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Dual-null dynamics

by

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ABSTRACT. – Generalised Lagrangian and Hamiltonian theories of dynamics with two evolution directions are presented. The corresponding evolution problem involves prescribing initial data on two intersecting surfaces. In the dual-null case, where the initial surfaces are null (or characteristic), and a certain functional is invertible, the initial data are the appropriate momentum field on each surface, and the configuration field on the intersection. The theory is applied to the Klein-Gordon and Maxwell fields, and the relativistic string.

RÉSUMÉ. – Nous présentons des théories Lagrangiennes et Hamiltoniennes généralisées à deux directions d'évolution. Le problème d'évolution correspondant nécessite de préciser les conditions initiales sur deux surfaces s'intersectant. Dans le cas dual-nul où les surfaces initiales sont caractéristiques, et une certaine fonctionnelle inversible, les données initiales sont les champs de moments appropriés sur chaque surface et le champ de configuration sur l'intersection. Nous appliquons cette théorie aux champs de Maxwell, de Klein-Gordon, et aux cordes relativistes.

0. INTRODUCTION AND SUMMARY

A large part of theoretical physics can be summarised as the development of Newtonian dynamics through Lagrangian and Hamiltonian dynamics into quantum dynamics. These may all be classified as theories of temporal dynamics, in that they all describe a spatial world evolving in time. This emphasis reflects the role of time in human consciousness, but is not particularly natural in Relativity, which describes the universe in a four-dimensionally covariant way. Moreover, null (or characteristic) surfaces are more relevant to the causal structure of Relativity than spatial (or Cauchy) surfaces, and so it is natural to consider null evolution as an alternative to temporal evolution.

For such reasons, it is desirable to have a formulation of null dynamics similar to the Lagrangian and Hamiltonian formulations of temporal dynamics, and eventually to find a representation of quantum theory which respects null evolution. In this paper, a formalism is developed to describe dual-null dynamics, which concerns the evolution of initial data prescribed on two intersecting null surfaces. By introducing apex conditions, the formalism may be modified to describe light-cone dynamics, where data are prescribed on a light-cone, or on past null infinity.

The starting point is a generalised Lagrangian theory which has two evolution directions instead of one, and consequently two velocity fields for each configuration field. At this stage the initial surfaces may have any signature. In the corresponding Hamiltonian theory, there is an integrability condition for each field which is particularly simple in the dual-null case, where the initial surfaces are null. This is related to the simplification of hyperbolic equations obtained by taking characteristic coordinates. In the absence of constraints, the initial data are the appropriate momentum field on each surface and the configuration field on the intersection.

The theory is applied to the examples of the Klein-Gordon field, the Maxwell field and the relativistic string in flat spacetime. The application to General Relativity has already led to the general solution to the Einstein equations on a null surface, and a necessary and sufficient condition for a light-cone to be trapped in terms of concentrations of matter of gravitational radiation (Hayward, 1992). One noteworthy point is that the dual-null formulation of a physical theory usually has fewer constraints than the corresponding temporal formulation. In particular, the Einstein gravitational field has no constraints in dual-null form. This is of considerable advantage analytically, and potentially for a quantum theory of gravity. It appears that there is no natural canonical quantisation scheme for dual-null dynamics, but that path-integral quantisation is feasible.

1. STRUCTURE AND NOTATION

The geometrical structures used are defined in Abraham & Marsden (1978) or Spivak (1970). The formalism is concerned with a compact orientable n-dimensional manifold S, of codimension 1 in spacetime for temporal dynamics, and codimension 2 for dual-null dynamics. Compactness of S is required for technical reasons only, and the dynamical equations derived apply equally to the non-compact case.

The ring of smooth functions (or scalars) from S to $\mathbf{R}$ is denoted by $\mathcal{F}$ S. The trivial line bundle is denoted by LS, the tangent bundle by TS, the bilinear (or metric) bundle of symmetric (0,2) tensors by BS, and the n-form bundle by DS. The module of smooth sections of a bundle is denoted by a C (for C^∞), e.g. CLS $= \mathcal{F}$ S for scalars, CTS for vector fields, and CDS $= \mathcal{F}_D$ S for n-forms (or scalar densities). The Lebesgue measure on S obtained by replacing n-forms with infinitesimals is denoted by $\int_S : \mathcal{F}_D S \to \mathbf{R}$. Multiplication by a nowhere-zero volume form $\mu \in \mathcal{F}_D S$ defines an isomorphism from $\mathcal{F}$ S to $\mathcal{F}_D$ S which preserves linearity and multiplication by scalars, and hence defines a Lebesgue measure $\int_S \mu : \mathcal{F} S \to \mathbf{R}$.

For a vector bundle Q over S, the dual bundle is denoted by Q^*, e.g. the cotangent bundle $T^* S$. Also needed is the density dual Q^*_D, whose fibre at a point consists of the linear maps from the corresponding fibre of Q to the fibre of DS; e.g. the cotangent density bundle $T^*_D S$. Multiplication by a volume form defines an isomorphism from CQ^* to CQ^*_D. One also has $Q_D = (Q^*_D)^*$, e.g. the tangent density bundle $T_D S$. The Whitney fibre sum of two bundles is denoted by $\oplus$, and multiple sums by powers, e.g. $Q^2 = Q \oplus Q$. In a similar way, TCQ, $T^*_D CQ$, BCQ etc denote the tangent bundle, cotangent density bundle, metric bundle etc of CQ.

Variations are defined in the usual way: given $q \in CQ$, consider a smooth 1-parameter family of fields $\tilde{q} : I \to CQ$, where I is a real interval around 0 and $\tilde{q}(0) = q$; then the variation $\delta q \in CQ$ of q with respect to this family is

$$\delta q = \frac{d}{d\varepsilon} \tilde{q}(\varepsilon) \bigg|_{\varepsilon = 0}.$$

For a functional $f : CQ \to \mathcal{F}_D S$, variations $\delta f : CQ \to \mathcal{F}_D S$ are similarly defined, and satisfy the chain rule

$$\delta f = \frac{\delta f}{\delta q} (\delta q),$$

in terms of the functional derivative $\delta f / \delta q \in CQ^*_D$.

An index-free notation has been adopted. A single symmetric contraction is denoted by a dot (.), a double symmetric contraction by a colon (:), a symmetric tensor product by $\otimes$, and an antisymmetric (exterior) product by $\wedge$.

2. TEMPORAL DYNAMICS

It is appropriate to begin with a brief review of the standard Lagrangian and Hamiltonian formalisms for temporal dynamics, roughly following Fischer & Marsden (1979), Abraham & Marsden (1978) and Arnold (1989). Since much of the structure is similar but simpler in the temporal case, this serves as an introduction to the methods, notation and problems to be encountered in dual-null dynamics.

2.1. Lagrangian dynamics

The basic kinematical space is a vector bundle Q over S, the configuration bundle. The configuration fields are the smooth sections $q \in \mathrm{CQ}$, and the velocity fields may be defined as $(q, \dot{q}) \in \mathrm{TCQ}$. The evolution space is an interval $\mathscr{T} = [0, \mathrm{T})$, with evolution parameter (time-coordinate) $\tau \in \mathscr{T}$. All fields on S are extended to $\mathrm{S} \times \mathscr{T}$ by taking $\dot{q} = dq/d\tau$.

Dynamics are determined by a Lagrangian (density) $\mathscr{L} : \mathrm{TCQ} \to \mathscr{F}_{\mathrm{D}}\mathrm{S}$, which is extended to $\mathrm{S} \times \mathscr{T}$ by invariance on $\mathscr{T}$. The Euler-Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\delta \mathscr{L}}{\delta \dot{q}}\right) = \frac{\delta \mathscr{L}}{\delta q}$$

are obtained from the principle of stationary action $\delta \int_{\mathscr{T}} \int_{\mathrm{S}} \mathscr{L}\, d\tau = 0$.

If $\mathscr{L}$ is independent of a particular velocity $\dot{c}$, then the corresponding Euler-Lagrange equation is a constraint,

$$0 = \frac{\delta \mathscr{L}}{\delta c},$$

rather than an evolution equation. Thus the evolution of c is not determined, and it may be described as not evolved. If it is possible to solve the constraint for c, for instance if $\mathscr{L}$ is quadratic in c, then c is spurious in the sense that the Lagrangian may be redefined without it. If c is not determined by the constraint, for instance if $\mathscr{L}$ is linear in c, then c must be specified at every event. In this case, c is a gauge field, representing coordinate freedom or some other internal freedom of the theory. In general, the number of evolved fields (and hence evolution equations) is the number of independent velocities in the Lagrangian, and the number of gauge fields (and hence constraints) is the number of remaining redundant velocities.

2.2. Hamiltonian dynamics

The momentum fields may be defined by $(q, \bar{q}) \in T^*_D CQ$. The Lagrangian defines a Lagrange transformation $\Lambda: TCQ \to T^*_D CQ$ by

$$(q, \bar{q}) = \Lambda(q, \dot{q}) = \left(q, \frac{\delta \mathscr{L}}{\delta \dot{q}}\right).$$

In practice, Lagrangians of interest are quadratic in the velocities $\dot{q}$, so that Λ is a linear transformation, with rank equal to the number of evolved fields, and nullity equal to the number of gauge fields. If there is an inverse Λ^{-1} then Λ is a Legendre transformation, and a Hamiltonian (density) $\mathscr{H}: T^*_D CQ \to \mathscr{F}_D S$ may be defined by

$$\mathscr{H}(q, \bar{q}) = ((0, \bar{q}) - \mathscr{L}) \Lambda^{-1}(q, \bar{q}),$$

or, with the usual abuse of notation,

$$\mathscr{H} = \bar{q}(\dot{q}) - \mathscr{L}.$$

In this case the second-order Euler-Lagrange equations are equivalent to the first-order Hamilton equations

$$\frac{dq}{d\tau} = \frac{\delta \mathscr{H}}{\delta \bar{q}}, \qquad \frac{d\bar{q}}{d\tau} = -\frac{\delta \mathscr{H}}{\delta q}.$$

The initial data are then $(q, \bar{q})$ on $S_0 = S \times \{0\}$.

If Λ is not invertible, a reduced Hamiltonian is defined on the constraint submanifold $\Lambda(TCQ)$ of $T^*_D CQ$ using the invertible part of Λ, and a reduced set of Hamilton equations is obtained by considering the non-invertible velocities as Lagrange multipliers. In practice, if variables $q = (e, c)$ are found which separate gauge fields c from evolved fields e, i.e. if $\delta \mathscr{L}/\delta \dot{c}$ vanishes and $\delta \mathscr{L}/\delta \dot{e}$ has maximal rank, then the reduced Hamiltonian $\mathscr{H}: \Lambda(T^*_D CQ) \to \mathscr{F}_D S$ is given by

$$\mathscr{H} = \bar{e}(\dot{e}) - \mathscr{L},$$

and the reduced Hamilton equations are

$$\frac{de}{d\tau} = \frac{\delta \mathscr{H}}{\delta \bar{e}}, \qquad \frac{d\bar{e}}{\delta \tau} = -\frac{\delta \mathscr{H}}{\delta e}, \qquad 0 = \frac{\delta \mathscr{H}}{\delta c}.$$

The initial data are then $(e, \bar{e})$ on S_0 satisfying the constraints, with c being prescribed over the whole neighbourhood $S \times \mathscr{T}$. In general, the constraint is not automatically preserved, so that

$$\frac{d}{d\tau}\left(\frac{\delta \mathscr{H}}{\delta c}\right) = 0$$

is an independent constraint. However, the constraint is automatically preserved if $\mathscr{H}$ is a contraction $\mathscr{H} = c(\mathscr{I})$ of c with a functional $\mathscr{I}(e, \bar{e})$, as occurs for the Einstein field.

The more general definition of Hamiltonian dynamics on a symplectic manifold appears to have no useful analogue in dual dynamics (§ 3.7).

3. DUAL-NULL DYNAMICS

Dynamics with two evolution directions, or dual dynamics, is essentially analogous to dynamics with one evolution direction, but requires additional consideration of integrability conditions. For each configuration field, there are two velocity fields, and consequently two momentum fields. The initial surfaces are two intersecting surfaces of codimension 1, S^+ and S^-, and their intersection S_0, defined subsequently. The basic theory of dual dynamics may be applied to any surfaces S^+ and S^-, but the integrability conditions take a particularly simple form in the dual-null case (§ 3.3). This is related to the simplification obtained by taking characteristic coordinates for hyperbolic kinetic-potential systems (§ 3.4).

3.1. Lagrangian dynamics

As for temporal dynamics, the configuration bundle Q is a vector bundle over a compact orientable manifold S, which is now of codimension 2 in spacetime. The configuration fields are the smooth sections $q \in \mathrm{CQ}$, and the velocity fields are $(q, q^+, q^-) \in (\mathrm{TCQ})^2 = \mathrm{TCQ} \oplus \mathrm{TCQ}$. The evolution space is a product of intervals $\mathscr{U} \times \mathscr{V} = [0, \mathrm{U}) \times [0, \mathrm{V})$, with evolution parameters $\xi \in \mathscr{U}$ and $\eta \in \mathscr{V}$. Fields on S are extended to $\mathrm{S} \times \mathscr{U} \times \mathscr{V}$ by taking $q^+ = \partial q/\partial\xi$ and $q^- = \partial q/\partial\eta$.

Dynamics are determined by a dual Lagrangian $\mathscr{L} : (\mathrm{TCQ})^2 \to \mathscr{F}_{\mathrm{D}}\mathrm{S}$, extended to $\mathrm{S} \times \mathscr{U} \times \mathscr{V}$ by invariance on $\mathscr{U} \times \mathscr{V}$. The principle of stationary action gives

$$
\begin{aligned}
0 &= \delta \int_{\mathscr{V}} \int_{\mathscr{U}} \int_{\mathrm{S}} \mathscr{L}\, d\xi\, d\eta \\
&= \int_{\mathscr{V}} \int_{\mathscr{U}} \int_{\mathrm{S}} \left(\frac{\delta \mathscr{L}}{\delta q}(\delta q) + \frac{\delta \mathscr{L}}{\delta q^+}(\delta q^+) + \frac{\delta \mathscr{L}}{\delta q^-}(\delta q^-) \right) d\xi\, d\eta \\
&= \int_{\mathscr{V}} \int_{\mathscr{U}} \int_{\mathrm{S}} \left(\frac{\delta \mathscr{L}}{\delta q} - \frac{d}{d\xi}\left(\frac{\delta \mathscr{L}}{\delta q^+}\right) - \frac{d}{d\eta}\left(\frac{\delta \mathscr{L}}{\delta q^-}\right) \right)(\delta q)\, d\xi\, d\eta \\
&\quad + \int_{\mathscr{V}} \int_{\mathrm{S}} \left[\frac{\delta \mathscr{L}}{\delta q^+}(\delta q) \right]_0^{\mathrm{U}} d\eta + \int_{\mathscr{U}} \int_{\mathrm{S}} \left[\frac{\delta \mathscr{L}}{\delta q^-}(\delta q) \right]_0^{\mathrm{V}} d\xi,
\end{aligned}
$$

and for variations δq vanishing at the boundaries of $S \times \mathscr{U} \times \mathscr{V}$, the dual Euler-Lagrange equations

$$\frac{\partial}{\partial \xi}\left(\frac{\delta \mathscr{L}}{\delta q^+}\right)+\frac{\partial}{\partial \eta}\left(\frac{\delta \mathscr{L}}{\delta q^-}\right)=\frac{\delta \mathscr{L}}{\delta q}$$

are obtained.

3.2. Hamiltonian dynamics

The momentum fields are $(q, \bar{q}, \hat{q}) \in (T^*_D CQ)^2 = T^*_D CQ \oplus T^*_D CQ$. The Lagrange transformation $\Lambda : (TCQ)^2 \to (T^*_D CQ)^2$ is given by

$$(q, \bar{q}, \hat{q})=\Lambda(q, q^+, q^-)=\left(q, \frac{\delta \mathscr{L}}{\delta q^+}, \frac{\delta \mathscr{L}}{\delta q^-}\right).$$

If Λ is invertible, the dual Hamiltonian $\mathscr{H} : (T^*_D CQ)^2 \to \mathscr{F}_D S$ is given by

$$\mathscr{H}(q, \bar{q}, \hat{q})=((0, \bar{q}, \hat{q})-\mathscr{L})\Lambda^{-1}(q, \bar{q}, \hat{q}),$$

or, with a useful abuse of notation,

$$\mathscr{H}=\bar{q}(q^+)+\hat{q}(q^-)-\mathscr{L}.$$

The dual Hamilton equations are

$$\frac{\partial q}{\partial \xi}=\frac{\delta \mathscr{H}}{\delta \bar{q}}, \qquad \frac{\partial q}{\partial \eta}=\frac{\delta \mathscr{H}}{\delta \hat{q}}, \qquad \frac{\partial \bar{q}}{\partial \xi}+\frac{\partial \hat{q}}{\partial \eta}=-\frac{\delta \mathscr{H}}{\delta q}.$$

The non-invertible case is more complex than for temporal dynamics, and is treated in § 3.5. The other qualitatively new feature of dual dynamics is that the above Hamilton equations do not by themselves give the full first-order field equations, but need to be supplemented by an integrability condition. A simple example is given in § 4.2.

3.3. Integrability conditions: the dual-null condition

Because of the way the evolution space $\mathscr{U} \times \mathscr{V}$ has been defined, the evolution derivatives commute:

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}=\frac{\partial}{\partial \eta}\frac{\partial}{\partial \xi}.$$

Applying this identity to q and using the Hamilton equations yields the condition

$$\frac{\partial}{\partial \xi}\left(\frac{\delta \mathscr{H}}{\delta \hat{q}}\right)=\frac{\partial}{\partial \eta}\left(\frac{\delta \mathscr{H}}{\delta \bar{q}}\right),$$

which expands to

$$\left(\frac{\delta\mathcal{H}}{\delta\bar{q}}\left(\frac{\delta}{\delta q}\right)\right)\frac{\delta\mathcal{H}}{\delta\hat{q}}+\left(\frac{\partial\bar{q}}{\partial\xi}\left(\frac{\delta}{\delta\bar{q}}\right)\right)\frac{\delta\mathcal{H}}{\delta\hat{q}}+\left(\frac{\partial\hat{q}}{\partial\xi}\left(\frac{\delta}{\delta\hat{q}}\right)\right)\frac{\delta\mathcal{H}}{\delta\hat{q}}$$
$$=\left(\frac{\delta\mathcal{H}}{\delta\hat{q}}\left(\frac{\delta}{\delta q}\right)\right)\frac{\delta\mathcal{H}}{\delta\bar{q}}+\left(\frac{\partial\bar{q}}{\partial\eta}\left(\frac{\delta}{\delta\bar{q}}\right)\right)\frac{\delta\mathcal{H}}{\delta\bar{q}}+\left(\frac{\partial\hat{q}}{\partial\eta}\left(\frac{\delta}{\delta\hat{q}}\right)\right)\frac{\delta\mathcal{H}}{\delta\bar{q}},$$

using the Hamilton equations again. This is an integrability condition for the Hamilton equations, relating the four derivatives of $\bar{q}$ and $\hat{q}$ along ξ and η in terms of the momentum fields $(q, \bar{q}, \hat{q})$. The character of the integrability condition is critical to the evolution problem, as follows. Bilinear functionals $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}:(\mathrm{T}^*_\mathrm{D}\mathrm{CQ})^2\to\mathrm{B}^*_\mathrm{D}\mathrm{CQ}$ may be defined by

$$\mathcal{A}=\left(\frac{\delta}{\delta\bar{q}}\otimes\frac{\delta}{\delta\bar{q}}\right)\mathcal{H},\qquad \mathcal{B}=\left(\frac{\delta}{\delta\bar{q}}\otimes\frac{\delta}{\delta\hat{q}}\right)\mathcal{H},\qquad \mathcal{C}=\left(\frac{\delta}{\delta\hat{q}}\otimes\frac{\delta}{\delta\hat{q}}\right)\mathcal{H}.$$

Inspecting the integrability condition, note that the derivatives $\partial\bar{q}/\partial\xi$ and $\partial\hat{q}/\partial\eta$ already occur in the Hamilton equations, but that $\partial\hat{q}/\partial\xi$ and $\partial\bar{q}/\partial\eta$ do not. The presence of the latter terms makes it difficult to formulate an evolution problem, since there are not enough equations to determine all four derivatives uniquely, and yet it would be unnatural to give such derivatives as initial data. Consequently, attention is restricted to the case

$$\mathcal{A}=\mathcal{C}=0,$$

which is the definition of the dual-null condition. The condition is not as strong as it might appear, since all Hamiltonians of interest are quadratic in the momenta, and consequently the condition reduces to the vanishing of the appropriate coefficients of the quadratic, as is shown in § 3.4. For spacetime examples, these coefficients are precisely those which vanish when the initial surfaces $\mathrm{S}^+=\mathrm{S}\times\mathcal{U}\times\{0\}$ and $\mathrm{S}^-=\mathrm{S}\times\{0\}\times\mathcal{V}$ are null in spacetime, as in the examples of § 4.

In the dual-null case, the integrability condition and the final Hamilton equation are simultaneous equations for $\partial\bar{q}/\partial\xi$ and $\partial\hat{q}/\partial\eta$ as functions of the momentum fields $(q, \bar{q}, \hat{q})$. These are uniquely solvable for $\partial\bar{q}/\partial\xi$ and $\partial\hat{q}/\partial\eta$ if and only if $\mathcal{B}$ has an inverse $\mathcal{B}^{-1}$: $(\mathrm{T}^*_\mathrm{D}\mathrm{CQ})^2\to\mathrm{B}_\mathrm{D}\mathrm{CQ}$. In this invertible case, the dynamical equations – Hamilton equations plus integrability condition – are equations determining $\partial q/\partial\xi$, $\partial q/\partial\eta$, $\partial\bar{q}/\partial\xi$ and $\partial\hat{q}/\partial\eta$ as functions of $(q, \bar{q}, \hat{q})$. Thus the inital data are q on $\mathrm{S}_0=\mathrm{S}\times\{0\}\times\{0\}$, $\bar{q}$ on S^-, and $\hat{q}$ on S^+, and the evolution equations may be expected to determine a solution in a neighbourhood $\mathrm{S}\times\mathcal{U}\times\mathcal{V}$.

In the non-invertible case, $\mathcal{B}$ may be decomposed into an invertible part and a zero part, with the integrability condition for the zero part being a constraint. The constraint may be trivial or non-trivial, but either way the evolution problem is now different, since one of the corresponding momenta must now be prescribed everywhere. For examples see § 4.

3.4. Kinetic-potential systems

The natural selection of the dual-null case is most easily seen for kinetic-potential systems, where the Lagrangian is the sum of a kinetic term quadratic in the velocities and a potential term independent of the velocities. To define quadratics requires a configuration metric (density) $h^* \in \mathrm{B_D CQ}$ for the configuration bundle Q, defining a linear operation $*: \mathrm{CQ} \to \mathrm{CQ}^*_\mathrm{D}$ denoted by $q^* = h^*(q)$, with the inverse metric $h_* \in \mathrm{B}^*_\mathrm{D}\mathrm{CQ}$ defining a linear operation $_*: \mathrm{CD}^*_\mathrm{D} \to \mathrm{CQ}$ denoted by $p_* = h_*(p)$. If Q is a sum of tensor bundles over S, the metric $h \in \mathrm{CBS}$ on S induces a natural metric h^* on Q, e. g. for a vector $s \in \mathrm{CTS}$, it is natural to take $h^*(s) = \mu\, h.s$, where $\mu \in \mathscr{F}_\mathrm{D}\mathrm{S}$ is the volume form of h. Using this structure, a symmetric kinetic-potential Lagrangian $\mathscr{L}: (\mathrm{TCQ})^2 \to \mathscr{F}_\mathrm{D}\mathrm{S}$ is one of the form

$$\mathscr{L} = \frac{1}{2} c (q^+)^* (q^+) - b (q^+)^* (q^-) + \frac{1}{2} a (q^-)^* (q^-) - \mathrm{V},$$

where a, b, c and the potential V depend only on q, i.e. $\mathrm{V}: \mathrm{CQ} \to \mathscr{F}_\mathrm{D}\mathrm{S}$ and a, b, $c: \mathrm{CQ} \to \mathscr{F}\mathrm{S}$. The only other quadratic structure which occurs in practice is an antisymmetric quadratic form induced from the alternating form associated with 2-dimensional h, which gives non-standard dynamics (§ 3.5).

The dual Euler-Lagrange equations

$$\frac{\partial}{\partial \xi} (c q^+ - b q^-)^* + \frac{\partial}{\partial \eta} (-b q^+ + a q^-)^* = \frac{\delta \mathscr{L}}{\delta q}$$

may be written as a quasi-linear second-order partial differential equation for q:

$$c \frac{\partial^2 q}{\partial \xi^2} - 2 b \frac{\partial^2 q}{\partial \xi \, \partial \eta} + a \frac{\partial^2 q}{\partial \eta^2} = f,$$

where $f: (\mathrm{TCQ})^2 \to \mathrm{CQ}$, so that f is first-order in q, and it is assumed that h^* is either included in the configuration bundle or fixed. The system may be classified as hyperbolic, parabolic or elliptic as $b^2 - ac$ is positive, zero or negative respectively. For the hyperbolic case, the linear transformation freedom in (ξ, η) may be used to select characteristic coordinates (ξ, η) for which $a = c = 0$, so that the equation takes the canonical form of the wave equation:

$$\frac{\partial^2 q}{\partial \xi \, \partial \eta} = -\frac{1}{2} b^{-1} f,$$

which may be taken as the definition of the dual-null case for such systems.

The Lagrange transformation $\Lambda : (\mathrm{TCQ})^2 \to (\mathrm{T}^*_{\mathrm{D}}\mathrm{CQ})^2$ is given by

$$\bar{q} = (cq^+ - bq^-)^* \in \mathrm{CQ}^*_{\mathrm{D}},\ \hat{q} = (-bq^+ + aq^-)^* \in \mathrm{CQ}^*_{\mathrm{D}},$$

and if $b^2 \neq ac$ then Λ is invertible and the Hamiltonian $\mathscr{H} : (\mathrm{T}^*_{\mathrm{D}}\mathrm{CQ})^2 \to \mathscr{F}_{\mathrm{D}}\mathrm{S}$ is given by

$$\mathscr{H} = -\frac{a\bar{q}(\bar{q}_*) + 2b\bar{q}(\hat{q}_*) + c\hat{q}(\hat{q}_*)}{2(b^2 - ac)} + \mathrm{V}.$$

Thus the bilinear functionals defined in § 3.3 are given by

$$\mathscr{A} = -\frac{ah_*}{b^2 - ac}, \qquad \mathscr{B} = -\frac{bh_*}{b^2 - ac}, \qquad \mathscr{C} = -\frac{ch_*}{b^2 - ac},$$

so that the definition $a = c = 0$ of the dual-null case in terms of characteristic coordinates agrees with the general definition $\mathscr{A} = \mathscr{C} = 0$, with invertibility of $\mathscr{B}$ following from the symmetry assumption. A simple example is provided by the relativistic string (§ 4.4).

The dual-null condition puts the Lagrange transformation in the canonical form

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -bh^* \\ 0 & -bh^* & 0 \end{pmatrix}.$$

so that $(\bar{q}, \hat{q}) = -b(q^-, q^+)^*$. This is just the canonical form of a kinetic quadratic which is hyperbolic.

3.5. Constraints

If the Lagrange transformation Λ is not invertible, there are constraints on momentum space (primary constraints). For dual dynamics, this does not necessarily mean that the corresponding Hamilton equations reduce to constraint equations (secondary constraints) for the remaining fields on S, since the two conjugate momenta need not both be zero; instead, there are intermediate possibilities of evolution equations with a different structure.

The procedure for dealing with momentum constraints is the same as in temporal dynamics. A reduced Hamiltonian is defined on the constraint submanifold $\Lambda(\mathrm{TCQ})^2)$ of $(\mathrm{T}^*_{\mathrm{D}}\mathrm{CQ})^2$ using the invertible part of Λ. The reduced Hamilton equations then follow by considering the non-invertible velocities as Lagrange multipliers of the constrained momenta.

The example which occurs in practice is that of a kinetic-potential system where the kinetic term is an antisymmetric quadratic,

$$\mathscr{L}=\frac{1}{2}(\psi^{+}-\phi^{-})^{*}(\psi^{+}-\phi^{-})-V$$

for configuration fields $(\phi, \psi)\in CQ$. Thus there is only one independent momentum $\bar{\psi}=(\psi^{+}-\phi^{-})^{*}\in CQ_{D}^{*}$, with the other three momenta constrained: $\hat{\psi}=\phi=\hat{\phi}+\bar{\psi}=0$.The reduced Hamiltonian is of the form $\mathscr{H}=\bar{\psi}(\psi^{+}-\phi^{-})-\mathscr{L}$, with a total Hamiltonian $\mathscr{H}_{T}=\mathscr{H}+\hat{\psi}(\rho)+\phi(\sigma)+(\phi+\bar{\psi})(\tau)$ containing Lagrange multipliers ρ, σ and $\tau\in CQ$. The Hamilton equations for $\mathscr{H}_{T}$ are

$$\frac{\partial\phi}{\partial\xi}=\sigma,\qquad \frac{\partial\phi}{\partial\eta}=\tau,\qquad \frac{\partial\bar{\phi}}{\partial\xi}+\frac{\partial\hat{\phi}}{\partial\eta}=-\frac{\delta\mathscr{H}}{\partial\phi},$$

$$\frac{\partial\psi}{\partial\xi}=\frac{\delta\mathscr{H}}{\delta\bar{\psi}}+\tau,\qquad \frac{\partial\psi}{\partial\eta}=\rho,\qquad \frac{\partial\bar{\psi}}{\partial\xi}+\frac{\partial\hat{\psi}}{\partial\eta}=-\frac{\delta\mathscr{H}}{\delta\psi},$$

which, on eliminating the multipliers and applying the constraints, reduce to

$$\frac{\partial\bar{\psi}}{\partial\xi}=-\frac{\delta\mathscr{H}}{\delta\psi},\qquad \frac{\partial\bar{\psi}}{\partial\eta}=\frac{\delta\mathscr{H}}{\delta\phi},\qquad \frac{\partial\psi}{\partial\xi}-\frac{\partial\phi}{\partial\eta}=\frac{\delta\mathscr{H}}{\delta\bar{\psi}}.$$

In this non-standard case, the integrability condition is got by applying the commutator to the momentum $\bar{\psi}$, the dual-null condition is

$$\left(\frac{\delta}{\delta\phi}\otimes\frac{\delta}{\delta\phi}\right)\mathscr{H}=\left(\frac{\delta}{\delta\psi}\otimes\frac{\delta}{\delta\psi}\right)\mathscr{H}=0,$$

and it is possible to solve uniquely for $\partial\psi/\partial\xi$ and $\partial\phi/\partial\eta$ if and only if

$$\left(\frac{\delta}{\delta\phi}\otimes\frac{\delta}{\delta\psi}\right)\mathscr{H}$$

is invertible. In this case, the initial data are $\bar{\psi}$ on S_0, ϕ on S^{+} and ψ on S^{-}. In the non-invertible case, a constraint is obtained, and either ϕ or ψ must be specified everywhere, and so may be interpreted as a gauge variable. The Maxwell field (§ 4.3) provides an example. Note that this is equivalent to a standard system with $(q, \bar{q}, \hat{q})=(\bar{\psi}, -\psi, \phi)$.

If both momenta $\bar{c}$ and $\hat{c}$ of a field c vanish, then the corresponding Hamilton equation is a constraint,

$$0=\frac{\delta\mathscr{H}}{\delta c},$$

and c may be interpreted as a gauge variable. There are three types of constraint: constraints on S_0, which involve only the initial data on S_0, and are preserved along ξ and η; constraints on S^{+}, which involve the initial data on S^+ and are preserved along η; and similar constraints on S^-.

3.6. Numerical considerations

One approach to evolution problems which appear to be analytically intractable in general is a programme of numerical integration of the equations. The dual-null evolution problem has certain advantages over the Cauchy problem in terms of numerical methods, which will be briefly indicated here.

A numerical model for a dual-null system may be constructed by analogy to the familiar grid model for the Cauchy problem. The neighbourhood $S \times \mathcal{U} \times \mathcal{V}$ is modelled by a Cartesian grid for $\mathcal{U} \times \mathcal{V}$, and a grid for S which represents some coordinate system. The fields $(q, \bar{q}, \hat{q})$ are replaced by their values on the grid, and the equations are modelled by finite difference equations along the grid lines. In the standard invertible case, where there are no constraints, the initial data are the values of q on S_0, $\hat{q}$ on S^+, and $\bar{q}$ on S^-. There are three integration routines to perform: integrate q and $\bar{q}$ in the ξ-direction up S^+; integrate q and $\hat{q}$ in the η-direction up S^-; and integrate q and $\bar{q}$ in the ξ-direction and q and $\hat{q}$ in the η-direction into $S \times \mathcal{U} \times \mathcal{V}$.

This gives two estimates q_1 and q_2 of q at each point. The double estimate has no analogue in the Cauchy problem, and has two noteworthy advantages. Firstly, by simply taking the average of the two estimates, a more accurate estimate is obtained. Secondly, this provides a good error measure

$$\frac{\| q_1 - q_2 \|_c}{\| q_1 + q_2 \|_c} \in \mathbf{R}$$

in terms of the configuration norm $\| \quad \|_c : CQ \to \mathbf{R}$ defined using the configuration metric of § 3.4:

$$\| q \|_c = \sqrt{\int_S q^* (q)}.$$

Alternatively, the dynamic (energy) norm $\| \quad \|_d$: $(T_D^* CQ)^2 \to \mathbf{R}$ given by

$$\| (q, \bar{q}, \hat{q}) \|_d = \sqrt{\int_S \mathcal{H} (q, \bar{q}, \hat{q})}$$

may be used to define an error

$$\frac{\| (q_1 - q_2, \bar{q}, \hat{q}) \|_d}{\| (q_1 + q_2, \bar{q}, \hat{q}) \|_d} \in \mathbf{R},$$

provided that $H=\int_S \mathscr{H}(q, \bar{q}, \hat{q})$ is non-negative. This has the status of a positive-energy condition for the field considered, with H being a measure of the energy of the field q on S. By comparison, the existence of $\|q\|_c$ depends only whether h^* is positive-definite, which follows if h^* is derived from a positive-definite metric $h \in \text{CBS}$.

Constraints $C(q, \bar{q}, \hat{q})=0$ may be treated similarly to constraints for the Cauchy problem, though there are now three types of constraint. Constraints may also be used to give more accurate estimates, though in a considerably more complicated way than that provided by double estimates. Given an estimate $(q, \bar{q}, \hat{q})_0$, it is required to find the closest solution satisfying the constraint, in terms either of the dynamic norm or the kinetic norm $\| \quad \|_k : (T^*_D CQ)^2 \to \mathbf{R}$ given by

$$\|q, \bar{q}, \hat{q}\|_k = \sqrt{\int_S (q^*(q) + \bar{q}_*(\bar{q}) + \hat{q}_*(\hat{q}))}.$$

This itself is a difficult problem. The minimum distance gives some sort of measure of the accuracy of the original estimate, but does not provide a good error measure since there is no natural scaling in general, corresponding to the fact that an estimate of zero is being calculated.

One other important advantage of a dual-null approach over the Cauchy approach is that in the latter, radiation propagates only approximately at light-speed, due to numerical error, whilst in the former the correct null propagation in the normal directions is forced by the structure. Thus a dual-null numerical model may be expected to be more accurate for radiation problems. It is difficult to formulate a precise comparison, since the initial surfaces are different.

3.7. Formal developments

The notation may be compacted by combining the velocities and momenta into single quantities $\dot{q}=(q^+, q^-)$ and $p=(\bar{q}, \hat{q})$, and the evolution derivatives into a single operator $\Delta=(\partial/\partial\xi, \partial/\partial\eta)$. The dual Euler-Lagrange equations are then

$$\Delta . \left(\frac{\delta \mathscr{L}}{\delta \dot{q}}\right) = \frac{\delta \mathscr{L}}{\delta q},$$

the Lagrange transformation is given by

$$p = \frac{\delta \mathscr{L}}{\delta \dot{q}},$$

and the dual Hamilton equations are

$$\Delta q = \frac{\delta \mathcal{H}}{\delta p}, \qquad \Delta . p = -\frac{\delta \mathcal{H}}{\delta q}.$$

A symplectic form J must be introduced in order to state the integrability condition:

$$\mathrm{J}(\Delta, \Delta)\, q = 0.$$

However, in practice this approach is not particularly advantageous, since the velocities or momenta must be separated to make sense of the initial data.

These seems to be no useful generalisation of Poisson brackets for dual dynamics. This is because, in the expansion of the derivative of a functional,

$$\frac{\partial f}{\partial \xi} = \frac{\delta f}{\delta q}\left(\frac{\partial q}{\partial \xi}\right) + \frac{\delta f}{\delta \bar{q}}\left(\frac{\partial \bar{q}}{\partial \xi}\right) + \frac{\delta f}{\delta \hat{q}}\left(\frac{\partial \hat{q}}{\partial \xi}\right),$$

the term in $\partial \hat{q}/\partial \xi$ is not determined by the field equations, but is part of the initial data. Consequently, evolution derivatives cannot be expressed solely in terms of the Hamiltonian, because there are always terms explicitly dependent on the initial data. Thus there seems little hope for a canonical quantisation scheme for dual-null dynamics. Path-integral quantisation is still possible, as for example for the relativistic string (Green, Schwarz & Witten, 1987).

4. EXAMPLES

Although the main aim of the dual-null formalism is to provide an approach to General Relativity, it also provides a convenient formulation of other physical theories. The examples of the Klein-Gordon field, the Maxwell field and the relativistic string have been chosen partly because of their intrinsic interest, and partly because they illustrate the key features of the formalism.

4.1. Fields on flat spacetime

For a flat spacetime (M, g), a dual-null basis (u, v; e) may be taken in which the metric $g \in$ CBM is given by

$$g = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix},$$

where u, $v \in \mathrm{CTM}$, $e \in (\mathrm{CTS})^2$ is a basis for the flat 2-surface S, and $\delta \in \mathrm{CBS}$ is the 2-dimensional Kronecker delta. For consistency with the general theory, S is a torus rather than a plane, though this is irrelevant to the equations. The evolution parameters $\xi \in \mathscr{U}$ and $\eta \in \mathscr{V}$ are given by $u = \partial/\partial\xi$ and $v = \partial/\partial\eta$. The initial surfaces are $S^+ = S \times \mathscr{U} \times \{0\}$, $S^- = S \times \{0\} \times \mathscr{V}$ and $S_0 = S \times \{0\} \times \{0\}$. The coordinate derivative ∇ on M decomposes into the coordinate derivative $\mathscr{D}$ on S and the evolution derivatives $\partial/\partial\xi$ and $\partial/\partial\eta$:

$$\nabla = \left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}; \mathscr{D}\right).$$

The fields must similarly be decomposed into fields q on S. The spacetime action $I = \int_M \mathscr{L}_0$ should then be converted to the form

$$I = \int_{\mathscr{V}} \int_{\mathscr{U}} \int_S \mathscr{L}(q, q^+, q^-)\, d\xi\, d\eta,$$

which may involve removing total divergences.

For any theory on a fixed spacetime background, the distinction between functions $\mathscr{F}$ S and n-forms $\mathscr{F}_D$S in the dynamics is no longer necessary, since the area form $\mu \in \mathscr{F}_D S$ is fixed. Consequently, in such cases it is sensible to modify the general theory of § 3 by omitting the subscripted Ds, and omitting μ from $\mathscr{L}$. Also, index raising ($\mathrm{CT^*S} \to \mathrm{CTS}$) and lowering ($\mathrm{CTS} \to \mathrm{CT^*S}$) with the identity metric δ is trivial, and will not be denoted.

4.2. The Klein-Gordon field

A simple example is provided by the Klein-Gordon scalar field ϕ, defined by the spacetime action

$$I = -\frac{1}{2} \int_M \nu \left(g^{ab} (\nabla_a \phi)(\nabla_b \phi) + m^2 \phi^2\right),$$

where $\nu \in \mathscr{F}_D M$ is the volume form of the metric g. The dual-null configuration bundle is just the line bundle $Q = LS$, with field $q = \phi \in \mathrm{CLS}$, and the dual-null Lagrangian $\mathscr{L} : (\mathrm{TCQ})^2 \to \mathscr{F}$ S is

$$\mathscr{L}(\phi, \phi^+, \phi^-) = \phi^+ \phi^- - \frac{1}{2} \mathscr{D}\phi . \mathscr{D}\phi - \frac{1}{2} m^2 \phi^2,$$

in the dual-null basis of 4.1. The Euler-Lagrange equations give the dual-null form of the Klein-Gordon equation:

$$2\frac{\partial^2 \phi}{\partial\xi\,\partial\eta}=\mathcal{D}^2\phi-m^2\phi,$$

where $\mathcal{D}^2=\mathcal{D}.\mathcal{D}$ is the Laplacian on S.

The momentum fields are

$$\bar{\phi}=\frac{\delta\mathcal{L}}{\delta\phi^+}=\phi^-\in\text{CLS},\qquad \hat{\phi}=\frac{\delta\mathcal{L}}{\delta\phi^-}=\phi^+\in\text{CLS},$$

and the Hamiltonian $\mathcal{H}:(\text{T}^*\,\text{CQ})^2\to\mathcal{F}\,\text{S}$ is

$$\mathcal{H}(\phi,\bar{\phi},\hat{\phi})=\bar{\phi}\phi^+ +\hat{\phi}\phi^- -\mathcal{L}=\bar{o}\hat{\phi}+\frac{1}{2}\mathcal{D}\phi.\mathcal{D}\phi+\frac{1}{2}m^2\phi^2.$$

The Hamilton equations are

$$\frac{\partial\phi}{\partial\xi}=\frac{\delta\mathcal{H}}{\delta\bar{\phi}}=\hat{\phi},\qquad \frac{\partial\phi}{\partial\eta}=\frac{\delta\mathcal{H}}{\delta\hat{\phi}}=\bar{\phi},\qquad \frac{\partial\bar{\phi}}{\partial\xi}+\frac{\partial\hat{\phi}}{\partial\eta}=-\frac{\delta\mathcal{H}}{\partial\phi}=\mathcal{D}^2\phi-m^2\phi,$$

and the integrability condition

$$\frac{\partial^2\phi}{\partial\xi\,\partial\eta}=\frac{\partial^2\phi}{\partial\eta\,\partial\xi}$$

is invertible and gives

$$\frac{\partial\bar{\phi}}{\partial\xi}=\frac{\partial\hat{\phi}}{\partial\eta}.$$

Thus the first-order equations are

$$\frac{\partial\phi}{\partial\xi}=\hat{o},\qquad \frac{\partial\phi}{\partial\eta}=\bar{\phi},\qquad \frac{\partial\bar{\phi}}{\partial\xi}=\frac{\partial\hat{\phi}}{\partial\eta}=\frac{1}{2}(\mathcal{D}^2\phi-m^2\phi),$$

and the initial data are ϕ on So, $\hat{\phi}$ on S^+ and $\bar{\phi}$ on S^-. This illustrates the standard structure of the dual-null equations and initial data.

4.3. The Maxwell field

A more interesting example is the Maxwell electromagnetic field, defined by the spacetime action

$$\text{I}=-\int_{\text{M}}\nu\,(g^{ac}g^{bd}(\nabla_{[a}\text{A}_{b]})(\nabla_{[c}\text{A}_{d]})+\text{A}_a\text{J}^a)$$

in terms of the covector potential $A \in CT^*M$, with the current vector $J \in CTM$ being regarded not as a dynamical field, but as a fixed background field. The physically measurable quantity is not the potential A, but the electromagnetic field tensor $F = 2\nabla \wedge A$.

Using the dual-null basis of § 4.1, the potential and the current are decomposed into

$$A = (\phi, \psi; a), \qquad J = (\rho, \sigma; j),$$

and so the configuration bundle is $Q = (LS)^2 \oplus T^*S$, with fields $q = (\phi, \psi; a) \in CQ$, and the background bundle is $P = (LS)^2 \oplus TS$, with fields $(\rho, \sigma; j) \in CP$. The dual-null Lagrangian $\mathscr{L} : (TCQ)^2 \rightarrow \mathscr{F} S$ is found to be

$$\mathscr{L} = (a^+ - \mathscr{D}\phi).(a^- - \mathscr{D}\psi) + \frac{1}{2}(\psi^+ - \phi^-)^2 - (\mathscr{D} \wedge a) : (\mathscr{D} \wedge a) - \phi\rho - \psi\sigma - a.j.$$

The independent momenta are

$$\bar{a} = a^- - \mathscr{D}\psi \in CTS, \qquad \hat{a} = a^+ - \mathscr{D}\phi \in CTS, \qquad \bar{\psi} = \psi^+ - \phi^- \in CLS,$$

and the momentum constraints are

$$\hat{\psi} = \bar{\phi} = \hat{\phi} + \bar{\psi} = 0.$$

Using the result for antisymmetric constraints (§ 3.5), there is a reduced Hamiltonian $\mathscr{H} : \Lambda((TCQ)^2) \rightarrow \mathscr{F} S$ of the form

$$\mathscr{H} = \bar{a}.a^+ + \hat{a}.a^- + \bar{\psi}(\psi^+ - \phi^-) - \mathscr{L},$$

and reduced Hamilton equations of the form

$$\frac{\partial a}{\partial \xi} = \frac{\delta \mathscr{H}}{\partial \bar{a}}, \qquad \frac{\partial a}{\partial \eta} = \frac{\delta \mathscr{H}}{\delta \hat{a}}, \qquad \frac{\partial \bar{a}}{\partial \xi} + \frac{\partial \hat{a}}{\partial \eta} = -\frac{\delta \mathscr{H}}{\delta a},$$

$$\frac{\partial \bar{\psi}}{\partial \xi} = -\frac{\delta \mathscr{H}}{\delta \psi}, \qquad \frac{\partial \bar{\psi}}{\partial \eta} = \frac{\delta \mathscr{H}}{\delta \phi}, \qquad \frac{\partial \psi}{\partial \xi} - \frac{\partial \phi}{\partial \eta} = \frac{\delta \mathscr{H}}{\delta \bar{\psi}}.$$

Explicitly, these are

$$\mathscr{H} = \bar{a}.\hat{a} + \frac{1}{2}\bar{\psi}^2 + \bar{a}.\mathscr{D}\phi + \hat{a}.\mathscr{D}\psi + (\mathscr{D} \wedge a) : (\mathscr{D} \wedge a) + \phi\rho + \psi\sigma + a.j$$

and

$$\frac{\partial a}{\partial \xi} = \hat{a} + \mathscr{D}\phi, \qquad \frac{\partial a}{\partial \eta} = \bar{a} + \mathscr{D}\psi, \qquad \frac{\partial \bar{a}}{\partial \xi} + \frac{\partial \hat{a}}{\partial \eta} = -2\mathscr{D}.(\mathscr{D} \otimes a) + 2\mathscr{D}^2 a - j,$$

$$\frac{\partial \bar{\psi}}{\partial \xi} = \mathscr{D}.\hat{a} - \sigma, \qquad \frac{\partial \bar{\psi}}{\partial \eta} = -\mathscr{D}.\bar{a} + \rho, \qquad \frac{\partial \psi}{\partial \xi} - \frac{\partial \phi}{\partial \eta} = \bar{\psi}.$$

The integrability condition

$$\frac{\partial^2 a}{\partial \xi \, \partial \eta} = \frac{\partial^2 a}{\partial \eta \, \partial \xi}$$

is invertible and yields

$$\frac{\partial\bar{a}}{\partial\xi}-\frac{\partial\hat{a}}{\partial\eta}=-\mathscr{D}\bar{\psi},$$

and the integrability condition

$$\frac{\partial^2\bar{\psi}}{\partial\xi\,\partial\eta}=\frac{\partial^2\bar{\psi}}{\partial\eta\,\partial\xi}$$

is non-invertible and gives charge conservation

$$0=\frac{\partial\rho}{\partial\xi}+\frac{\partial\sigma}{\partial\eta}+\mathscr{D}.j,$$

which is to be regarded as a consistency condition on the background fields, and in the absence of sources in trivial.

Recalling that the physically measurable quantity is the field tensor F, the potential a is replaced by the 2-form $b=2\mathscr{D}\wedge a\in\mathrm{CDS}$, so that F is given in the dual-null basis by

$$\mathrm{F}=\begin{pmatrix}0 & \bar{\psi} & \hat{a}\\ -\bar{\psi} & 0 & \bar{a}\\ -\hat{a} & -\bar{a} & b\end{pmatrix}.$$

The dynamical equations then become

$$\frac{\partial b}{\partial\xi}=2\mathscr{D}\wedge\hat{a},\qquad \frac{\partial\bar{a}}{\partial\xi}=\frac{1}{2}(-\mathscr{D}\bar{\psi}-\mathscr{D}.b-j),\qquad \frac{\partial\bar{\psi}}{\partial\xi}=\mathscr{D}.\hat{a}-\sigma,$$

$$\frac{\partial b}{\partial\eta}=2\mathscr{D}\wedge\bar{a},\qquad \frac{\partial\hat{a}}{\partial\eta}=\frac{1}{2}(\mathscr{D}\bar{\psi}-\mathscr{D}.b-j),\qquad \frac{\partial\bar{\psi}}{\partial\eta}=-\mathscr{D}.\bar{a}+\rho,$$

together with charge conservation. These are the Maxwell equations in dual-null form.

Thus the electromagnetic initial data are b and $\bar{\psi}$ on S_0, $\hat{a}$ on S^+, and $\bar{a}$ on S^-. For sources, one can prescribe j and σ everywhere and ρ on S^-, or j and ρ everywhere and σ on S^+. The 2-vectors $\bar{a}$ and $\hat{a}$ may be interpreted as encoding the electromagnetic radiation propagating in the u and v directions respectively, since they constitute the free data on S^- and S^+. The fields $\bar{\psi}$ and b represent respectively the components of the electric and magnetic fields normal to S. Note that there are no constraints, which constitutes a significant advantage over the classical temporal form of the Maxwell equations in terms of the electric and magnetic fields.

4.4. The relativistic string

In the previous examples, the dual-null case was selected at the outset by coordinate choice. This is not possible if the variables encoding the coordinate freedom are also included in the dynamics, as occurs in Relativity. In this case, all variables must be retained in the Lagrangian or Hamiltonian, with the Lagrange or Hamilton equations being evaluated for the dual-null case, since otherwise dynamical equations will be missed.

The simplest such example is the relativistic string (Green, Schwarz & Witten, 1987), defined by the string action

$$\mathrm{I}=-\frac{1}{2}\int_{\mathrm{M}} \nu\, g_{\rho\sigma} h^{ab} (\mathscr{D}_a x^\rho)(\mathscr{D}_b x^\sigma),$$

where x^ρ are coordinates and $g_{\rho\sigma}$ is a flat metric on spacetime N, and h_{ab} is the metric on the string M, with covariant derivative $\mathscr{D}$ and area form ν. This string theory is different to the preceding spacetime theories in that the dynamical arena M is the string itself, and the variables x which are usually interpreted as spacetime coordinates are just fields on the string.

For dual dynamics, S is just a point, and the evolution vectors u and $v \in \mathrm{CTM}$ provide a basis (u, v) for the string, in terms of which the string metric $h \in \mathrm{CBM}$ can be decomposed as

$$h=\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Thus the configuration bundle is $\mathrm{Q}=\mathbf{R}^3 \oplus \mathrm{V}$, where V is a vector space with the same dimension as spacetime, and the configuration fields are $q=(a, b, c;\, x) \in \mathrm{Q}$. The dual Lagrangian $\mathscr{L}$ should satisfy

$$\mathrm{I}=\int_{\mathscr{V}}\int_{\mathscr{U}} \mathscr{L}\, d\xi\, d\eta,$$

where $u=\partial/\partial\xi$ and $v=\partial/\partial\eta$. Since $\nu=\sqrt{b^2-ac}\, d\xi\, d\eta$, the Lagrangian $\mathscr{L}:(\mathrm{TQ})^2 \to \mathbf{R}$ is found to be

$$\mathscr{L}=\frac{c x^+ . x^+_\flat - 2 b x^+ . x^-_\flat + a x^- . x^-_\flat}{2\sqrt{b^2-ac}},$$

where the flat ($^\flat$) denotes index lowering with the spacetime metric $g \in \mathrm{CBN}$.

The independent momenta are

$$\bar{x}=\frac{c x^+_\flat - b x^-_\flat}{\sqrt{b^2-ac}} \in \mathrm{V}^*, \qquad \hat{x}=\frac{-b x^+_\flat + a x^-_\flat}{\sqrt{b^2-ac}} \in \mathrm{V}^*,$$

where V* is the dual of V, and the constrained momenta are

$$\bar{a}=\hat{a}=\bar{b}=\hat{b}=\bar{c}=\hat{c}=0,$$

so that a, b and c are gauge variables. The reduced Hamiltonian $\mathscr{H}:\Lambda((\mathrm{T}^*\mathrm{Q})^2) \to \mathbf{R}$ is of the form

$$\mathscr{H}=\bar{x}.x^+ + \hat{x}.x^- - \mathscr{L},$$

and the reduced Hamilton equations are of the form

$$\frac{\partial x}{\partial \xi}=\frac{\delta \mathscr{H}}{\delta \bar{x}}, \qquad \frac{\partial x}{\partial \eta}=\frac{\delta \mathscr{H}}{\partial \hat{x}},$$
$$\frac{\partial \bar{x}}{\partial \xi}+\frac{\partial \hat{x}}{\partial \eta}=-\frac{\delta \mathscr{H}}{\delta x}, \qquad 0=\frac{\delta \mathscr{H}}{\delta a}=\frac{\delta \mathscr{H}}{\delta b}=\frac{\delta \mathscr{H}}{\delta c}.$$

Explicitly, the Hamiltonian is

$$\mathscr{H}=-\frac{a\bar{x}.\bar{x}^{\#}+2b\bar{x}.\hat{x}^{\#}+c\hat{x}.\hat{x}^{\#}}{2\sqrt{b^2-ac}},$$

where the sharp (#) denotes index raising with the inverse metric $g^{-1}\in \mathrm{CB^*\,N}$; and the Hamilton equations are

$$\frac{\partial x}{\partial \xi}=-\frac{a\bar{x}^{\#}+b\hat{x}^{\#}}{\sqrt{b^2-ac}}, \quad \frac{\partial x}{\partial \eta}=-\frac{b\bar{x}^{\#}+c\hat{x}^{\#}}{\sqrt{b^2-ac}}, \quad \frac{\partial \bar{x}}{\partial \xi}+\frac{\partial \hat{x}}{\partial \eta}=0,$$
$$0=b\bar{x}.\bar{x}^{\#}+c\bar{x}.\hat{x}^{\#}=a\bar{x}.\hat{x}^{\#}+b\hat{x}.\hat{x}^{\#},$$

where only two of the three original constraints are independent, indicating that one combination of a, b and c is irrelevant to the dynamics, corresponding to the Weyl symmetry of the string.

The integrability condition

$$\frac{\partial^2 x}{\partial \xi\, \partial \eta}=\frac{\partial^2 x}{\partial \eta\, \partial \xi}$$

is in general rather lengthy, involving all ten velocity fields. However, the cordinates may now be fixed by taking

$$a=c=0,$$

which is dual-null in all the senses of § 3: it means that the evolution directions u and v are null in spacetime; it means that x explicitly satisfies the wave equation

$$\frac{\partial^2 x}{\partial \xi\, \partial \eta}=0;$$

and it means that the integrability condition is invertible and simplifies dramatically to

$$\frac{\partial \bar{x}}{\partial \xi}=\frac{\partial \hat{x}}{\partial \eta}.$$

Thus the dual-null equations are simply

$$\frac{\partial x}{\partial \xi}=-\hat{x}^{\#}, \qquad \frac{\partial x}{\partial \eta}=-\bar{x}^{\#},$$

$$\frac{\partial \bar{x}}{\partial \xi} = \frac{\partial \hat{x}}{\partial \eta} = 0, \qquad 0 = \bar{x} . \bar{x}^{\#} = \hat{x} . \hat{x}^{\#},$$

and it is to be noted that b has disappeared from the equations. Apart from the constraints, which indicate that S^+ and S^- are null is spacetime, this is just the first-order dual-null form of the wave equation for x, this being the archetypical hyperbolic equation. The initial data are x on S_0, $\hat{x}$ on S^+ satisfying $\hat{x} . \hat{x}^{\#} = 0$, and $\bar{x}$ on S^- satisfying $\bar{x} . \bar{x}^{\#} = 0$. Interpreting the variables, $\bar{x}$ and $\hat{x}$ represent the wave profiles moving in either direction on the string, which propagate unchanged, with the general solution being just a sum of the initial data. This may be compared favourably with the temporal formulation. The main lesson is how the evolution problem simplifies in the dual-null case, which also occurs in Relativity.

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