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Smooth static solutions of the spherically symmetric Vlasov-Einstein system

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ABSTRACT. – We consider the Vlasov-Einstein system in a spherically symmetric setting and prove the existence of globally defined, smooth, static solutions with isotropic pressure which are asymptotically flat and have a regular center, finite total mass and finite extension of the matter.

RÉSUMÉ. – Nous considérons le système de Vlasov-Einstein dans le cas d'une symétrie sphérique et démontrons l'existence de solutions globalement définies, lisses et statiques à pression isotrope qui sont asymptotiquement plates et possèdent un centre régulier, une masse totale finie et une extension de la matière finie.

0. INTRODUCTION

It is well known that the only static, spherically symmetric vacuum solutions of Einstein's field equations are the Schwarzschild solutions which possess a spacetime singularity (or are identically flat). In the present note we couple Einstein's equations with the Vlasov or Liouville equation for a static, spherically symmetric distribution function f of identical particles (stars in a galaxy, galaxies in a galaxy cluster etc.) on phase space. This results in the following system of equations:

$$\frac{v}{\sqrt{1+v^2}} \cdot \partial_x f - \sqrt{1+v^2}\, \mu' \frac{x}{r} \cdot \partial_v f = 0, x, v \in \mathbb{R}^3, r := |x|,$$

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2 \rho,$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 p,$$

where

$$\rho(x) = \rho(r) := \int_{\mathbb{R}^3} f(x, v) \sqrt{1+v^2}\, dv,$$

$$p(x) = p(r) := \int_{\mathbb{R}^3} f(x, v) \left(\frac{x \cdot v}{r}\right)^2 \frac{dv}{\sqrt{1+v^2}}.$$

Here the prime denotes derivative with respect to r, and spherical symmetry of f means that $f(\mathrm{A}x, \mathrm{A}v) = f(x, v)$ for every orthogonal matrix A and $x, v \in \mathbb{R}^3$. If we let $x = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ then the spacetime metric is given by

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2).$$

Asymptotic flatness and a regular center at $r=0$ are guaranteed by the following boundary conditions:

$$\lim_{r\to\infty} \lambda(r) = \lim_{r\to\infty} \mu(r) = \lambda(0) = 0.$$

We refer to [6] on the particular choice of coordinates on phase space leading to the above formulation of the system. It should be pointed out that the above equation for f is only equivalent to the Vlasov equation if f is spherically symmetric.

A brief overview of the literature related to the present investigation seems in order. In [6] the authors investigate the initial value problem for the corresponding time dependent system. In [7] it is shown that the classical Vlasov-Poisson system is the Newtonian limit of the Vlasov-Einstein system. The existence of static, spherically symmetric solutions of the Vlasov-Poisson system is established in [3]. Taking the distribution function f as a function of the local energy and the angular momentum, which are conserved along characteristics, the problem is reduced to solving a nonlinear Poisson problem. For certain cases corresponding to the so-called polytropes it can be shown that this leads to solutions with finite radius and finite total mass, *cf.* also [4]. Cylindrically symmetric, static solutions of the Vlasov-Poisson system are investigaged in [2]. In [10] it is shown that coupling Einstein's equations with a Yang-Mills field can lead to static, singularity-free solutions with finite total mass. Static solutions of general-relativistic elasticity have been studied in [5]. A system in some sense intermediate between the Newtonian and the general-relativistic setting is the relativistic Vlasov-Poisson system, where similar results as for the nonrelativistic version hold, *cf.* [1]. For further references, especially on the astrophysical literature, we refer to [9].

The present investigation proceeds as follows: In the next section we give two conserved quantities, which correspond to the local energy and angular momentum in the Newtonian limit, and reduce the existence problem for static solutions of the Vlasov-Einstein system to solving a nonlinear integrodifferential equation for μ. The fact that the distribution function f indeed has to be a function of the local energy and angular momentum (known as Jeans' Theorem) is rigorously established for the Vlasov-Poisson system in [3] but not known in the present situation. In other words, it is not clear whether our ansatz for f covers all possible spherically symmetric, static solutions. The existence of solutions to the remaining equation for μ is investigated in the third section under the additional assumption that f depends on the local energy only. Here we profit from the investigation in [8] when showing that μ exists globally in r. In the last section we consider a particular ansatz for f corresponding to the polytropes in the classical case and show that this leads to solutions with finite mass and finite radius. This is done by treating the Vlasov-Einstein system as a perturbation of the Vlasov-Poisson system and using the criteria for a finite radius from the classical Lane-Emden-Fowler equation.

1. CONSERVED QUANTITIES AND REDUCTION OF THE PROBLEM

As explained above, a key point in our investigation is to reduce the full system to a nonlinear integrodifferential equation for μ. This is achieved by making the ansatz that the distribution function depends only on certain integrals of the characteristic system. In the coordinates used above, the characteristic system corresponding to the above equation for f reads

$$\dot{x}=\frac{v}{\sqrt{1+v^2}},$$
$$\dot{v}=-\sqrt{1+v^2}\,\mu'(r)\frac{x}{r};$$

note that these equations are not the geodesic equations. One immediately checks that the quantities

$$\mathrm{E} := e^{\mu(r)} \sqrt{1+v^2}, \qquad \mathrm{F} := x^2 v^2 - (x.v)^2 = |x \times v|^2$$

are conserved; in the last section the relation of E to the classical local energy will become apparent, F can be interpreted as the modulus of the angular momentum. If we take f to be of the form

$$f(x, v) = \phi(\mathrm{E}, \mathrm{F})$$

for some function ϕ, the Vlasov equation is automatically satisfied. Inserting this into the definitions of ρ and p we obtain, after a transformation of variables,

$$\rho(r) = \frac{2\pi}{r^2} \int_1^\infty \int_0^{r^2(\varepsilon^2-1)} \phi(e^{\mu(r)}\varepsilon, \mathrm{F}) \frac{\varepsilon^2}{\sqrt{\varepsilon^2 - 1 - \mathrm{F}/r^2}} d\mathrm{F}\, d\varepsilon,$$

$$p(r) = \frac{2\pi}{r^2} \int_1^\infty \int_0^{r^2(\varepsilon^2-1)} \phi(e^{\mu(r)}\varepsilon, \mathrm{F}) \sqrt{\varepsilon^2 - 1 - \mathrm{F}/r^2}\, d\mathrm{F}\, d\varepsilon.$$

Inserting these into the field equations would result in a nonlinear system for λ and μ. However, we are only able to treat this system under the assumption that

$$f(x, v) = \phi(\mathrm{E}),$$

which implies that the pressure is isotropic. In this case the integration with respect of F can be carried out explicitly to obtain the following relations:

$$\rho(r) = g_\phi(\mu(r)), \qquad p(r) = h_\phi(\mu(r)),$$

where

$$g_\phi(u) := 4\pi \int_1^\infty \phi(\varepsilon e^u)\, \varepsilon^2 (\varepsilon^2 - 1)^{1/2}\, d\varepsilon, \tag{1.1}$$

$$h_\phi(u) := \frac{4\pi}{3} \int_1^\infty \phi(\varepsilon e^u) (\varepsilon^2 - 1)^{3/2}\, d\varepsilon \tag{1.2}$$

for $u \in \mathbb{R}$; in the next section we shall give a class of functions ϕ which lead to well behaved functions g_ϕ and h_ϕ. The Vlasov-Einstein system is now reduced to the field equations in the form

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2 g_\phi(\mu), \tag{1.3}$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 h_\phi(\mu). \tag{1.4}$$

Using the boundary condition at $r = 0$ the first equation can be integrated to give

$$e^{-2\lambda(r)} = 1 - \frac{8\pi}{r} \int_0^r s^2 g_\phi(\mu(s))\, ds. \tag{1.5}$$

If we insert this into the second field equation we obtain the desired equation for μ:

$$\mu'(r)=\frac{4\pi}{1-\frac{8\pi}{r}\int_0^r s^2 g_\phi(\mu(s))\,ds}\left(rh_\phi(\mu(r))+\frac{1}{r^2}\int_0^r s^2 g_\phi(\mu(s))\,ds\right). \quad (1.6)$$

Thus the problem of finding static solutions of the Vlasov-Einstein system is reduced to solving the above equation globally in $r \geqq 0$. This will be done in the next section. Once a solution μ is obtained, one has to check whether it leads to a static solution of Vlasov-Einstein with finite total mass.

2. EXISTENCE OF SOLUTIONS

As a first step we show that the functions g_ϕ and h_ϕ are well defined and well behaved for a sufficiently large class of functions ϕ. More precisely:

LEMMA 2.1. – *Let* $\phi:]0, \infty[\to [0, \infty[$ *be measurable with*

$$\phi(E) \leqq C(1+E)^{-\alpha}, \qquad E>0,$$

for some constants $C>0$ *and* $\alpha>4$. *Then Eqns.* (1.1) *and* (1.2) *define decreasing functions* g_ϕ, $h_\phi \in C^1(\mathbb{R})$ *and*

$$h'_\phi(u) = -(h_\phi(u)+g_\phi(u)), \qquad u \in \mathbb{R}.$$

Proof. – The integrals defining g_ϕ and h_ϕ exist by the decay property of ϕ. A transformation of variables shows that

$$g_\phi(u)=4\pi e^{-4u}\tilde{g}(e^u), \qquad h_\phi(u)=\frac{4\pi}{3}e^{-4u}\tilde{h}(e^u),$$

where

$$\tilde{g}(t)=\int_t^\infty \phi(E)\,E^2(E^2-t^2)^{1/2}\,dE,$$

$$\tilde{h}(t)=\int_t^\infty \phi(E)(E^2-t^2)^{3/2}\,dE, \; t>0.$$

Lebesgue's dominated convergence theorem implies that the functions $\tilde{g}$ and $\tilde{h}$ and thus also g_ϕ and h_ϕ are continuous. For $t>0$ and $\Delta t>0$ such that $t-\Delta t>0$ we have

$$\frac{1}{\Delta t}(\tilde{h}(t-\Delta t)-\tilde{h}(t))=\frac{1}{\Delta t}\int_{t-\Delta t}^{t}\phi(\mathrm{E})(\mathrm{E}^2-(t-\Delta t)^2)^{3/2}\,d\mathrm{E}$$
$$+\int_t^\infty \phi(\mathrm{E})\frac{1}{\Delta t}((\mathrm{E}^2-(t-\Delta t)^2)^{3/2}-(\mathrm{E}^2-t^2)^{3/2})\,d\mathrm{E}$$
$$=:\mathrm{I}_1+\mathrm{I}_2.$$

Obviously, $\mathrm{I}_1\to 0$ as $\Delta t\to 0$. The term I_2 has a limit as $\Delta t\to 0$ by Lebesgue's theorem. Thus, the function $\tilde{h}$ is left-differentiable with

$$\frac{d^-}{dt}\tilde{h}(t)=-3\,t\int_t^\infty \phi(\mathrm{E})(\mathrm{E}^2-t^2)^{1/2}\,d\mathrm{E}.$$

Again by Lebesgue's theorem this function is continuous, and a continuous and continuously left-differentiable function is continuously differentiable. Similarly, we see that

$$\frac{d^-}{dt}\tilde{g}(t)=-t\int_t^\infty \phi(\mathrm{E})\,\mathrm{E}^2(\mathrm{E}^2-t^2)^{-1/2}\,d\mathrm{E},$$

which is again continuous. Thus, g_ϕ and h_ϕ are C^1 and obviously decreasing, and a short calculation proves the relation for h'_ϕ. □

The regularity of the functions g_ϕ and h_ϕ being established, we can now prove the global existence of λ and μ.

THEOREM 2.2. — *Let* $\phi:]0,\infty[\to[0,\infty[$ *be measurable with*

$$\phi(\mathrm{E})\leqq \mathrm{C}(1+\mathrm{E})^{-\alpha},\qquad \mathrm{E}>0,$$

for some constants $\mathrm{C}>0$ *and* $\alpha>4$ *and let* g_ϕ, h_ϕ *be defined by Eqns.* (1.1) *and* (1.2). *Then for every* $\mu_0\in\mathbb{R}$ *there exists a unique solution* λ, $\mu\in\mathrm{C}^2([0,\infty[)$ *of the Eqns.* (1.3) *and* (1.4) *with*

$$\lambda(0)=0,\qquad \mu(0)=\mu_0.$$

If we define

$$\rho(r):=g_\phi(\mu(r)),\qquad p(r):=h_\phi(\mu(r)),\qquad m(r):=4\pi\int_0^r s^2\rho(s)\,ds,\qquad r\geqq 0,$$

then ρ, $p\in\mathrm{C}^1([0,\infty[)$ *are decreasing with*

$$\lim_{r\to\infty}\rho(r)=\lim_{r\to\infty}p(r)=0,$$

and

$$0\leqq\frac{m(r)}{r}<\frac{1}{2},\ r>0.$$

Proof. — We start by proving local existence in a neighbourhood of $r=0$. This is not a completely trivial matter due to the singularity of the

Eqns. (1.3) and (1.4) at $r=0$. Integrating Eqn. (1.6) subject to the initial condition $\mu(0)=\mu_0$ we obtain the following fixed point problem for μ:

$$\mu(r)=(\mathrm{T}\mu)(r), r\geqq 0$$

where

$$(\mathrm{T}\mu)(r):=\mu_0+\int_0^r \frac{4\pi}{1-\dfrac{8\pi}{s}\displaystyle\int_0^s \sigma^2 g_\phi(\mu(\sigma))\,d\sigma} \times\left(sh_\phi(\mu(s))+\frac{1}{s^2}\int_0^s \sigma^2 g_\phi(\mu(\sigma))\,d\sigma\right)ds.$$

A lengthy but straight forward argument shows that T acts as a contraction with respect to the norm $\|\,.\,\|_\infty$ on the set

$$\mathrm{M}:=\left\{\mu:[0,\delta]\to\mathbb{R}\,|\,\mu(0)=\mu_0,\ \mu_0\leqq\mu(r)\leqq\mu_0+1,\ \frac{8\pi}{r}\int_0^r s^2 g_\phi(\mu(s))\,ds\leqq\frac{1}{2},\ r\in[0,\delta]\right\}$$

if $\delta>0$ is chosen small enough. This gives the existence of a solution of Eqn. (1.6) on the interval $[0, \delta]$, and on this interval we define λ by Eqn. (1.5) and obtain a local solution $\lambda, \mu\in C^1([0, \delta])$ of (1.3), (1.4). Once we are away from the singularity $r=0$, we can apply standard existence and uniqueness results to the system (1.3), (1.4) to extend λ, μ to a maximal solution on an interval $[0, R[$. Obviously, the boundary conditions at $r=0$ are satisfied. The regularity of the functions g_ϕ and h_ϕ implies that $\lambda, \mu\in C^2(]0, R[)$, and it can be shown that the second derivatives continuously extend to $r=0$ and $\lambda'(0)=\mu'(0)=0$. Finally, if $R<\infty$ then

$$\limsup_{r\to R}\frac{8\pi}{r}\int_0^r s^2 g_\phi(\mu(s))\,ds=1$$

or

$$\limsup_{r\to R}\mu(r)=\infty.$$

To prove that $R=\infty$ we therefore have to exclude these two possibilities. Defining $\rho(r):=g_\phi(\mu(r))$ and $p(r):=h_\phi(\mu(r))$ for $r\in[0, R[$ and using Lemma 2.1 we obtain the following system of equations on $[0, R[$:

$$e^{-2\lambda}(2r\lambda'-1)+1=8\pi r^2\rho(r), \tag{2.1}$$

$$e^{-2\lambda}(2r\mu'+1)-1=8\pi r^2 p(r), \tag{2.2}$$

$$p'(r)=-\mu'(r)(p(r)+\rho(r)). \tag{2.3}$$

Note that since ρ and p are strictly decreasing on their support, we could write ρ as a function of p so that we are in the situation of [8], *i.e.*

the matter model is that of a fluid body with a not explicitly known equation of state. Thus, the analysis is now very similar to [8], and we only indicate the main steps for the convenience of the reader. If we integrate Eqn. (2.1) and introduce the functions

$$m(r):=4\pi\int_0^r s^2\rho(s)\,ds, \qquad w(r):=2\frac{m(r)}{r^3},$$
$$y(r):=e^{-\lambda(r)}, \qquad z(r):=e^{\mu(r)}$$

we obtain the equations

$$p(r)=\frac{2}{r}y^2(r)\frac{z'(r)}{z(r)}-w(r), \tag{2.4}$$

$$p'(r)=-\frac{z'(r)}{z(r)}(p(r)+3w(r)+rw'(r)); \tag{2.5}$$

note that $w\in C([0,R[)\cap C^1(]0,R[)$. Now we use $x:=r^2$ as radial variable and by abuse of notation use the same names for p, ρ etc. as functions of x. Then Eqs. (2.4) and (2.5) read

$$p(x)=4y^2(x)\frac{z'(x)}{z(x)}-w(x), \tag{2.6}$$

$$p'(x)=-\frac{z'(x)}{z(x)}(p(x)+3w(x)+2xw'(x)). \tag{2.7}$$

From the Eqns. (2.6) and (2.7) p can be eliminated which yields

$$(y(x)z'(x))'-\frac{1}{4}\frac{z(x)}{y(x)}w'(x)=0, x\in]0,R^2[. \tag{2.8}$$

Since g_ϕ is decreasing and μ is increasing, the function ρ is decreasing, and thus

$$w'(r)=-\frac{6}{r^4}\int_0^r s^2\rho(s)\,ds+\frac{2}{r}\rho(r)\leqq-\frac{6}{r^4}\rho(r)\int_0^r s^2\,ds+\frac{2}{r}\rho(r)=0.$$

This implies that w is decreasing as a function of r as well as x, and thus Eqn. (2.8) yields

$$(y(x)z'(x))'\leqq 0. \tag{2.9}$$

Together with Eqn. (2.4) this implies the estimate

$$\frac{z(r)}{y(r)}(p(r)+w(r))\leqq\frac{z(0)}{y(0)}(p(0)+w(0))$$
$$=z(0)\left(p(0)+\frac{2}{3}\rho(0)\right), \qquad r\in[0,R[,$$

and thus, by the monotonicity of z,

$$y(r) \geqq \frac{p(r)+w(r)}{p(0)+w(0)}, \qquad r \in [0, R[.$$

Now assume that $R < \infty$. Then

$$w(r) \geqq \lim_{s \to R} w(s) = 2\frac{m(R)}{R^3},$$

where the latter constant is positive or our solution is trivial and therefore global. Without loss of generality we obtain the existence of a constant $C > 0$ such that

$$y^2(r) = e^{-2\lambda(r)} = 1 - 2\frac{m(r)}{r} > C^2, \qquad r \in [0, R[. \tag{2.10}$$

Together with the monotonicity of p and ρ this in turn yields the estimate

$$\mu'(r) \leqq C^2 (4\pi r p(0) + 4\pi r \rho(0)) \leqq C' r.$$

Therefore, μ is bounded on the interval $[0, R[$ as well, and together with Eqn. (2.10) this contradicts the maximality of R. Thus, we have shown that $R = \infty$.

It remains to prove the asymptotic behaviour of p and ρ at infinity. By the monotonicity, the limits exist in $\mathbb{R}$. Assume that $\rho(\infty) > 0$ and thus $\rho(r) \geqq \rho(\infty) > 0$ for $r \geqq 0$. Then

$$\frac{m(r)}{r} \geqq 4\pi\rho(\infty)\frac{r^2}{3} > 1$$

for sufficiently large r, a contradiction to the above. Furthermore,

$$p(r) = \frac{4\pi}{3}\int_1^\infty \phi(\varepsilon\mu)(\varepsilon^2-1)^{3/2}\, d\varepsilon \leqq 4\pi \int_1^\infty \phi(\varepsilon\mu)\,\varepsilon^2(\varepsilon^2-1)^{1/2}\, d\varepsilon = \rho(r),$$

and the proof is complete.

Remark. – $(f := \phi(E), \lambda, \mu)$ is a static solution of the Vlasov-Einstein system in the sense that f is constant along characteristics and leads to functions ρ, $p \in C^1([0, \infty[)$, and the field equations hold classically. Moreover, since $\lambda'(0) = \mu'(0) = 0$ we have by abuse of notation λ, $\mu \in C^2(\mathbb{R}^3)$ and ρ, $p \in C^1(\mathbb{R}^3)$. If ϕ is continuous or continuously differentiable then in addition f has the same regularity, and in the latter case satisfies Vlasov's equation classically.

3. THE PROBLEM OF FINITE MASS AND FINITE RADIUS

In order to decide whether a solution obtained in the previous section has finite total mass or whether $\rho(r)$ vanishes for r large, one has to have rather detailed information on the behaviour of the function μ. Due to the complexity of Eqn. (1.6) we have not been able to decide these questions directly, even for simple examples of ϕ. However, it is possible to show that the solutions obtained above converge to solutions of the corresponding Newtonian problem as the speed of light tends to infinity. It is then possible to use the results on finite mass and finite radius which are known in the Newtonian case for the so-called polytropes to obtain solutions with the same properties for the Vlasov-Einstein system.

To carry out this program we introduce the parameter $\gamma := \frac{1}{c^2}$ where c denotes the speed of light, define $\nu := \frac{1}{\gamma}\mu$, and recall from [7] that the Vlasov-Einstein system with γ inserted in the appropriate places reads

$$\frac{v}{\sqrt{1+\gamma v^2}} \cdot \partial_x f - \sqrt{1+\gamma v^2}\, \nu' \frac{x}{r} \cdot \partial_v f = 0,$$
$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi\gamma r^2 \rho,$$
$$e^{-2\lambda}(2r\nu' + 1/\gamma) - 1/\gamma = 8\pi\gamma r^2 p,$$

where

$$\rho(x) := \int_{\mathbb{R}^3} f(x, v) \sqrt{1+\gamma v^2}\, dv,$$
$$p(x) := \int_{\mathbb{R}^3} f(x, v) \left(\frac{x \cdot v}{r}\right)^2 \frac{dv}{\sqrt{1+\gamma v^2}}.$$

The conserved quantity E now becomes

$$\sqrt{1+\gamma v^2}\, e^{\gamma\nu(x)},$$

but in order to obtain the correct limit as $\gamma \to 0$, we rewrite our ansatz for the distribution function in the following form:

$$f(x, v) = \phi\left(\frac{1}{\gamma}\sqrt{1+\gamma v^2}\, e^{\gamma\nu(r)} - \frac{1}{\gamma}\right). \tag{3.1}$$

As in Section 1 the Vlasov-Einstein system then reduces to a single equation for ν, namely

$$\nu'(r)=\frac{4\pi}{1-\frac{8\pi}{r}\gamma\int_0^r s^2 g_{\phi,\gamma}(\nu(s))\,ds}\times\left(\gamma r h_{\phi,\gamma}(\nu(r))+\frac{1}{r^2}\int_0^r s^2 g_{\phi,\gamma}(\nu(s))\,ds\right), \quad (3.2)$$

where

$$g_{\phi,\gamma}(u):=4\pi e^{-4\gamma u}\int_{(e^{\gamma u}-1)/\gamma}^{\infty}\phi(\mathrm{E})(1+\gamma\mathrm{E})^2\left(\frac{1}{\gamma}(1+\gamma\mathrm{E})^2-\frac{1}{\gamma}e^{2\gamma u}\right)^{1/2}d\mathrm{E}, \quad (3.3)$$

$$h_{\phi,\gamma}(u):=\frac{4\pi}{3}e^{-4\gamma u}\int_{(e^{\gamma u}-1)/\gamma}^{\infty}\phi(\mathrm{E})\left(\frac{1}{\gamma}(1+\gamma\mathrm{E})^2-\frac{1}{\gamma}e^{2\gamma u}\right)^{3/2}d\mathrm{E}, \quad (3.4)$$

so that

$$\rho(r)=g_{\phi,\gamma}(\nu(r)), \qquad p(r)=h_{\phi,\gamma}(\nu(r)). \quad (3.5)$$

For the Newtonian case the corresponding ansatz

$$f(x,v)=\phi\left(\frac{v^2}{2}+\mathrm{U}(r)\right)$$

reduces the Vlasov-Poisson system to the equation

$$\mathrm{U}'(r)=\frac{4\pi}{r^2}\int_0^r s^2 g_0(\mathrm{U}(s))\,ds, \quad (3.6)$$

where

$$g_0(u):=4\pi\int_u^{\infty}\phi(\mathrm{E})\sqrt{2(\mathrm{E}-u)}\,d\mathrm{E}.$$

Let us define

$$\phi(\mathrm{E}):=\begin{cases}(-\mathrm{E})^k, & \mathrm{E}\leqq 0\\ 0, & \mathrm{E}>0\end{cases}, \qquad k\in[0, 7/2[\quad (3.7)$$

and fix $\nu_0<0$. Clearly, the results of the previous section apply so that for every $\gamma>0$ Enq. (3.2) has a unique, nontrivial, global solution with $\nu(0)=\nu_0$. On the other hand, Eqn. (3.6) has a unique, nontrivial, global solution with $\mathrm{U}(0)=\nu_0$. Furthermore, this solution has the property that $\mathrm{U}(\mathrm{R})=0$ for some $\mathrm{R}>0$ which means that the corresponding density $g_0(\mathrm{U}(r))$ vanishes for $r\geqq\mathrm{R}$ and the total mass $4\pi\int_0^{\infty} r^2 g_0(\mathrm{U}(r))\,dr$ is finite, *cf.* [3]. We shall now prove that ν converges to U as $\gamma\to 0$, more precisely:

LEMMA 3.1. — *For every* $R>0$ *there exist constants* $C>0$ *and* $\gamma_0>0$ *such that for every* $\gamma\in]0, \gamma_0]$ *the solution* v *of* (3.2) *with* $v(0)=v_0$ *satisfies the estimate*

$$|v(r)-U(r)|\leqq C\sqrt{\gamma}, \quad r\in[0, R],$$

where U *is the solution of* (3.6) *with* $U(0)=v_0$.

Proof. — We have the estimate

$$|v'(r)-U'(r)|\leqq I_1+I_2+I_3,$$

where

$$I_1:=\frac{1}{1-\frac{8\pi}{r}\gamma\int_0^r s^2 g_{\phi,\gamma}(v(s))\,ds}4\pi\gamma\, h_{\phi,\gamma}(v(r)),$$

$$I_2:=\left|\frac{1}{1-\frac{8\pi}{r}\gamma\int_0^r s^2 g_{\phi,\gamma}(v(s))\,ds}-1\right|\frac{4\pi}{r^2}\int_0^r s^2 g_{\phi,\gamma}(v(s))\,ds,$$

$$I_3:=\frac{4\pi}{r^2}\int_0^r s^2\,|g_{\phi,\gamma}(v(s))-g_0(U(s))|\,ds.$$

Since both v and U are increasing and $\phi(E)=0$ for $E>0$, it is easily seen that

$$g_{\phi,\gamma}(v(r))\leqq C, \qquad h_{\phi,\gamma}(v(r))\leqq C, \qquad r\geqq 0, \qquad \gamma\in]0, 1],$$

where the constant C depends only on ϕ and v_0. Thus, for $R>0$ fixed and $r\in[0, R]$ we obtain the estimate

$$I_2+I_2\leqq\frac{1}{1-CR^2\gamma_0}C(1+R)^2\gamma,$$

for $\gamma\in]0, \gamma_0]$ and $\gamma_0\in]0, 1]$ such that $1-CR^2\gamma_0>0$. Next we estimate

$$|g_{\phi,\gamma}(v(r))-g_0(U(r))|\leqq J_1+J_2+J_3,$$

where

$$J_1:=4\pi|e^{-4\gamma v}-1|\int_{(e^{\gamma v}-1)/\gamma}^{\infty}\phi(E)(1+\gamma E)^2\left(\frac{1}{\gamma}(1+\gamma E)^2-\frac{1}{\gamma}e^{2\gamma v}\right)^{1/2}dE,$$

$$J_2:=4\pi\int_{(e^{\gamma v}-1)/\gamma}^{\infty}\phi(E)(1+\gamma E)^2\left(\frac{1}{\gamma}(1+\gamma E)^2-\frac{1}{\gamma}e^{2\gamma v}\right)^{1/2}dE,$$

$$J_3:=4\pi\int_U^{\infty}\phi(E)\left|(1+\gamma E)^2\left(\frac{1}{\gamma}(1+\gamma E)^2-\frac{1}{\gamma}e^{2\gamma v}\right)^{1/2}-(2(E-U))^{1/2}\right|dE.$$

Here we have assumed that $\frac{e^{\gamma v}-1}{\gamma} \leqq U(r)$, the other case being completely analogous. Obviously,

$$J_1 \leqq C\gamma.$$

and

$$\begin{aligned} J_2 &\leqq C\left| U(r) - \frac{e^{\gamma v(r)}-1}{\gamma}\right| \leqq C\,|U(r)-v(r)| + C\left| v(r) - \frac{e^{\gamma v(r)}-1}{\gamma}\right| \\ &\leqq C\,|U(r)-v(r)| + C\gamma. \end{aligned}$$

To estimate J_3 observe that

$$J_3 \leqq C \int_{\min(U(r),\,0)}^{0} (K_1 + K_2 + K_3)\, dE,$$

where

$$\begin{aligned} K_1 &:= |(1+\gamma E)^2 - 1| \left| \frac{1}{\gamma}(1+\gamma E)^2 - \frac{1}{\gamma} e^{2\gamma v}\right|^{1/2}, \\ K_2 &:= \left| \left(\frac{1}{\gamma}(1+\gamma E)^2 - \frac{1}{\gamma} e^{2\gamma v}\right)^{1/2} - (2(E - v(r)))^{1/2}\right|, \\ K_3 &:= |\sqrt{E - v(r)} - \sqrt{E - U(r)}|. \end{aligned}$$

Now on the domain of integration we have

$$K_1 \leqq C\gamma \left| 1 + \frac{1 - e^{2\gamma v}}{\gamma}\right|^{1/2} \leqq C\gamma$$

and

$$\begin{aligned} K_2 &\leqq \left| \frac{1}{\gamma}(1+\gamma E)^2 - \frac{1}{\gamma} e^{2\gamma v} - 2(E - v(r))\right|^{1/2} \\ &\leqq \left| \gamma E^2 + \frac{1 + 2\gamma v(r) - e^{2\gamma v}}{\gamma}\right|^{1/2} \\ &\leqq C\gamma^{1/2}, \end{aligned}$$

whereas distinguishing the cases $v(r) \leqq U(r)$ and $v(r) > U(r)$ immediately yields

$$\int_{\min(U(r),\,0)}^{0} K_3\, dE \leqq C\,|U(r) - v(r)|.$$

Collecting all estimates we obtain the inequality

$$|U'(r)-v'(r)| \leqq C\sqrt{\gamma}+\frac{C}{r^2}\int_0^r s^2 |U(s)-v(s)|\,ds$$
$$\leqq C\sqrt{\gamma}+C\int_0^r |U(s)-v(s)|\,ds, \qquad r\in[0, R],$$

and thus, after integrating,

$$|U(r)-v(r)| \leqq C\sqrt{\gamma}+C\int_0^r |U(s)-v(s)|\,ds, \qquad r\in[0, R],$$

where the constant C depends on ϕ, v_0, and R. The usual Gronwall argument completes the proof.

From [3, 5.4] we know that the Newtonian potential U has a zero, which is also the radius where the density vanishes. Since by the assumption $v_0<0$ the solution is nontrivial, U has to strictly increase so that there exists $R>0$ such that $U(R)>0$. The above lemma then tells us that for all $\gamma>0$ sufficiently small, $v(R)>0$ which by the definition of ϕ and Eq. (3.5) implies that $\rho(r)=p(r)=0$ for $r\geqq R$. Thus we have proved the following theorem:

THEOREM 3.2. – *Let ϕ be defined by Eq.* (3.7) *and take* $v_0<0$. *Then for all* $\gamma>0$ *sufficiently small the corresponding solution* v *has a zero, the density* ρ *and radial pressure* p *as defined by Eqs.* (3.5), (3.3), *and* (3.4) *have finite support, and*

$$0<4\pi\int_0^\infty s^2\rho(s)\,ds<\infty$$

i.e. *the solution is nontrivial with finite total mass.*

Final Remarks:

1. In this section we have obtained solutions of the static Vlasov-Einstein system with $\gamma>0$ small. However, if f, λ, v is such a solution then $\gamma^{-3/2}f(\gamma^{-1/2}\cdot, \gamma^{-1/2}\cdot)$, $\lambda(\gamma^{-1/2}\cdot)$, $\gamma v(\gamma^{-1/2}\cdot)$ is a solution of the system with $\gamma=1$, which again has finite radius and finite total mass.

2. If ϕ is defined by Eq. (3.7) then at least $f\in C^1(\mathrm{supp} f)$, and $f\in C(\mathbb{R}^6)$ or $f\in C^1(\mathbb{R}^6)$ if $k>0$ or $k>1$ respectively.

3. The finiteness of the total mass implies that $\lambda(r)\to 0$ as $r\to 0$ and v and μ have a finite limit as $r\to\infty$. Since only the derivatives of these functions appear in the equations, we may add some constant to obtain the limit zero also for v and μ, thus satisfying the condition for asymptotic flatness as stated in the introduction.

4. Our solutions are not only global in the coordinates which we used, but are singularity-free in the sense that the corresponding spacetime is timelike and null geodesically complete.

REFERENCES

[1] J. BATT, Steady State Solutions of the Relativistic Vlasov-Poisson System, in *Proceedings of the Fifth Marcel Grossmann Meeting on General Relativity*, R. RUFFINI Ed., 1988.

[2] J. BATT, H. BERESTYCKI, P. DEGOND and B. PERTHAME, Some Families of Solutions of the Vlasov-Poisson System, *Arch. Rational Mech. Anal.*, Vol. **104**, 1988, pp. 79-103.

[3] J. BATT, W. FALTENBACHER and E. HORST, Stationary Spherically Symmetric Models in Stellar Dynamics, *Arch. Rational Mech. Anal.*, Vol. **93**, 1986, pp. 159-183.

[4] J. BATT and K. PFAFFELMOSER, On the Radius Continuity of the Models of Polytropic Gas Spheres which Correspond to the Positive Solutions of the Generalized Emden-Fowler Equation, *Math. Meth. in the Appl. Sci.*, Vol. **10**, 1988, pp. 499-516.

[5] J. KIJOWSKI and G. MAGLI, A Generalization of the Relativistic Equilibrium Equation for a Nonrotating Star, *General Relativity and Gravitation*, Vol. **24**, 1992, pp. 139-158.

[6] G. REIN and A. RENDALL, Global Existence of Solutions of the Spherically Symmetric Vlasov-Einstein System with Small Initial Data, *Commun. Math. Phys.*, Vol. **150**, 1992, pp. 561-583.

[7] G. REIN and A. RENDALL, The Newtonian limit of the Spherically Symmetric Vlasov-Einstein System, *Commun. Math. Phys.*, Vol. **150**, 1992, pp. 585-591.

[8] A. RENDALL and B. SCHMIDT, Existence and Properties of Spherically Symmetric Static Fluid Bodies with a Given Equation of State, *Class. Quantum Grav.*, Vol. **8**, 1991, pp. 985-1000.

[9] S. L. SHAPIRO and S. A. TEUKOLSKY, Relativistic Stellar Dynamics on the Computer, I. Motivation and Numerical Method, *Astrophysical Journal*, Vol. **298**, 1985, pp. 34-57.

[10] J. SMOLLER, A. WASSERMAN, S.-T. YAU and J. MCLEOD, Smooth Static Solutions of the Einstein/Yang-Mills Equations, *Commun. Math. Phys.*, Vol. **143**, 1991, pp. 115-147.

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