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Instability of standing waves for the generalized Davey-Stewartson system

by

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ABSTRACT. – In this paper we study the instability of standing wave $u_\omega(t, x) = e^{i\omega t}\varphi_\omega(x)$ for the following equation:

$$iu_t + \Delta u + a\,|\,u\,|^{p-1}u + E_1(|\,u\,|^2)\,u = 0, \qquad t \geqq 0, \quad x \in \mathbb{R}^n,$$

where $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3. We prove that if $p \geqq 1 + 4/n$, then u_ω is unstable for any $\omega \in (0, \infty)$. This is an extension of the previous result due to Cipolatti [4], who proved that if $p \geqq 3$, then u_ω is unstable for any $\omega \in (0, \infty)$. We show a different criterion of instability from that in [4], which can cover the case of $1 + 4/3 \leqq p < 3$ for $n = 3$. Furthermore, we prove that if $n = 3$ and $1 < p < 1 + 4/3$, then there exists $\omega_0 = \omega_0(a, p) > 0$ such that u_ω is unstable for any $\omega \in (\omega_0, \infty)$.

RÉSUMÉ. – Dans cet article on étudie l'instabilité de la solution $u_\omega(t, x) = e^{i\omega t}\varphi_\omega(x)$ pour l'équation suivante:

$$iu_t + \Delta u + a\,|\,u\,|^{p-1}u + E_1(|\,u\,|^2)\,u = 0, \qquad t \geqq 0, \quad x \in \mathbb{R}^n,$$

où $a > 0$, $1 < p < 1 + 4/(n-2)$ et $n = 2$ ou 3. On prouve que si $p \geqq 1 + 4/n$, u_ω est instable pour tout $\omega \in (0, \infty)$. C'est une extension du résultat précédent par Cipolatti [4], qui prouva que si $p \geqq 3$, u_ω est instable pour tout $\omega \in (0, \infty)$. On montre un critère d'instabilité différente de celui de [4], qui peut couvrir le cas de $1 + 4/3 \leqq p < 3$ pour $n = 3$. En outre, on prouve que si $n = 3$ et $1 < p < 1 + 4/3$, il existe $\omega_0 = \omega_0(a, p) > 0$ telle que u_ω est instable pour tout $\omega \in (\omega_0, \infty)$.

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1. INTRODUCTION AND RESULTS

In the present paper we consider the instability of standing waves for the following nonlinear Schrödinger equation:

$$iu_t + \Delta u + a\,|\,u\,|^{p-1}\,u + E_1\,(|\,u\,|^2)\,u = 0, \qquad t \geqq 0, \quad x \in \mathbb{R}^n, \qquad (1.1)$$

where $a > 0$, $1 < p < 1 + 4/(n-2)$, $n = 2$ or 3 and E_1 is the singular integral operator with symbol $\sigma_1\,(\xi) = \xi_1^2/|\,\xi\,|^2$, $\xi \in \mathbb{R}^n$.

The equation (1.1) has its origin in fluid mechanics where, for $n = 2$ and $p = 3$, it describes the evolution of weakly nonlinear water waves that travel predominantly in one direction. More precisely, (1.1) is the n-dimensional extension of the generalized Davey-Stewartson system in the elliptic-elliptic case, namely

$$\left.\begin{array}{c} iu_t + \lambda\,u_{xx} + u_{yy} + a\,|\,u\,|^{p-1}\,u + uv_x = 0, \\ v_{xx} + \mu\,v_{yy} = (|\,u\,|^2)_x, \end{array}\right\}$$

where λ, $\mu > 0$ (*see* [5]).

By a standing wave, we mean a solution of (1.1) with the form

$$u\,(t,\,x) = e^{i\omega t}\,\varphi_\omega\,(x),$$

where $\omega > 0$ and φ_ω is a ground state of the following stationary problem:

$$\left.\begin{array}{c} -\Delta\psi + \omega\psi - a\,|\,\psi\,|^{p-1}\,\psi - E_1\,(|\,\psi\,|^2)\,\psi = 0, \quad x \in \mathbb{R}^n, \\ \psi \in H^1\,(\mathbb{R}^n), \quad \psi \not\equiv 0. \end{array}\right\} \qquad (1.2_\omega)$$

DEFINITION 1. – We define the following notations.

$$S_\omega\,(v) = \frac{1}{2}\,|\,\nabla\,v\,|_2^2 + \frac{\omega}{2}\,|\,v\,|_2^2 - \frac{a}{p+1}\,|\,v\,|_{p+1}^{p+1} - \frac{1}{4}\int |\,v\,|^2\,E_1\,(|\,v\,|^2)\,dx,$$

$$\begin{aligned} \chi_\omega &= \text{the set of solutions for } (1.2_\omega) \\ &= \{\psi \in H^1\,(\mathbb{R}^n) : S'_\omega\,(\psi) = 0, \quad \psi \neq 0\}, \\ \mathcal{G}_\omega &= \text{the set of ground states for } (1.2_\omega) \\ &= \{\varphi \in \chi_\omega : S_\omega\,(\varphi) \leqq S_\omega\,(\psi) \text{ for all } \psi \in \chi_\omega\}, \end{aligned}$$

$$(\tau_y\,v)\,(x) = v\,(x-y), \qquad v \in H^1\,(\mathbb{R}^n), \quad x,\,y \in \mathbb{R}^n.$$

Remark 1. – Cipolatti [3] showed that if $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3, then $\mathcal{G}_\omega$ is not empty for any $\omega \in (0,\,\infty)$.

DEFINITION 2. – We shall say that the standing wave $u_\omega\,(t) = e^{i\omega t}\,\varphi_\omega$ is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

If $u_0 \in H^1(\mathbb{R}^n)$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $\| u_0 - \varphi_\omega \|_{H^1} < \delta$, then

$$\sup_{0 \leqq t < \infty} \inf_{\theta \in \mathbb{R},\, y \in \mathbb{R}^n} \| u(t) - e^{i\theta} \tau_y \varphi_\omega \|_{H^1} < \varepsilon.$$

Otherwise, u_ω is said to be *unstable.*

Remark 2. – The unique local existence of H^1 solution for (1.1) was established by Ghidaglia and Saut [6]: If $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3, then for any $u_0 \in H^1(\mathbb{R}^n)$ there exist $T > 0$ and a unique solution $u(\cdot) \in C([0, T); H^1(\mathbb{R}^n))$ of (1.1) with $u(0) = u_0$. Furthermore, $u(t)$ satisfies:

$$| u(t) |_2 = | u_0 |_2, \tag{1.3}$$

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0), \tag{1.4}$$

for all $t \in [0, T)$, where $\mathcal{E}$ is defined on $H^1(\mathbb{R}^n)$ by

$$\mathcal{E}(v) = \frac{1}{2} | \nabla v |_2^2 - \frac{a}{p+1} | v |_{p+1}^{p+1} - \frac{1}{4} \int | v |^2 E_1(| v |^2)\, dx.$$

Cipolatti [4] has proved that if $a > 0$, $3 \leqq p < 1 + 4/(n-2)$ and $n = 2$ or 3, then the standing wave $e^{i\omega t} \varphi_\omega$ is unstable for any $\omega \in (0, \infty)$. He has applied a sufficient condition for the instability essentially due to Gonçalves Ribeiro [7], and has constructed the unstable flow by using the Pohozaev multiplier $x \cdot \nabla \varphi_\omega$. For $p \geqq 3$, the sufficient condition of instability in [4] is satisfied. On the other hand, under the assumption that the ground state of (1.2_ω) is unique, up to a translation and a phase change, the author [8] showed that if $a > 0$, $1 < p < 1 + 4/n$ and $n = 2$ or 3, then there exists a sequence (ω_k) such that $\omega_k > 0$, $\omega_k \to 0$ and $e^{i\omega_k t} \varphi_{\omega_k}$ is stable. When $n = 3$, the result due to Cipolatti [4] is not optimal, that is, we obtain the following results.

THEOREM 1. – *If $a > 0$, $1 + 4/n \leqq p < 1 + 4/(n-2)$ and $n = 2$ or 3, then the standing wave $e^{i\omega t} \varphi_\omega$ is unstable for any $\omega \in (0, \infty)$.*

THEOREM 2. – *If $a > 0$, $n = 3$ and $1 < p < 1 + 4/3$, then there exists $\omega_0 = \omega_0(a, p) > 0$ such that $e^{i\omega t} \varphi_\omega$ is unstable for any $\omega \in (\omega_0, \infty)$.*

Remark 3. – As stated above, if $a > 0$, $1 < p < 1 + 4/n$ and $n = 2$ or 3, then there exist stable standing waves for ω close to 0 (*see* [8]). Thus, it is natural that the exponent $p = 1 + 4/n$ should appear in Theorems 1 and 2.

Remark 4. – The exponent $p = 1 + 4/n$ is the critical one for the case of the single power nonlinearity:

$$iu_t + \Delta u + | u |^{p-1} u = 0, \qquad t \geqq 0, \quad x \in \mathbb{R}^n.$$

That is, it is well known that if $1 < p < 1 + 4/n$, then all standing waves are stable, and if $1 + 4/n \leqq p < 1 + 4/(n-2)$, then all standing waves are unstable (*see* [1], [2], [10]).

This paper is organized as follows. In Section 2 we first state Theorem 3, which gives a sufficient condition for the instability. Next we prove Theorems 1 and 2 by using Theorem 3. In Section 3 we give the proof of Theorem 3. We should mention that the proof of Theorem 3 is based on the ideas of Shatah and Strauss [9]. In particular, *see* Section 4 in [9]. In the proof of Theorem 3, we give the unstable direction explicitly by using the scaling $\varphi_\omega^\lambda(x) = \lambda^{n/2}\varphi_\omega(\lambda x)$, $\lambda > 0$, which associates with the pseudo-conformal identity [*see* (3.4) in Section 3]. In the proof of Theorem 1, we can easily check our sufficient condition $\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} < 0$ by a simple computation. This enables us to remove the restriction of $p \geqq 3$ for $n = 3$, which was assumed in the paper [4] by Cipolatti. In the proof of Theorem 2, we compare the norms of the ground states with the case of $a = 0$ in (1.1) by using the variational characterization of the ground states [*see* Lemma 1 in Section 2]. For the case of $a = 0$, we can use the scaling argument and can estimate the norms of the ground states [*see* (2.11) and under (2.14)]. The analogous method has been used in [8] to show the existence of the stable standing waves.

In what follows, we omit the integral variables with respect to the spatial variable x, and we omit the integral region when it is the whole space $\mathbb{R}^n$. We denote the norms of $L^q(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ by $|\cdot|_q$ and $\|\cdot\|_{H^1}$, respectively.

2. PROOF OF THEOREMS 1 AND 2

In this section we prove Theorems 1 and 2 by using the following Theorem 3, which will be proved in the next section.

THEOREM 3. – *Suppose that* $a > 0$, $1 < p < 1 + 4/(n-2)$ *and* $n = 2$ *or* 3. *If* $\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, *then the standing wave* $e^{i\omega t}\varphi_\omega$ *is unstable, where* $v^\lambda(x) = \lambda^{n/2} v(\lambda x)$, $\lambda > 0$.

Proof of Theorem 1. – A simple computation shows

$$\mathcal{E}(\varphi_\omega^\lambda) = \frac{\lambda^2}{2}|\nabla\varphi_\omega|_2^2 - \frac{a}{p+1}\lambda^{\frac{n}{2}(p-1)}|\varphi_\omega|_{p+1}^{p+1} - \frac{1}{4}\lambda^n \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\,dx, \tag{2.1}$$

$$\partial_\lambda \mathcal{E}(\varphi_\omega^\lambda)\,|_{\lambda=1} = |\nabla \varphi_\omega|_2^2 - \frac{n(p-1)}{2(p+1)}\, a\, |\varphi_\omega|_{p+1}^{p+1} - \frac{n}{4}\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\,dx, \tag{2.2}$$

$$\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)\,|_{\lambda=1} = |\nabla \varphi_\omega|_2^2 - \frac{n(p-1)}{2(p+1)}\left(\frac{n}{2}(p-1)-1\right) a\, |\varphi_\omega|_{p+1}^{p+1} - \frac{n}{4}(n-1)\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\,dx. \tag{2.3}$$

Here, we have used the fact that $E_1(v(\lambda\cdot))(x) = E_1(v)(\lambda x)$, $\lambda > 0$ (*see* [3], Lemma 2.1). Since $|\varphi_\omega^\lambda|_2 = |\varphi_\omega|_2$ and $\varphi_\omega \in \chi_\omega$, we have

$$\partial_\lambda \mathcal{E}(\varphi_\omega^\lambda)\,|_{\lambda=1} = \partial_\lambda S_\omega(\varphi_\omega^\lambda)\,|_{\lambda=1} = \langle S_\omega'(\varphi_\omega),\, \partial_\lambda \varphi_\omega^\lambda\,|_{\lambda=1}\rangle = 0. \tag{2.4}$$

Substituting (2.2) and (2.4) into (2.3), we have

$$\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)\,|_{\lambda=1} = \left(\frac{n}{2}\right)^2 \frac{p-1}{p+1}\left(1+\frac{4}{n}-p\right) a\, |\varphi_\omega|_{p+1}^{p+1} + \frac{n}{4}(2-n)\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\,dx. \tag{2.5}$$

From the definition of E_1 and the Parseval identity, we have

$$\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\,dx = \int |\varphi_\omega|^2 \mathcal{F}^{-1}\sigma_1(\xi)\,\mathcal{F}(|\varphi_\omega|^2)\,dx = \int \sigma_1(\xi)\,|\mathcal{F}(|\varphi_\omega|^2)|^2\,d\xi > 0,$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$. Therefore, from (2.5), we have

$$\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)\,|_{\lambda=1} < 0, \tag{2.6}$$

if $n = 2$ and $p > 1 + 4/n$ or $n = 3$ and $p \geqq 1 + 4/n$. The proof follows from (2.6) and Theorem 3, except the case $n = 2$ and $p = 1 + 4/n$. For this special case, *see* Proposition 3.16 in [4]. □

Before giving the proof of Theorem 2, we make some preparations.

DEFINITION 3. – For any $v \in H^1(\mathbb{R}^n)$, we define that

$$K_\omega(v) = S_\omega(v) - \frac{1}{n}\,|\nabla v|_2^2$$
$$= \left(\frac{1}{2} - \frac{1}{n}\right)|\nabla v|_2^2 + \frac{\omega}{2}\,|v|_2^2$$
$$- \frac{a}{p+1}\,|v|_{p+1}^{p+1} - \frac{1}{4}\int |v|^2 E_1(|v|^2)\,dx,$$
$$P(v) = |\nabla v|_2^2 - \frac{n(p-1)}{2(p+1)}\,a\,|v|_{p+1}^{p+1} - \frac{n}{4}\int |v|^2 E_1(|v|^2)\,dx.$$

Remark 5. – The functional K_ω is called the Pohozaev functional. Since $K_\omega(v) = \frac{1}{n}\,\partial_\lambda S_\omega(v(\cdot/\lambda))\,|_{\lambda=1}$ for $v \in H^1(\mathbb{R}^n)$, we have $K_\omega(\psi) = 0$ for $\psi \in \chi_\omega$. Moreover, from (2.2), we have $P(v) = \partial_\lambda \mathcal{E}(v^\lambda)\,|_{\lambda=1}$ for $v \in H^1(\mathbb{R}^n)$. The functional P is closely related to the pseudo-conformal conservation law [*see*, e.g., (3.4) in Section 3].

LEMMA 1. – *Assume that* $a > 0$, $1 < p < 1 + 4/(n-2)$ *and* $n = 2$ *or* 3.

(1) *If* $v \in H^1(\mathbb{R}^n)$ *satisfies* $K_\omega(v) < 0$, *then there exists* $\tilde{v} \in H^1(\mathbb{R}^n)$ *such that* $K_\omega(\tilde{v}) = 0$, $\tilde{v} \neq 0$ *and* $|\nabla \tilde{v}|_2 < |\nabla v|_2$,

(2) $K_\omega(\varphi_\omega) = 0$ *and*

$$|\nabla \varphi_\omega|_2^2 = \inf\{\,|\nabla v|_2^2 : v \in H^1(\mathbb{R}^n),\ v \neq 0,\ K_\omega(v) = 0\},$$

(3) $S_\omega(\varphi_\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^n),\ |\nabla v|_2 = |\nabla \varphi_\omega|_2\}$.

Proof. – (1) Let $K_\omega(v) < 0$. For any $\eta > 0$, we have

$$K_\omega(\eta v) = \left(\frac{1}{2} - \frac{1}{n}\right)\eta^2\,|\nabla v|_2^2 + \frac{\omega}{2}\,\eta^2\,|v|_2^2 - \frac{a}{p+1}\,\eta^{p+1}\,|v|_{p+1}^{p+1}$$
$$- \frac{1}{4}\,\eta^4 \int |v|^2 E_1(|v|^2)\,dx.$$

Thus, we have $K_\omega(\eta v) > 0$ for η small enough. Therefore, there exists $\eta_0 \in (0, 1)$ such that $K_\omega(\eta_0 v) = 0$. Taking $\tilde{v} = \eta_0 v$, we have $|\nabla \tilde{v}|_2 = \eta_0 |\nabla v|_2 < |\nabla v|_2$. This implies (1).

(2) *See* Lemma 2.1 in [8].

(3) Consider any $v \in H^1(\mathbb{R}^n)$ such that $|\nabla v|_2 = |\nabla \varphi_\omega|_2$.

Then, it follows from (1) and (2) that $K_\omega(v) \geqq 0$. Thus, we have

$$S_\omega(v) = \frac{1}{n} |\nabla v|_2^2 + K_\omega(v) \geqq \frac{1}{n} |\nabla \varphi_\omega|_2^2 = S_\omega(\varphi_\omega),$$

which implies (3). □

Remark 6. – We note that even if there are many distinct ground states, we have

$$|\nabla \varphi_\omega^{(1)}|_2^2 = n\, S_\omega(\varphi_\omega^{(1)}) = n\, S_\omega(\varphi_\omega^{(2)}) = |\nabla \varphi_\omega^{(2)}|_2^2, \ \varphi_\omega^{(1)},\ \varphi_\omega^{(2)} \in \mathcal{G}_\omega.$$

Proof of Theorem 2. – Let $a > 0$, $n = 3$ and $1 < p < 1 + 4/3$. From (2.5), we have $\partial_\lambda^2 \mathcal{E}(\varphi_\omega^\lambda)|_{\lambda=1} < 0$, provided that

$$\frac{|\varphi_\omega|_{p+1}^{p+1}}{\displaystyle\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\, dx} < \frac{p+1}{3(p-1)(1+4/3-p)\, a}. \tag{2.7}$$

If we have

$$\lim_{\omega \to \infty} \frac{|\varphi_\omega|_{p+1}^{p+1}}{\displaystyle\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\, dx} = 0, \tag{2.8}$$

then there exists $\omega_0 = \omega_0(a, p) > 0$ such that (2.7) holds for any $\omega \in (\omega_0, \infty)$. Thus, the proof follows from Theorem 3. Therefore, it is enough to show (2.8). Let $\tilde\varphi_\omega$ be a ground state of

$$-\Delta \psi + \omega\psi - E_1(|\psi|^2)\psi = 0, \quad x \in \mathbb{R}^n, \tag{2.9}$$

and let

$$\tilde K_\omega(v) = \frac{1}{6} |\nabla v|_2^2 + \frac{\omega}{2} |v|_2^2 - \frac{1}{4} \int |v|^2 E_1(|v|^2)\, dx.$$

From Lemma 1 (1) and (2), we have

$$K_\omega(\tilde\varphi_\omega) < \tilde K_\omega(\tilde\varphi_\omega) = 0 \quad \text{and} \quad |\nabla \varphi_\omega|_2^2 < |\nabla \tilde\varphi_\omega|_2^2. \tag{2.10}$$

We set $\tilde\varphi(x) = (1/\sqrt{\omega})\,\tilde\varphi_\omega(x/\sqrt{\omega})$. Then, $\tilde\varphi$ is a ground state of (2.9) with $\omega = 1$. If we put $d_1 = |\nabla \tilde\varphi|_2^2$, from Remark 6, we have $d_1 = |\nabla \varphi|_2^2$ for all ground states φ of (2.9) with $\omega = 1$. By the change of variables, we also have $|\nabla \tilde\varphi_\omega|_2^2 = d_1 \sqrt{\omega}$. Therefore, from (2.10), we obtain

$$|\nabla \varphi_\omega|_2^2 \leqq d_1 \sqrt{\omega} \quad \text{for any } \omega \in (0, \infty). \tag{2.11}$$

From the definition of P and (2.4), we have $P(\varphi_\omega) = 0$, which together with (2.11) implies

$$|\varphi_\omega|_{p+1}^{p+1} + \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\, dx \leqq C_1 \sqrt{\omega}, \quad \omega \in (0, \infty), \tag{2.12}$$

for some $C_1 = C_1(a, p) > 0$.

Also, from $K_\omega(\varphi_\omega) = 0$ and (2.12), we have

$$\frac{\omega}{2} |\varphi_\omega|_2^2 \leqq \frac{a}{p+1} |\varphi_\omega|_{p+1}^{p+1} + \frac{1}{4} \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\, dx \leqq C_2 \sqrt{\omega}, \quad \omega \in (0, \infty),$$

for some $C_2 = C_2(a, p) > 0$.

Thus, we have

$$|\varphi_\omega|_2^2 \leqq 2 C_2 \omega^{-1/2}, \quad \omega \in (0, \infty). \tag{2.13}$$

Next, we shall estimate $|\nabla \varphi_\omega|_2^2$ from below. Let $\hat{\varphi}_\omega$ be a ground state of

$$-\Delta\psi + \omega\psi - 2 E_1(|\psi|^2)\psi = 0, \quad x \in \mathbb{R}^n,$$

and let

$$\hat{K}_\omega(v) = \frac{1}{6} |\nabla v|_2^2 + \frac{\omega}{2} |v|_2^2 - \frac{1}{2} \int |v|^2 E_1(|v|^2)\, dx.$$

If we have

$$\frac{2a}{p+1} |\varphi_\omega|_{p+1}^{p+1} < \frac{1}{6} |\nabla \varphi_\omega|_2^2 + \frac{\omega}{2} |\varphi_\omega|_2^2, \quad \omega > \omega_1, \tag{2.14}$$

for some $\omega_1 > 0$, then we obtain $\hat{K}_\omega(\varphi_\omega) < 2 K_\omega(\varphi_\omega)$, for $\omega > \omega_1$, and in the same way as (2.11) we can also show that $d_2 \sqrt{\omega} \leqq |\nabla \varphi_\omega|_2^2$ for $\omega > \omega_1$, where $d_2 = |\nabla \hat{\varphi}_1|_2^2$. In fact, from the Gagliardo and Nirenberg inequality, we have

$$\begin{aligned} \int |\varphi_\omega|^{p+1}\, dx &\leqq \int |\varphi_\omega|^{2+4/3}\, dx + \int |\varphi_\omega|^2\, dx \\ &\leqq C \left(\int |\varphi_\omega|^2\, dx \right)^{2/3} \int |\nabla \varphi_\omega|^2\, dx + \int |\varphi_\omega|^2\, dx, \end{aligned} \tag{2.15}$$

for some $C > 0$. It follows from (2.13) and (2.15) that there exists $\omega_1 = \omega_1(a, p) > 0$ such that (2.14) holds for any $\omega \in (\omega_1, \infty)$.

Therefore, we have

$$d_2 \sqrt{\omega} \leqq |\nabla \varphi_\omega|_2^2, \quad \omega \in (\omega_1, \infty). \tag{2.16}$$

Furthermore, from (2.11), (2.13) and (2.15), we have

$$|\varphi_\omega|_{p+1}^{p+1} \leqq C_3 (\omega^{-1/3} \cdot \omega^{1/2} + \omega^{-1/2}) \leqq C_4 \omega^{1/6}, \quad \omega \in (\omega_2, \infty), \tag{2.17}$$

where positive constants ω_2, C_3 and C_4 depend only on a and p.

Also, from (2.16), (2.17) and $P(\varphi_\omega) = 0$, we have

$$\begin{aligned} \frac{3}{4} \int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\,dx &= |\nabla \varphi_\omega|_2^2 - \frac{3(p-1)}{2(p+1)} a |\varphi_\omega|_{p+1}^{p+1} \\ &\geqq C_5 \omega^{1/2} - C_6 \omega^{1/6} \geqq C_7 \omega^{1/2}, \quad \omega \in (\omega_3, \infty), \end{aligned} \tag{2.18}$$

where positive constants ω_3, C_5, C_6 and C_7 depend only on a and p.

From (2.17) and (2.18), we have

$$\frac{|\varphi_\omega|_{p+1}^{p+1}}{\displaystyle\int |\varphi_\omega|^2 E_1(|\varphi_\omega|^2)\,dx} \leqq C_8 \omega^{-1/3}, \quad \omega \in (\omega_3, \infty),$$

for some $C_8 = C_8(a, p) > 0$, which implies (2.8). □

3. PROOF OF THEOREM 3

In this section we first prepare three lemmas and next prove Theorem 3. Throughout this section, we assume that $a > 0$, $1 < p < 1 + 4/(n-2)$ and $n = 2$ or 3. Moreover, since we fix the parameter ω, we drop the subscript ω. Thus, we write φ for φ_ω, S for S_ω, and so on.

LEMMA 2. – *For any $\varepsilon > 0$, there exists $\delta > 0$ and a mapping*

$$\lambda : \quad N_\varepsilon(\varphi) \to (1-\delta, 1+\delta)$$

$$\textit{such that } |\nabla v^{\lambda(v)}|_2 = |\nabla \varphi|_2 \quad \textit{for} \quad v \in N_\varepsilon(\varphi),$$

where $N_\varepsilon(\varphi) = \{v \in H^1(\mathbb{R}^n) : \inf_{\theta \in \mathbb{R},\, y \in \mathbb{R}^n} \| v - e^{i\theta} \tau_y \varphi \|_{H^1} < \varepsilon\}$.

Proof. – Since $|\nabla v^\lambda|_2 = \lambda |\nabla v|_2$ for any $\lambda > 0$ and $v \in H^1(\mathbb{R}^n)$, it suffices to take $\lambda(v) = |\nabla \varphi|_2 / |\nabla v|_2$. □

LEMMA 3. – *If* $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$, *then there exist* $\varepsilon_1,\ \delta_1 > 0$ *with the following property: for any* $v \in N_{\varepsilon_1}(\varphi)$ *satisfying* $|v|_2 = |\varphi|_2$, *there exists* $\lambda(v) \in (1-\delta_1,\ 1+\delta_1)$ *such that* $\mathcal{E}(\varphi) \leqq \mathcal{E}(v) + (\lambda(v)-1)P(v)$, *where* P *is defined as in Definition* 3.

Proof. – From the assumption $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$ and the continuity of $\partial_\lambda^2 \mathcal{E}(v^\lambda)$ in λ and v, there exist $\varepsilon_1,\ \delta_1 > 0$ such that $\partial_\lambda^2 \mathcal{E}(v^\lambda)| < 0$ for any $\lambda \in (1-\delta_1,\ 1+\delta_1)$ and $v \in N_{\varepsilon_1}(\varphi)$. Since $\partial_\lambda \mathcal{E}(v^\lambda)|_{\lambda=1} = P(v)$, the Taylor expansion at $\lambda = 1$ gives

$$\mathcal{E}(v^\lambda) \leqq \mathcal{E}(v) + (\lambda-1)P(v),\ \lambda \in (1-\delta_1,\ 1+\delta_1), \quad v \in N_{\varepsilon_1}(\varphi). \tag{3.1}$$

From Lemma 2, if we take ε_1 small enough, for any $v \in N_{\varepsilon_1}(\varphi)$, there exists $\lambda(v) \in (1-\delta_{1,},1+\delta_1)$ such that $|\nabla v^{\lambda(v)}|_2 = |\nabla\varphi|_2$. Furthermore, from Lemma 1 (3), If $|v|_2 = |\varphi|_2$, we have

$$\mathcal{E}(v^{\lambda(v)}) = S(v^{\lambda(v)}) - \frac{\omega}{2}|v^{\lambda(v)}|_2^2 \geqq S(\varphi) - \frac{\omega}{2}|\varphi|_2^2 = \mathcal{E}(\varphi). \tag{3.2}$$

Therefore, from (3.1) and (3.2), we have

$$\mathcal{E}(\varphi) \leqq \mathcal{E}(v) + (\lambda(v)-1)P(v)$$

for any $v \in N_{\varepsilon_1}(\varphi)$ satisfying $|v|_2 = |\varphi|_2$. □

DEFINITION 4. – Put

$$\mathcal{A} = \{v \in N_{\varepsilon_1}(\varphi) : \mathcal{E}(v) < \mathcal{E}(\varphi),\ |v|_2 = |\varphi|_2,\ P(v) < 0\},$$

and for any $u_0 \in N_{\varepsilon_1}(\varphi)$, we define the exit time from $N_{\varepsilon_1}(\varphi)$ as follows:

$$T(u_0) = \sup\{T > 0 : u(t) \in N_{\varepsilon_1}(\varphi),\ 0 \leqq t \leqq T\},$$

where $u(t)$ is the solution of (1.1) with $u(0) = u_0$.

LEMMA 4. – *If* $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$, *then for any* $u_0 \in \mathcal{A}$ *there exists* $\delta_0 = \delta_0(u_0) > 0$ *such that* $P(u(t)) \leqq -\delta_0$ *for* $0 \leqq t < T(u_0)$.

Proof. – Take $u_0 \in \mathcal{A}$ and put $\delta_2 = \mathcal{E}(\varphi) - \mathcal{E}(u_0) > 0$. From Lemma 3 and the conservation laws (1.3) and (1.4), we have

$$\delta_2 \leqq (\lambda(u(t))-1)P(u(t)), \quad 0 \leqq t < T(u_0). \tag{3.3}$$

Thus, we have $P(u(t)) \neq 0$ for $0 \leqq t < T(u_0)$. Since the mapping $t \mapsto P(u(t))$ is continuous and $P(u_0) < 0$, we have $P(u(t)) < 0$ for $0 \leqq t < T(u_0)$. Therefore, from Lemma 3 and (3.3), we have

$$-P(u(t)) \geqq \frac{\delta_2}{1-\lambda(u(t))} \geqq \frac{\delta_2}{\delta_1}, \quad 0 \leqq t < T(u_0).$$

Hence, putting $\delta_0 = \delta_2/\delta_1$, we have

$$P(u(t)) \leqq -\delta_0 \quad \text{for} \quad 0 \leqq t < T(u_0). \quad \square$$

Proof of Theorem 3. – Since $\partial_\lambda \mathcal{E}(\varphi^\lambda)|_{\lambda=1} = 0$, $\partial_\lambda^2 \mathcal{E}(\varphi^\lambda)|_{\lambda=1} < 0$ and $P(\varphi^\lambda) = \lambda \partial_\lambda \mathcal{E}(\varphi^\lambda)$, we have $\mathcal{E}(\varphi^\lambda) < \mathcal{E}(\varphi)$ and $P(\varphi^\lambda) < 0$ for $\lambda > 1$ sufficiently close to 1. Furthermore, since $|\varphi^\lambda|_2 = |\varphi|_2$ and $\lim_{\lambda \to 1} \|\varphi^\lambda - \varphi\|_{H^1} = 0$, we have $\varphi^\lambda \in \mathcal{A}$ for $\lambda > 1$ sufficiently close to 1. Since it follows from Theorem 2.4 in [3] that $\int |x|^2 |\varphi^\lambda(x)|^2 dx < \infty$, we have

$$\frac{d^2}{dt^2} \int |x|^2 |u_\lambda(t, x)|^2 dx = 8 P(u_\lambda(t)), \quad 0 \leqq t < T(\varphi^\lambda), \tag{3.4}$$

where $u_\lambda(t)$ is the solution of (1.1) with $u_\lambda(0) = \varphi^\lambda$. From Lemma 4, there exists $\delta_\lambda > 0$ such that

$$P(u_\lambda(t)) \leqq -\delta_\lambda, \quad 0 \leqq t < T(\varphi^\lambda). \tag{3.5}$$

Hence, from (3.4) and (3.5), we can conclude that $T(\varphi^\lambda) < \infty$.

Since $\lim_{\lambda \to 1} \|\varphi^\lambda - \varphi\|_{H^1} = 0$, the proof is completed. $\square$

Remark 7. – Cipolatti also showed in [3] the existence of ground states of (1.2_ω) in the case when $a < 0$, $1 < p \leqq 3$ and $n = 2$ or 3, and proved in [4] that all standing waves are unstable in that case. We note that our proof is applicable to that case.

Remark 8. – It is an open problem whether the above u_λ blows up in finite time or not. For the case of local nonlinearity, *see* Berestycki and Cazenave [1].

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