda).u \quad \text{in }]-a+c, a+c[\qquad (\mathrm{A}\,5)_1$$

$$z'(\pm a+c)=0 \qquad (\mathrm{A}\,5)_2$$

$$(z, u)_{\mathrm{L}^2(]-a+c, a+c[)}=0. \qquad (\mathrm{A}\,5)_3$$

Here z is related to w by:

$$z(y)=w(y-c),\ y\in]-a+c, a+c[.$$

Step 1. – Let us first construct an approximation of u denoted by u^a. The proof is the same as in $[\mathrm{Bo}]_2$, although the interval is not symmetric with respect to 0.

Let us consider the function f_1 defined in $\mathbb{R}$ by:

$$f_1(y)=\exp(-y^2/2), \qquad y\in\mathbb{R} \qquad (\mathrm{A}\,6)$$

f_1 is a solution of:

$$-f''+y^2 f-f=0 \quad \text{in } \mathbb{R} \qquad (\mathrm{A}\,7)$$

Furthermore (*see* [Si], Theorems 6.1 and 7.1), there exists a function denoted f_2 s.t. $\{f_1, f_2\}$ is a basis of the set of the solutions of equation (A 7) and s.t. the function f_2 satisfies:

$$\left.\begin{array}{c} f_2(y)=\exp(y^2/2)(1/y)(1+1/(2y^2)+O(1/y^4)) \\ \text{when } y\to+\infty \end{array}\right\} \qquad (\mathrm{A}\,8)$$

and

$$\left.\begin{array}{c} f'_2(y)=\exp(y^2/2)(1-1/(2y^2)+O(1/y^4)) \\ \text{when } y\to+\infty \end{array}\right\} \qquad (A\,9)$$

We are looking for an approximation u^a of u of the following form:

$$u^a(y, a, c)=\sigma(a, c).[f_1(y)+\delta(a, c) f_2(y)\Xi(y/\beta) \\ +\gamma(a, c) f_2(-y)\Xi(-y/\alpha)] \qquad (A\,10)_1$$

where $\sigma(a, c)$, $\delta(a, c)$ and $\gamma(a, c)$ are chosen s. t.

$$(\partial_c u^a)(\pm a+c, a, c)=0 \qquad (A\,10)_2$$

$$\|u^a\|_{L^2(]-a+c,\, a+c[)}=1 \qquad (A\,10)_3$$

and Ξ is a cutoff function s. t.:

$$\Xi(x)=0 \quad \text{if } x\leqq 1/2, \qquad \Xi(x)=1 \quad \text{if } x\geqq 3/4$$

For $B>0$, there exists $A>0$ s. t. for $(a, c)\in b(A, B)$, the condition:

$$(\partial_c u^a)(a+c)=0 \text{ is equivalent to:} \\ \delta(a, c)=-f'_1(a+c)/f'_2(a+c) \qquad (A\,11)$$

then, using (A 6) and (A 9):

$$\delta(a, c)=(a+c)\exp(-(a+c)^2).(1+1/(2(a+c)^2)+O(1/a^4)) \qquad (A\,12)$$

where O is uniform with respect to (a, c) satisfying $(a, c)\in b(A, B)$.

For $(a, c)\in b(a, c)$, the condition $(\partial_c u^a)(-a+c)=0$ is equivalent to:

$$\gamma(a, c)=f'_1(-a+c)/f'_2(a-c) \qquad (A\,13)$$

then:

$$\gamma(a, c)=(a-c)\exp(-(a-c)^2).(1+1/(2(a-c)^2)+O(1/a^4)) \qquad (A\,14)$$

Let us define $\alpha=a-c$ and $\beta=a+c$.

$\sigma(a, c)$ is determined by the normalization condition $(A\,10)_3$, say:

$$\sigma(a, c)^2.[\pi^{1/2}-\int_{a+c}^{\infty}\exp(-y^2)\,dy-\int_{-\infty}^{-a+c}\exp(-y^2)\,dy \\ +2\delta(a, c)\int_{\beta/2}^{\beta} f_1(y) f_2(y)\Xi(y/\beta)\,dy \\ +2\gamma(a, c)\int_{-\alpha}^{-\alpha/2} f_1(y) f_2(-y)\Xi(-y/\beta)\,dy \\ +2\delta(a, c)^2\int_{\beta/2}^{\beta} f_2(y)^2\,\Xi(y/\beta)^2\,dy \\ +2\gamma(a, c)\int_{-\alpha}^{-\alpha/2} f_2(-y)^2\,\Xi(-y/\beta)^2\,dy]=1 \qquad (A\,15)$$

We have, as in $[\mathrm{Bo}]_2$, for $(a, c) \in b(\mathrm{A}, \mathrm{B})$:

$$\sigma(a, c) = \pi^{-1/4} + O(a . \exp(-a^2)) \tag{A 16}$$

It results that for $\mathrm{B}>0$, there exists $\mathrm{A}>0$ and $\mathrm{C}>0$ s.t. uniformly for $(a, c) \in b(\mathrm{A}, \mathrm{B})$:

$$\| u(., a, c) - u^a \|_{\mathrm{C}^1([-\alpha, \beta])} \leq \mathrm{C} . \exp(-5 a^2/8) \tag{A 17}$$

and:

$$|\lambda(a, c) - 1| \leq \mathrm{C} . \mathrm{a} . \exp(-a^2) \tag{A 18}$$

We have in particular:

LEMMA A 1. — *For* $(a, c) \in b(\mathrm{A}, \mathrm{B})$:

$$u(\beta, a, c) = 2\sigma(a, c) \exp(-\beta^2/2) . [1 + 1/(2\beta^2) + O(1/a^4)] \tag{A 19}$$
$$u(-\alpha, a, c) = 2\sigma(a, c) \exp(-\alpha^2/2) . [1 + 1/(2\alpha^2) + O(1/a^4)] \tag{A 20}$$

Let us first remark that:

$$|\partial_c \lambda(a, c)| \leq \mathrm{C} . \exp(-a^2) \quad \text{for} \quad (a, c) \in b(\mathrm{A}, \mathrm{B}) \tag{A 21}$$

Indeed (*see* [Da-He], and (6.5), (6.13)):

$$\partial_c \lambda(a, c) = \Phi(c) - \Phi(-c)$$

with:

$$\Phi(c) = (\alpha^2 - \lambda(a, c)) u^2(\alpha, a, c)$$

Therefore, (A 21) results from (A 19) and (A 20).

Step 2. — We construct an approximation of $\partial_c v$ by differentiation of u^a with respect to c.

As for the derivative of u, we consider the translated problem in the variable x. An approximation of v, solution of (A 3) is given by:

$$v^a(x, a, c) = u^a(x + c, a, c) \quad \text{for} \quad x \in [-a, a]$$

The approximation v^a satisfies the following equation:

$$-v^{a''} + (x+c)^2 v^a - \lambda v^a = r \quad \text{in }]-a, a[\tag{A 22$_1$}$$
$$v^{a'}(\pm a) = 0 \tag{A 22$_2$}$$
$$\| v^a \|_{\mathrm{L}^2(]-a, a[)} = 1 \tag{A 22$_3$}$$

with

$$\begin{aligned} r(x, a, c) = {} & (\sigma(a, c) . \delta(a, c)/\beta) [2 f_2'(x+c) \Xi'((x+c)/\beta) \\ & + (1/\beta) f_2(x+c) \Xi''((x+c)/\beta)] \\ & + (\sigma(a, c) . \gamma(a, c)/\alpha) [2 f_2'(-x-c) \Xi'(-(x+c)/\alpha) \\ & + (1/\alpha) f_2(-x-c) \Xi''(-(x+c)/\alpha)] \\ & - (\lambda - 1) u^a \end{aligned} \tag{A 23}$$

and:

$$\|r\|_{L^2(]-a,a[)} \leqq C.\exp(-(11a^2)/16) \quad \text{(A 24)}$$

The approximation v^a of v verifies the same estimations as u^a in the corresponding spaces.

Let $\partial_c v^a$ be the derivative of v^a with respect to c, then:

$$\begin{aligned}
\partial_c v^a(x) = \; & \sigma(a, c)(-(x+c).\exp(-(x+c)^2/2) \\
& + \partial_c \sigma(a, c).\exp(-(x+c)^2/2) \\
& + \partial_c \sigma(a, c).\delta(a, c).f_2(x+c)\,\Xi((x+c)/\beta) \\
& + \sigma(a, c).\partial_c \delta(a, c).f_2(x+c)\,\Xi((x+c)/\beta) \\
& + \sigma(a, c)\delta(a, c).f'_2(x+c)\,\Xi((x+c)/\beta) \\
& + (\sigma(a, c).\delta(a, c)/\beta).f_2(x+c).\Xi'((x+c)/\beta) \\
& + \partial_c \sigma(a, c).\gamma(a, c).f_2(-x-c).\Xi(-(x+c)/\alpha) \\
& + \sigma(a, c).\partial_c \gamma(a, c).f_2(-x-c).\Xi(-(x+c)/\alpha) \\
& - \sigma(a, c)\gamma(a, c).f'_2(-x-c)\,\Xi(-(x+c)/\alpha) \\
& - (\sigma(a, c).\gamma(a, c)/\alpha).f_2(-x-c)\,\Xi'(-(x+c)/\alpha) \qquad \text{(A 25)}
\end{aligned}$$

where $\partial_c \sigma(a, c)$, $\partial_c \delta(a, c)$, $\partial_c \gamma(a, c)$ are obtained by differentiation of (A 15), (A 11), (A 13) with respect to c; the derivative $f''_2(x+c)$ needed for this calculation is given by:

$$f''_2(x+c) = ((x+c)^2-1).f_2(x+c)$$

and we obtain easily the corresponding expansions:

We have in particular for $(a, c) \in b(A, B)$:

$$\partial_c \delta(a, c) = -2\beta^2.\exp(-\beta^2).(1+O(1/a^4)) \quad \text{(A 26)}$$
$$\partial_c \gamma(a, c) = 2\alpha^2.\exp(-\alpha^2).(1+O(1/a^4)) \quad \text{(A 27)}$$
$$\partial_c \sigma(a, c) = O(a.\exp(-a^2)) \quad \text{(A 28)}$$

In (A 25), we have also used the fact that the functions f_1 and f_2 are independent of c.

The derivative $\partial_c v^a$ satisfies the following equations, obtained by differentiation of (A 22) with respect to c:

$$-z'' + y^2 z - \lambda z = (\partial_c r) - 2(x+c)v^a + (\partial_c \lambda).v^a \quad \text{in }]-a, a[\quad \text{(A 29)}_1$$
$$z'(\pm a) = 0 \quad \text{(A 29)}_2$$

and

$$(z, v^a)_{L^2(]-a,a[)} = 0 \quad \text{(A 29)}_3$$

But the approximation w^a of $(\partial_c v)$ has to satisfy the orthogonalization condition:

$$(w^a, v)_{L^2(]-a,a[)} = 0 \quad \text{(A 30)}$$

Thus we look for w^a under the form:

$$w^a = (\partial_c v^a) + \tau(a, c)\, v$$

where $\tau(a, c)$ is chosen s. t. (A 30) is satisfied.

This condition is equivalent to:

$$\tau(a, c) = -(\partial_c v^a, v)_{L^2(]-a, a[)}/(v, v)_{L^2(]-a, a[)} \qquad \text{(A 31)}$$

We have:

$$\begin{aligned}(\partial_c v^a, v)_{L^2(]-a, a[)} &= (\partial_c v^a, v^a)_{L^2(]-a, a[)} + (\partial_c v^a, v - v^a)_{L^2(]-a, a[)} \\ &= (\partial_c v^a, v - v^a)_{L^2(]-a, a[)} \quad \text{from (A 29)}_3\end{aligned}$$

then:

$$|(\partial_c v^a, v)_{L^2(]-a, a[)}| \leqq \| \partial_c v^a \|_{L^2(]-a, a[)} \cdot \| v - v^a \|_{L^2(]-a, a[)}$$

but, as a tends to $+\infty$:

$$\| \sigma(a, c)\, f'_1 \|_{L^2(]-a, a[)} \leqq \sigma(a, c) . (a + c) \pi^{1/4} \leqq a(1 + O(\alpha \exp(-a^2))$$

and, for all $\varepsilon > 0$, the others terms of $\partial_c v^a$ are of order $O(\exp(-(1-\varepsilon) a^2/2))$ then, using (A 17) with the variable x:

$$|(\partial_c v^a, v)_{L^2(]-a, a[)}| \leqq C . a^2 \exp(-5 a^2/8)$$

we get:

$$|\tau(a, c)| \leqq C . a^2 \exp(-5 a^2/8) \qquad \text{(A.32)}$$

We have now to verify that w^a gives us a "good" approximation of $\partial_c v$. w^a satisfies the same equations $\text{(A 29)}_{1,2}$ (as $(\partial_c v^a)$) and (A 30). Thus $(w^a - \partial_c v)$ is a solution of:

$$\begin{aligned} -z'' + y^2 z - \lambda z = \partial_c r - 2(x + c)(v^a - v) & \\ + (\partial_c \lambda) . (v^a - v) \quad \text{in }]-a, a[& \qquad \text{(A 33)}_1 \\ z'(\pm a) = 0 & \qquad \text{(A 33)}_2 \\ (z, v)_{L^2(]-a, a[)} = 0 & \qquad \text{(A 33)}_3 \end{aligned}$$

We now have to prove that the r. h. s. of (A 33)_1 is sufficiently small.

$(\partial_c \lambda)$ satisfies (A 21) and v is normed in $L^2(]-a, a[)$. So, we only have to give estimations about $\partial_c r$; using preceding estimates (A 6), (A 7), (A 12), (A 14), (A 16), (A 26)-(A 28) about the functions f_1 and f_2, the coefficients $\delta(a, c)$, $\gamma(a, c)$, $\sigma(a, c)$ and their derivatives, we obtain that for $(a, c) \in b(A, B)$:

$$\| \partial_c r \|_{L^2(]-\alpha, \beta[)} \leqq C . \exp(-9 a^2/16)$$

It results that:

LEMMA A 2. – *$\forall B > 0$, $\exists A > 0$ and $\exists C$ s. t. uniformly for* $(a, c) \in b(A, B)$:

$$\| w^a - (\partial_c v)(. + c) \|_{C^1([-\alpha, \beta])} \leqq C . \exp(-9 a^2/16) \qquad \text{(A 33)}$$

Step 3. – In Lemma 6.2, we need an approximation of $(\partial_c u)(\beta, a, c)$.

We know that $(\partial_c u)(\beta, a, c) = (\partial_c v)(a, a, c)$ is approximated by $w^a(\beta, a, c)$.

Then, we have for $(a, c) \in b(A, B)$:

$$w^a(\beta, a, c) = -2\pi^{-1/4}\beta . \exp(-\beta^2/2).(1 + O(1/a)) \qquad (A\,34)$$

and

$$(\partial_c u)(\beta, a, c) = w^a(\beta, a, c) + O(\exp(-9a^2/16))$$

Therefore:

COROLLARY A 3. — $\forall B > 0$, $\exists A > 0$ s. t. for $(a, c) \in b(A, B)$:

$$(\partial_c u)(a + c, a, c) = -2\pi^{-1/4}(a + c).\exp(-(a + c)^2/2).(1 + O(1/a)) \qquad (A\,35)$$

$$(\partial_c u)(a + c, a, c)/u(a + c, a, c) = -(a + c)(1 + O(1/a)) \qquad (A\,36)$$

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