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## Mass generation for an interface in the mean field regime: Addendum.

by

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ABSTRACT. – We fill a gap in the proof of the main theorem in [DMR], and give some extensions of that theorem to more general potentials.

RÉSUMÉ. – Nous comblons une lacune dans la preuve du théorème principal de [DMR] et nous étendons son domaine de validité à des potentiels plus généraux.

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In [DMRR] and [DMR] we studied solid-on-solid models of interfaces corresponding mathematically to two dimensional massless field theories with an ultraviolet cutoff and a small attractive potential. Physically the massless Gaussian measure can mimic the surface tension and the potential an attracting wall, as occur e.g. in the problem of wetting. We considered the case of an even potential monotonous on  $\mathbb{R}^+$ . In [DMRR] we proved that on the lattice for any even monotonous non-constant such potential, the mean value of the field (interface height) at the origin is bounded in the thermodynamic limit. We used a large/small field analysis and correlation inequalities.

In [DMR] we considered the question of the exponential decrease of correlation functions (mass generation), using a cluster expansion together with a large/small field analysis. We stated this exponential decrease for the potential

$$V(\phi) = -\varepsilon \left[ e^{-\frac{\phi^2}{2a^2}} - 1 \right] \quad (1)$$

with  $a$  and  $\varepsilon$  both positive, in a certain regime of parameters  $a$  and  $\varepsilon$  in which the generated mass is well approximated by the curvature at the bottom of the potential. The method of cluster expansion allows a detailed estimate of the generated mass and clearly applies to smooth but not necessarily monotonous potentials and is not restricted to lattice cutoffs. However there is a gap in the proof of the theorem stated in [DMR] and the first purpose of this note is to fill this gap.

We also take advantage of this note to state that for a large class of monotonous even non-constant potential, mass generation can be proven by correlation inequalities methods combined with large/small field analysis. For instance one can combine the results of [BFS] and [DMRR] to get exponential decrease for any smooth *strictly* monotonous even potential\*. Indeed in [BFS] exponential decrease of correlation functions was proven for lattice models with strictly monotonous potentials which grow at infinity at least as

$$\alpha \log |\phi| \quad \text{with } \alpha > 1 \quad (2)$$

(so that  $\int e^{-V(\phi)} d\phi < \infty$  (condition (b) on page 137 of [BFS])). These potentials which grow at infinity are not suitable for the modelization of the attraction of a wall. However reading carefully [BFS] one finds that using the random walk representation [BFS] actually proves mass generation for any strictly monotonous potential for which one can by other means prove that the value of the field at the origin is not almost surely infinity in the thermodynamic limit. But the results of [DMRR] imply this last fact!

In fact using other correlation inequalities we can extend the proof of mass generation to a class of potentials which are not *strictly* monotonous. This is shown in part II of this note.

## I. THE RESULT OF [DMR] REVISITED

The proof of the main theorem (theorem II.1) in [DMR] is incomplete. On pages 351 and 352 it is explained how “large field squares) are rare in probability. This is correct but it is assumed without proof that the small probabilistic factors that one obtains for a set of “large field squares” are independent so that the probabilistic factor for a set of such cubes is essentially the small probabilistic factor for a single square to the power the number of squares.

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\* We thank J. Fröhlich for a remark which lead us to realize this fact.

This assumption is in fact subtle and although it is basically correct it is technically not obvious to justify it. The difficult case is when the small probabilistic factor does not come from the potential but from anomalously large gradients of the field. In that case we concluded that the best way to prove the approximate factorization of the small factors associated to each large field square is to use the fact that the massless propagator with many regularizing gradients has good decay properties. But to use this fact one has to reconstruct this massless propagator deep inside the large field regions, and this leads to a somewhat non-trivial restructuring of the expansion of [DMR] itself.

Therefore we have given a complete proof of a slightly revised version of the main theorem of [DMR] in [R]. But for completeness we state here this slightly revised theorem and give an outline of its proof.

A continuous solid-on-solid model of interface is a two dimensional field theory with ultraviolet cutoff. The field represents the interface height. To model the surface tension we consider a massless Gaussian measure in a finite volume  $\Lambda = [-L, L]^2$ . This massless Gaussian measure is formally proportional to

$$e^{-(1/2) \int (\nabla \phi)^2(x)} \prod_x d\phi(x), \tag{I.1}$$

However such an expression, even with an ultraviolet cutoff, is ill-defined since it is invariant under global translation of the variables  $\{\phi(x), x \in \Lambda\}$ . To have a true measure we must break this global invariance, using some kind of boundary condition at the border of  $\Lambda$ . A particularly convenient choice is to use free boundary conditions on the massive lattice propagator  $C$  defined by

$$C(x, y) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \frac{e^{ik \cdot (x-y)}}{m^2 + 2(2 - \cos k_1 - \cos k_2)} \tag{I.2}$$

(An other type of ultraviolet cutoff is used in [R], but this has no effect on the results). The value of the mass  $m$  is fixed to precisely the curvature expected at the minimum of our potential. We then make this propagator massless inside  $\Lambda$  by insertion of the suitable “mass counterterm”;

$$e^{+(1/2) \int_{x \in \Lambda} m^2 \phi^2(x)}. \tag{I.3}$$

This boundary condition is well adapted to cluster expansions. (Any set of bounded boundary conditions would here lead to the same thermodynamic limit, but in a less convenient way.)

Therefore we define:

$$d\mu_\Lambda = \frac{1}{Z_\Lambda} e^{+(1/2) \int_\Lambda m^2 \phi^2(x)} d\mu \quad (I.4)$$

where  $d\mu$  is the normalized measure with propagator  $C$  (this measure can be defined directly in the infinite volume limit). (An estimate of  $Z_\Lambda$  is given in [DMR], Lemma II.3).

By an easy Gaussian computation the mean value  $\langle \phi(x)^2 \rangle$  at any fixed site  $x$  diverges logarithmically as  $\Lambda \rightarrow \infty$ , *i.e.* as the thermodynamic limit is performed. We add now the small interacting potential (1). It tends to a constant when  $\phi^2 \rightarrow \infty$  and tends to confine  $\phi$  near 0.

We define the normalized measure:

$$d\mu_{V,\Lambda} = \frac{1}{Z_{V,\Lambda}} d\mu_\Lambda e^{-\int_\Lambda V(\phi(x))} \quad (I.5)$$

using the notation  $\langle \cdot \rangle_{V,\Lambda}$  for the expectation value with respect to this measure.

The regime of parameters which we study is  $a \gg 1$  and  $\varepsilon/a^2 \ll 1$ . We want to prove that in this regime the two point function decreases exponentially and to obtain an estimate of the corresponding mass gap. More precisely we prove:

**THEOREM I.** – *Let  $\Lambda = [-L, L]^2$ , and let  $\{\phi(x)\}_{x \in \Lambda}$  be distributed according to the probability measure (B.6b), *i.e.* the measure*

$$\frac{1}{Z_{V,\Lambda}} e^{+(1/2) \int_\Lambda m^2 \phi^2(x)} e^{-\int_\Lambda V(\phi(x))} d\mu(\phi) \quad (I.6)$$

where  $d\mu$  is the Gaussian measure of covariance  $C(x, y)$  given by (I.2),  $V(\phi)$  is given by (1) and  $m = \sqrt{\varepsilon}/a$ . Assume  $0 < \varepsilon \leq 1$ . We also assume that the parameters  $a$  and  $\varepsilon$  satisfy

$$K \cdot \log(1 + \varepsilon^{-1}) < a^{(1/2)-\eta} \quad (I.7)$$

where  $K$  is a sufficiently large constant and  $\eta$  is some fixed number strictly positive (this means that  $a$  is always large and that if  $\varepsilon \rightarrow 0$ ,  $a \rightarrow \infty$  in a certain way). Under these conditions the thermodynamic limit of the correlation functions exists and satisfy an exponential clustering property (the truncated correlation functions decrease exponentially). The decay rate or effective mass is close to the decay rate  $m = \sqrt{\varepsilon}/a$  of  $C$  when  $a$  is large. More precisely there exist positive constants  $K$  and  $c$  such that:

$$\lim_{\Lambda \rightarrow \mathbb{R}^2} \langle \phi(x) \phi(y) \rangle_{V,\Lambda} \leq K \log(a/\sqrt{\varepsilon}) e^{-c\sqrt{\varepsilon}|x-y|/a}. \quad (I.8)$$

Furthermore one can let  $c$  tends to 1 (uniformly in  $\varepsilon$ ) if  $a \rightarrow \infty$ .

The technical condition (I.7) restricts the model to the so-called “mean field regime” of wetting (in this regime the mass is very close to the value of the curvature at the bottom of the well, namely  $\sqrt{\varepsilon}/a$ . In [DMR] the same theorem was stated directly with  $\eta = 0$ . However in the corrected proof ([R]), the natural result is to take  $\eta > 0$  (or to take  $\eta = 0$  in (I.7) but to divide by a large power of  $\log a$ ). It seems to us that the theorem holds probably also for  $\eta = 0$  but that its proof presumably requires some heavier analysis which is perhaps not worth the trouble. A more interesting task would be to treat a model with a regime which is not mean-field, and then one certainly needs a multiscale analysis\*.

Following [R] we summarize now how to correct the proof of [DMR]. We analyze the theory with respect to a lattice  $\mathbf{D}$  which is a regular paving of  $\Lambda$  by squares  $\Delta$  of side  $l = a/\sqrt{\varepsilon}$ , namely the inverse of the expected mass. In the squares where the average value of  $\phi^4$  is less than  $a^2$ , which we call the small field region, the quadratic approximation to the potential which gives a mass  $m = \sqrt{\varepsilon}/a$  is valid, and the analysis of this region by a cluster expansion essentially follows [DMR]. However in the large field region in order to prove that the small probabilistic factors for each cube are nearly independent, we change the conditions A)B)C) on page 351 in [DMR], showing that either the potential itself gives the small factors (in which case they are obviously independent), or some norm of a high order derivative of the field (interface height) is anomalously large. Now the massless Gaussian propagator when regularized by sufficiently many derivatives has rather large power decay. Therefore one can prove by explicit Gaussian integration on these norms of high order derivatives an approximate factorization property. This proves that for a large field region of  $N$  cubes one really has an associated small factor to the power  $N$ , the missing element in [DMR].

To implement this simple idea there are however some technical complications described in [R]. First the use of high order derivatives (instead of the “ordinary derivative” like in [DMR] case C) page 351) creates somewhat longer formulas. More seriously to patch the analysis between the small field region where the measure is essentially massive, and the large field region where apart from a boundary condition it is essentially massless, one has to add some technicalities to the analysis of [DMR]. Essentially in [DMR] we introduced a corridor around the outside of the large field regions to test their couplings through the small field

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\* For a model with two linear exponentials (for which renormalization reduces to some explicit Wick ordering) we are presently attempting this multiscale analysis.

regions. In [R] we have to add a second corridor around the “inside” of the large field regions. Indeed to reconstruct the massless Gaussian measure deeply inside the large field region one needs to show that the effect of the boundary condition at its border is small, and this is done by some additional expansion which tests the effect of this boundary condition through this “inner” corridor and shows that deep inside the large field region this effect is small.

We refer the reader further interested by these technical aspects to [R].

## II. GENERATION OF MASS WITH LARGE/SMALL FIELD ANALYSIS PLUS CORRELATION INEQUALITIES.

We shall discuss here potentials of the following three types:

$$\left. \begin{aligned} U_{\varepsilon, a}(\phi) &= -\varepsilon & \text{if } |\phi| < a; \\ U_{\varepsilon, a}(\phi) &= 0 & \text{if } |\phi| \geq a, \end{aligned} \right\} \quad (\text{II.1})$$

$$V_{\varepsilon, a}(\phi) = -\varepsilon e^{-\frac{\phi^2}{2a^2}}, \quad (\text{II.2})$$

$$\left. \begin{aligned} W_{\varepsilon, a}(\phi) &= -\frac{\varepsilon}{4} e^{-\frac{2\phi^2}{a^2-\phi^2}} & \text{if } |\phi| < a; \\ W_{\varepsilon, a}(\phi) &= 0 & \text{if } |\phi| \geq a. \end{aligned} \right\} \quad (\text{II.3})$$

The analysis of [DMRR] can be applied to all these potentials for arbitrary  $\varepsilon > 0$  and  $a > 0$ , proving

$$\sup \langle |\phi(x)| \rangle_{\varepsilon, a} \leq 3a + K |\log(\bar{a}\bar{\varepsilon})|, \quad (\text{II.4})$$

where  $\bar{a} = \inf(a, 1)$  and  $\bar{\varepsilon} = \inf\left(\varepsilon, \frac{1}{2}\right)$ . Theorem 1 of this paper applies to  $V_{\varepsilon, a}(\phi)$  when condition (I.7) is satisfied, *i.e.*  $K \log(1 + \varepsilon^{-1}) < a^{\frac{1}{2}-\eta}$ . Theorem 1 could be extended to  $W_{\varepsilon, a}(\phi)$ , under the same hypothesis and with the same conclusion, namely

$$\langle \phi(x)\phi(y) \rangle_{\varepsilon, a}^W \leq K \log\left(\frac{a}{\sqrt{\varepsilon}}\right) e^{-c\sqrt{\varepsilon}|x-y|/a}. \quad (\text{II.5})$$

We shall now derive some comparison inequalities between expectations  $\langle \phi(x)\phi(y) \rangle_{\varepsilon, a}$  for different choices of  $\varepsilon, a$  and  $U_{\varepsilon, a}, V_{\varepsilon, a}$  or  $W_{\varepsilon, a}$ .

LEMMA II.1. – *Suppose  $b \geq a$  and  $\eta/b^2 \geq \varepsilon/a^2$ . Then*

$$\langle \phi(x)\phi(y) \rangle_{\varepsilon, a}^V \geq \langle \phi(x)\phi(y) \rangle_{\eta, b}^V \quad (\text{II.6})$$

and

$$\langle \phi(x) \phi(y) \rangle_{\varepsilon, a}^W \geq \langle \phi(x) \phi(y) \rangle_{\eta, b}^W. \tag{II.7}$$

*Proof.* – Let us first consider the potential  $V$ . It suffices to prove for all  $p, q$  odd that

$$\int_0^{+\infty} d\phi \int_0^\phi d\phi' (\phi + \phi')^p (\phi - \phi')^q e^{-4\phi^2 - 4(\phi')^2} \times (e^{-V_{\varepsilon, a}(\phi) - V_{\eta, b}(\phi')} - e^{-V_{\varepsilon, a}(\phi') - V_{\eta, b}(\phi)}) \geq 0, \tag{II.8}$$

which is true because  $V_{\varepsilon, a}(\phi) - V_{\eta, b}(\phi)$  is a decreasing function of  $\phi$  under the hypothesis of the lemma.

The same applies to  $W$ , but some analysis is required to check that  $W_{\varepsilon, a}(\phi) - W_{\eta, b}(\phi)$  is a decreasing function of  $\phi$ . We write

$$W_{\varepsilon, a}(\phi) = -\frac{\varepsilon}{4} e^2 \exp\left(-\frac{2}{1 - a^{-2}\phi^2}\right) \tag{II.9}$$

and consider  $f(\alpha, x) = -\varepsilon(\alpha) \exp\left(-\frac{\lambda}{1 - \alpha x}\right)$ . We compute

$$\frac{\partial^2 f}{\partial \alpha \partial x} = \frac{\lambda \exp\left(-\frac{\lambda}{1 - \alpha x}\right)}{(1 - \alpha x)^4} \cdot ((\varepsilon(\alpha) + \alpha \varepsilon'(\alpha))(1 - \alpha x)^2 + (2 - \lambda)\varepsilon(\alpha)\alpha x - 2\varepsilon(\alpha)\alpha^2 x^2) \tag{II.10}$$

which is negative for all  $x \in [0, \alpha^{-1}]$  if we choose  $\varepsilon(\alpha) + \alpha \varepsilon'(\alpha) = 0$  and  $\lambda = 2$ . Then

$$W_{\varepsilon(a^{-2}), a}(\phi) - W_{\varepsilon(b^{-2}), b}(\phi) = \frac{\varepsilon^2}{4} \int_{b^{-2}}^{a^{-2}} d\alpha \cdot \frac{\partial f}{\partial \alpha}(\alpha, \phi^2) \tag{II.11}$$

is a decreasing function of  $\phi^2$  if  $b \geq a$ . To compare  $\varepsilon(a^{-2})$  and  $\varepsilon(b^{-2})$ , we use  $\varepsilon(\alpha) + \alpha \varepsilon'(\alpha) = 0$ , which gives  $a^{-2} \varepsilon(a^{-2}) = b^{-2} \varepsilon(b^{-2})$ . The proof of Lemma II.1 is then concluded by noting that  $\langle \phi(x) \phi(y) \rangle_{\varepsilon, a}^W$  is a decreasing function of  $\varepsilon$  (this is true also with  $U$  or  $V$ ). ■

The next lemma, which is clearly not optimal, bounds a correlation function obtained with a square well potential  $U$  by that obtained with a smooth potential  $W$ :

LEMMA II.2. – Let  $\eta = \varepsilon e^{-4a^2}$ . Then

$$\langle \phi(x) \phi(y) \rangle_{\varepsilon, a}^U \leq \langle \phi(x) \phi(y) \rangle_{\eta, a}^W. \quad (\text{II.12})$$

*Proof.* – It suffices to prove that

$$\begin{aligned} I &= \int_0^{+\infty} d\phi \int_0^{+\infty} d\phi' (\phi + \phi')^p (\phi - \phi')^q \\ &\quad \times e^{-4\phi^2 - 4(\phi')^2} e^{-W(\phi) - U(\phi')} \geq 0 \end{aligned} \quad (\text{II.13})$$

for all  $p, q$  odd. Using the definition of  $U$  and  $W$ , this integral reduce to

$$\begin{aligned} I &= I_1 + I_2 \\ I_1 &= \int_a^{+\infty} d\phi \int_0^a d\phi' (\phi + \phi')^p (\phi - \phi')^q e^{-4\phi^2 - 4(\phi')^2} \\ &\quad \times (-e^\varepsilon + e^{-W(\phi')}) \\ I_2 &= e^\eta \int_0^a d\phi \int_0^\phi d\phi' (\phi + \phi')^p (\phi - \phi')^q e^{-4\phi^2 - 4(\phi')^2} \\ &\quad \times (e^{-W(\phi)} - e^{-W(\phi')}) \end{aligned}$$

Using  $-\eta \leq W(\phi) \leq 0$ , we have

$$\begin{aligned} I_1 &\geq (e^\varepsilon - e^\eta) \int_a^{+\infty} d\phi \int_0^a d\phi' (\phi + \phi')^p (\phi - \phi')^q e^{-4\phi^2 - 4(\phi')^2} \\ I_2 &\geq -e^\varepsilon (e^\eta - 1) \int_0^a d\phi \int_0^\phi d\phi' (\phi + \phi')^p (\phi - \phi')^q e^{-4\phi^2 - 4(\phi')^2} \end{aligned}$$

The worst case is for  $p = q = 1$ , where we find by explicit integration

$$\int_a^{+\infty} d\phi \int_0^a d\phi' (\phi + \phi') (\phi - \phi') e^{-4\phi^2 - 4(\phi')^2} = \frac{\sqrt{\pi}}{32} a e^{-4a^2},$$

and

$$\begin{aligned} &\int_0^a d\phi \int_0^\phi d\phi' (\phi + \phi') (\phi - \phi') e^{-4\phi^2 - 4(\phi')^2} \\ &\leq \int_0^\infty d\phi \int_0^\phi d\phi' (\phi + \phi') (\phi - \phi') e^{-4\phi^2 - 4(\phi')^2} = \frac{1}{64}, \end{aligned}$$

which proves the lemma. ■

We conclude that we can prove exponential decay for some non-smooth, non-strictly monotonous potentials, such as the square well potentials  $U_{\varepsilon, a}$  satisfying  $K \log(1 + \varepsilon^{-1} e^{4a^2}) < a^{(1/2)-\eta}$ . However remark that these wells have to deeper and deeper as  $a \rightarrow \infty$ . Therefore we consider that it is still an open problem to prove or disprove exponential decay of the interface correlation functions for any square well potential.

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