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Standard generalized vectors in the space of Hilbert-Schmidt operators

by

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ABSTRACT. – Given an O^* -algebra \mathfrak{N} acting in a Hilbert space \mathcal{K} , standard generalized vectors for \mathfrak{N} are a possible tool for setting up a Tomita-Takesaki theory of modular automorphisms on \mathfrak{N} , and thus for constructing KMS quasi-weights on \mathfrak{N} . If \mathfrak{N} is the observable algebra of a physical system, these quasi-weights may be interpreted as equilibrium states of the system. In this paper, we consider the case where \mathcal{K} is the space of Hilbert-Schmidt operators on some Hilbert space \mathcal{H} and \mathfrak{N} the natural representation $\pi(\mathfrak{M})$ on that space of a self-adjoint O^* -algebra \mathfrak{M} acting in \mathcal{H} . We show that every positive Hilbert-Schmidt operator on \mathcal{H} , and more generally every positive self-adjoint unbounded operator on \mathcal{H} , determines a standard generalized vector for $\pi(\mathfrak{M})$. Then we apply this machinery to several physical examples.

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RÉSUMÉ. – Étant donné une O^* -algèbre \mathfrak{N} agissant dans un espace de Hilbert \mathcal{K} , les vecteurs généralisés standard pour \mathfrak{N} représentant une voie possible pour l'élaboration d'une théorie des automorphismes modulaires sur \mathfrak{N} , au sens de Tomita-Takesaki, et donc pour la construction de quasi-poids KMS sur \mathfrak{N} . Si \mathfrak{N} est l'algèbre des observables d'un système physique, ces quasi-poids peuvent s'interpréter comme états d'équilibre du système. Dans ce travail, on considère le cas où \mathcal{K} est l'espace de Hilbert-Schmidt sur un espace de Hilbert \mathcal{H} et \mathfrak{N} la représentation naturelle $\pi(\mathfrak{M})$ sur cet espace d'une O^* -algèbre auto-adjointe \mathfrak{M} opérant dans \mathcal{H} . On montre que tout opérateur de Hilbert-Schmidt positif sur \mathcal{H} , et plus généralement tout opérateur auto-adjoint positif non borné sur \mathcal{H} , détermine un vecteur généralisé standard pour $\pi(\mathfrak{M})$. On applique ensuite cette construction à différents exemples physiques.

1. INTRODUCTION

Given a physical system, the first task of quantum statistical mechanics is to try and construct equilibrium states of the system. In the traditional algebraic formulation [1], the system is characterized by the algebra \mathfrak{A} of its observables, usually taken as an algebra of bounded operators. The latter in turn may be obtained by applying the well-known GNS construction defined by a state on some abstract $*$ -algebra. Then the standard treatment of the basic problem consists in applying to \mathfrak{A} the Tomita-Takesaki theory of modular automorphisms, which yields states on \mathfrak{A} that satisfy the KMS condition. The latter is a characteristic of equilibrium: Gibbs states do satisfy the KMS condition. For finite systems, the converse is also true ([1], [2]), whereas, for infinite systems, the KMS condition characterizes only the local thermodynamical stability [2]. For many models, the equality between the sets of KMS states and Gibbs equilibrium states persists also after the thermodynamical limit [1]. This fact suggests the general interpretation of KMS states as equilibrium states in the Gibbs formulation, at least if the system is described as a C^* - or W^* -dynamical system.

However, there are systems for which the standard approach fails, typically spin systems with long range interactions such as the BCS model of superconductivity and its relatives. For such systems, indeed, the thermodynamic limit does not converge in any norm topology. An elegant way of circumventing the difficulty consists in taking for observable

algebra an algebra of unbounded operators, namely an O^* -algebra on some dense invariant domain \mathcal{D} in the Hilbert space at hand [3]. The same technique may be applied when unbounded observables are considered, such as position and momentum in the CCR algebra [4] (then $\mathcal{H} = L^2(\mathbb{R}^3)$ and \mathcal{D} is Schwartz space $\mathcal{S}(\mathbb{R}^3)$).

Since the examples presented in this paper are of that nature, we will adopt the O^* -approach. This means that the observables of the system (either local or in the thermodynamical limit) are represented by the elements of an O^* -algebra \mathfrak{M} . Thus we are facing the same question as before: how does one construct KMS states on an O^* -algebra? A first answer was given by one of us in the following way ([5], [6]). Let \mathfrak{M} be an O^* -algebra on the dense domain \mathcal{D} , that is, \mathfrak{M} is an algebra of operators with invariant domain \mathcal{D} , whose adjoints have a domain containing \mathcal{D} . Then a Tomita-Takesaki theory may be derived for \mathfrak{M} if, among other conditions, \mathfrak{M} possesses a strongly cyclic vector $\xi_0 \in \mathcal{D}$. In that case, one obtains states on \mathfrak{M} (in the usual sense) that satisfy the KMS condition. However, the existence of the cyclic vector is a rather restrictive condition, that we want to avoid.

A hint toward a possible generalization is the observation that the O^* -approach, by necessity, involves singular elements. Indeed, when seen from \mathfrak{M} , a vector in $\mathcal{H} \setminus \mathcal{D}$ is a singular object. An interesting possibility is to consider it as a *generalized vector* [7], that is, a linear map λ from some left ideal $D(\lambda)$ of \mathfrak{M} into \mathcal{D} , satisfying the relation

$$\lambda(XA) = X\lambda(A), \quad \forall X \in \mathfrak{M}, \quad \forall A \in D(\lambda). \quad (1.1)$$

The main advantage of this interpretation is that generalized vectors (vectors in $\mathcal{H} \setminus \mathcal{D}$ are only the simplest case) are also closely related to the concept of *weights* and *quasi-weights* on O^* -algebras, that will be studied in Section 2 below.

Indeed, the aim of this paper is to explore further the O^* -approach to statistical mechanics, and this involves identifying suitable equilibrium states. It turns out that the notion of *quasi-weight* is a possible candidate. Roughly speaking, a quasi-weight on a O^* -algebra \mathfrak{A} is a linear functional that takes finite values only on certain positive elements of \mathfrak{A} . Thus, for a system whose observable algebra is assumed to be an O^* -algebra \mathfrak{M} , we will be able to show the existence of quasi-weights on \mathfrak{M} satisfying the KMS condition. In view of the discussion above, in a generalized setup where physical states of the system would be represented by quasi-weights on the algebra of observables, it is plausible that these KMS quasi-weights would represent equilibrium states.

Now the link between this result and the previous considerations is the observation that, under suitable restrictions, generalized vectors define quasi-weights, with the required KMS property. Actually arbitrary generalized vectors are too general for obtaining a Tomita-Takesaki theory, only the subclass of *standard* generalized vectors will do, as shown in [7]. However, their definition is rather restrictive and can be weakened to *essentially standard*, and even further to *modular* generalized vectors, while still reaching the original aim, in a restricted sense at least (the definitions will be given in Section 2).

For physical applications, it is customary to study the Hilbert space $\mathcal{C}^2 \mathcal{H} \equiv \mathcal{H} \otimes \overline{\mathcal{H}}$ of all Hilbert-Schmidt operators on \mathcal{H} , together with the natural representation π of \mathfrak{M} on $\mathcal{H} \otimes \overline{\mathcal{H}}$ (this representation, and its lack of appropriate cyclic vectors, was studied in previous papers ([5], [8], [9])). Thus one of the aims of the present paper is to construct and study generalized vectors for $\pi(\mathfrak{M})$ in $\mathcal{H} \otimes \overline{\mathcal{H}}$, in such a way that KMS quasi-weights on $\pi(\mathfrak{M})$ may be obtained by a suitable Tomita-Takesaki formalism.

The paper is organized as follows. In Section 2, we collect the relevant definitions about (quasi-)weights and generalized vectors for an O^* -algebra \mathfrak{M} , in particular regular, standard and modular ones. Then we establish the crucial result that a regular generalized vector on an O^* -algebra \mathfrak{M} generates a quasi-weight on \mathfrak{M} (Theorem 2.8). In Section 3, we show how every positive Hilbert-Schmidt operator Ω determines a generalized vector λ_Ω for $\pi(\mathfrak{M})$, and we investigate under which conditions λ_Ω is standard or modular. In Section 4, we prove that a positive self-adjoint unbounded operator Ω also defines a generalized vector λ_Ω , and we ask the same questions as before. In Section 5, finally, we turn to physical applications and we describe three examples of the construction of Section 4, namely the BCS-Bogolubov model of superconductivity, a class of interacting boson models in Fock space and finally quantum field theory.

In the case of the BCS model, a rigorous algebraic description, in the quasi-spin formulation, was given long ago by Thirring and Wehrl ([10], [11]). Using this formulation Lassner ([12], [13]) solved the problem of the thermodynamical limit discussed above by constructing a rather complicated topological quasi $*$ -algebra. We will show here that the existence of KMS quasi-weights may be obtained with a much simpler O^* -algebra, provided one uses appropriate generalized vectors, as described in Sections 2-4. Similar statements apply to the other examples discussed in Section 5.

2. GENERALIZED VECTORS AND WEIGHTS

In this section we state the definitions and the basic properties of generalized vectors [7] and weights on O^* -algebras, and define the new notion of quasi-weight. The main goal here is to construct (quasi-)weights from suitable generalized vectors (Theorem 2.8).

Let \mathcal{D} be a dense subspace in a Hilbert space \mathcal{H} . We denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators X from \mathcal{D} to \mathcal{D} such that $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a O^* -algebra under the usual operations: $X+Y$, λX , XY and the involution $X \rightarrow X^\dagger \equiv X^* \upharpoonright \mathcal{D}$. A $*$ -subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an O^* -algebra on \mathcal{D} in \mathcal{H} . In this paper we assume that an O^* -algebra has always the identity operator I . Let \mathfrak{M} be an O^* -algebra on \mathcal{D} in \mathcal{H} . We introduce on \mathcal{D} the graph topology $t_{\mathfrak{M}}$ defined by the family $\{\|\cdot\|_X; X \in \mathfrak{M}\}$ of seminorms: $\|\xi\|_X = \|X\xi\|$, $\xi \in \mathcal{D}$. If the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ equipped with the topology $t_{\mathfrak{M}}$ is complete, then \mathfrak{M} is said to be *closed*. We put

$$\tilde{\mathcal{D}}(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} \mathcal{D}(\overline{X}) \quad \text{and} \quad \tilde{\mathfrak{M}} = \{\tilde{X} \equiv \overline{X} \upharpoonright \tilde{\mathcal{D}}(\mathfrak{M}); X \in \mathfrak{M}\}.$$

Then $\tilde{\mathfrak{M}}$ is a closed O^* -algebra on $\tilde{\mathcal{D}}(\mathfrak{M})$ in \mathcal{H} which is the smallest closed extension of \mathfrak{M} , and so it is called the *closure* of \mathfrak{M} . It is clear that \mathfrak{M} is closed if and only if $\mathcal{D} = \tilde{\mathcal{D}}(\mathfrak{M})$. If $\mathcal{D} = \mathcal{D}^*(\mathfrak{M}) \equiv \bigcap_{X \in \mathfrak{M}} \mathcal{D}(X^*)$, then \mathfrak{M}

is said to be *self-adjoint*. We define the *weak commutant* \mathfrak{M}'_w of \mathfrak{M} by

$$\mathfrak{M}'_w = \{C \in \mathcal{B}(\mathcal{H}); (CX\xi | \eta) = (C\xi | X^\dagger \eta)\}$$

$$\text{for all } X \in \mathfrak{M} \text{ and } \xi, \eta \in \mathcal{D}\},$$

where $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on \mathcal{H} . Then \mathfrak{M}'_w is a weakly closed, $*$ -invariant subspace of $\mathcal{B}(\mathcal{H})$, but it is not necessarily an algebra. If \mathfrak{M} is self-adjoint, then $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$, and this condition implies that \mathfrak{M}'_w is a von Neumann algebra. Let \mathcal{A} be a $*$ -algebra. A $*$ -homomorphism π of \mathcal{A} onto an O^* -algebra on $\mathcal{D}(\pi)$ in \mathcal{H}_π is said to be a $*$ -representation of \mathcal{A} in \mathcal{H}_π . Let π be a $*$ -representation of \mathcal{A} in \mathcal{H}_π . If the O^* -algebra $\pi(\mathcal{A})$ is closed (resp. self-adjoint), then π is said to be *closed* (resp. *self-adjoint*). We put

$$\tilde{\mathcal{D}}(\tilde{\pi}) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\overline{\pi(x)}) \quad \text{and} \quad \tilde{\pi}(x) = \overline{\pi(x)} \upharpoonright \tilde{\mathcal{D}}(\tilde{\pi}), \quad x \in \mathcal{A}.$$

Then $\tilde{\pi}$ is a closed $*$ -representation of \mathcal{A} in \mathcal{H}_π which is the smallest closed extension of π , and it is called the *closure* of π . Further information about O^* -algebra and $*$ -representations may be found in [3], [14].

We next proceed to define the notion of generalized vectors for O^* -algebras.

DEFINITION 2.1. – A *generalized vector* for an O^* -algebra \mathfrak{M} on \mathcal{D} in \mathcal{H} is a linear map λ from a left ideal $D(\lambda)$ into \mathcal{D} such that $\lambda(AX) = A\lambda(X)$ for all $A \in \mathfrak{M}$ and $X \in D(\lambda)$.

EXAMPLE 2.2. – *Generalized vectors associated to vectors of $\mathcal{H} \setminus \mathcal{D}$.* Given an O^* -algebra \mathfrak{M} on \mathcal{D} in \mathcal{H} and $x \in \mathcal{H}$, define

$$\begin{cases} D(\lambda_x) = \{X \in \mathfrak{M}; x \in \mathcal{D}(X^{\dagger*}) \text{ and } X^{\dagger*}x \in \mathcal{D}\}, \\ \lambda_x(x) = X^{\dagger*}x, \quad X \in D(\lambda_x). \end{cases}$$

Then λ_x is a generalized vector for \mathfrak{M} . Note that $I \notin D(\lambda_x)$ unless $x \in \mathcal{D}$.

As explained in Section 1, one needs a suitable generalization of cyclic vectors. The following notions seem to be useful.

DEFINITION 2.3. – The generalized vector λ is said to be *cyclic* (resp. *strongly cyclic*) if $\lambda(D(\lambda))$ is dense in the Hilbert space \mathcal{H} (resp. the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$).

This is indeed a genuine extension, for there are O^* -algebras which have no strongly cyclic vector, but do have a strongly cyclic generalized vector of the above type. Here is an example. Let

$$\mathfrak{M} = \left\{ \sum_{k=0}^n f_k \left(\frac{d}{dt} \right)^k \mid C_0^\infty(\mathbb{R}); f_k \in C^\infty(\mathbb{R}), 0 \leq k \leq n, n \in \mathbb{N} \right\}.$$

Then \mathfrak{M} is a self-adjoint O^* -algebra on $C_0^\infty(\mathbb{R})$ which has no strongly cyclic vector for reasons of support. Let $\xi_0(t) = e^{-t^2}$, $t \in \mathbb{R}$. Then λ_{ξ_0} is a strongly cyclic generalized vector for \mathfrak{M} . In fact, $\mathfrak{N} \equiv \left\{ f \frac{d^n}{dt^n} \mid C_0^\infty(\mathbb{R}); F \in C_0^\infty(\mathbb{R}), n \in \mathbb{N} \cup \{0\} \right\} \subset D(\lambda_{\xi_0})$ and $\mathfrak{N}^{\dagger*} \xi_0 = C_0^\infty(\mathbb{R})$.

Furthermore, the concept of a strongly cyclic generalized vector is in a sense natural. Indeed if \mathfrak{M} is closed and the generalized vector λ is cyclic, the closure of $\mathfrak{M}[\lambda(D(\lambda))]$ is an O^* -algebra contained in \mathfrak{M} , but it is difficult to control their mutual relationship. On the contrary, if λ is strongly cyclic, these two O^* -algebras may be identified.

We introduce now the central concepts of this paper, namely weights and quasi-weights on O^* -algebras. Given an O^* -algebra \mathfrak{M} on $\mathcal{D} \subset \mathcal{H}$, define the two positive cones

$$\mathfrak{M}_+ = \{X \in \mathfrak{M}; X \geq 0\}$$

and

$$\mathcal{P}(\mathfrak{M}) = \left\{ \sum_{k=1}^n X_k^\dagger X_k; X_k \in \mathfrak{M}, (k = 1, 2, \dots, n), n \in \mathbb{N} \right\} \subset \mathfrak{M}_+.$$

DEFINITIONS 2.4. – (i) Let \mathfrak{M} be an O^* -algebra. A map φ of \mathfrak{M}_+ into $\mathbb{R}_+ \cup \{+\infty\}$ is said to be a *weight* on \mathfrak{M}_+ if

$$(W)_1 \quad \varphi(\alpha X) = \alpha \varphi(X), \quad X \in \mathfrak{M}_+, \quad \alpha \geq 0;$$

$$(W)_2 \quad \varphi(X + Y) = \varphi(X) + \varphi(Y), \quad X, Y \in \mathfrak{M}_+,$$

where $0 \cdot (+\infty) = 0$.

(ii) A map φ from $\mathcal{P}(\mathfrak{M})$ into $\mathbb{R}_+ \cup \{+\infty\}$ is said to be a *weight* on $\mathcal{P}(\mathfrak{M})$ if it satisfies the above conditions $(W)_1$ and $(W)_2$ for $\mathcal{P}(\mathfrak{M})$. It is clear that if φ is a weight on \mathfrak{M}_+ , then it is a weight on $\mathcal{P}(\mathfrak{M})$.

Notice that the definition (ii) applies as well to an abstract $*$ -algebra, not necessarily an O^* -algebra, since it is purely algebraic. However, according to these definitions, a weight may take infinite values, as in the case of bounded operator algebras [1]. This property may sometimes be inconvenient, in particular for defining a representation by a generalized GNS construction. It may be avoided by restriction to a proper subset of $\mathcal{P}(\mathfrak{M})$, and this leads to the definition of a quasi-weight on $\mathcal{P}(\mathfrak{M})$. Given a subspace \mathfrak{N} of \mathfrak{M} , we denote by $\mathcal{P}(\mathfrak{N})$ the positive cone generated by \mathfrak{N} , namely:

$$\mathcal{P}(\mathfrak{N}) = \left\{ \sum_{k=1}^n X_k^\dagger X_k; X_k \in \mathfrak{N}, (k = 1, 2, \dots, n), n \in \mathbb{N} \right\}.$$

DEFINITION 2.5. – Given an O^* -algebra \mathfrak{M} , we say that φ is a *quasi-weight* on $\mathcal{P}(\mathfrak{M})$ if there exists a left ideal \mathfrak{N}_φ of \mathfrak{M} such that φ is a map from $\mathcal{P}(\mathfrak{N}_\varphi)$ into \mathbb{R}_+ , satisfying the conditions $(W)_1$ and $(W)_2$ above, for all $X, Y \in \mathcal{P}(\mathfrak{N}_\varphi)$.

Clearly a weight on $\mathcal{P}(\mathfrak{M})$ generates a quasi-weight by restriction to the domain where it takes finite values. Indeed, given a weight φ on $\mathcal{P}(\mathfrak{M})$, let us put

$$\mathfrak{N}_\varphi = \{X \in \mathfrak{M}; \varphi((AX)^\dagger (AX)) < \infty \text{ for all } A \in \mathfrak{M}\}.$$

Then \mathfrak{N}_φ is a left ideal of \mathfrak{M} and the restriction $\varphi|_{\mathcal{P}(\mathfrak{N}_\varphi)}$ of φ to the positive cone $\mathcal{P}(\mathfrak{N}_\varphi)$ is a quasi-weight on $\mathcal{P}(\mathfrak{M})$, called the quasi-weight

on $\mathcal{P}(\mathfrak{M})$ generated by φ and denoted by φ_q . Conversely, however, a quasi-weight does not generate a weight in general.

EXAMPLE 2.6. – *Generalized vectors associated to weights and quasi-weights.* Given an O^* -algebra \mathfrak{M} and a quasi-weight φ on $\mathcal{P}(\mathfrak{M})$, with left ideal \mathfrak{N}_φ , we perform the familiar GNS construction. We denote by $D(\varphi)$ the subspace of \mathfrak{M} generated by $\{X^\dagger X; X \in \mathfrak{N}_\varphi\}$. Since \mathfrak{N}_φ is a left ideal of \mathfrak{M} , we have

$$D(\varphi) = \text{linear span of } \{Y^\dagger X; X, Y \in \mathfrak{N}_\varphi\},$$

and so each $\sum_k \alpha_k Y_k^\dagger X_k$ ($\alpha_k \in \mathbb{C}$, $X_k, Y_k \in \mathfrak{N}_\varphi$) may be represented as $\sum_j \beta_j Z_j^\dagger Z_j$ for some $\beta_j \in \mathbb{C}$ and $Z_j \in \mathfrak{N}_\varphi$. Then we can define a linear functional on $D(\varphi)$ by

$$\sum_k \alpha_k Y_k^\dagger X_k \mapsto \sum_j \beta_j \varphi(Z_j^\dagger Z_j)$$

and denote it with the same symbol φ . It is easily shown that

$$|\varphi(Y^\dagger X)|^2 \leq \varphi(Y^\dagger Y) \varphi(X^\dagger X)$$

for all $X, Y \in \mathfrak{N}_\varphi$. We put

$$\mathfrak{K}_\varphi = \{X \in \mathfrak{N}_\varphi; \varphi(X^\dagger X) = 0\},$$

$$\lambda_\varphi(X) = X + \mathfrak{K}_\varphi \in \mathfrak{N}_\varphi / \mathfrak{K}_\varphi, \quad X \in \mathfrak{N}_\varphi.$$

Then \mathfrak{K}_φ is a left ideal of \mathfrak{N}_φ and $\lambda_\varphi(\mathfrak{N}_\varphi) \equiv \mathfrak{N}_\varphi / \mathfrak{K}_\varphi$ is a pre-Hilbert space with the inner product

$$(\lambda_\varphi(X) | \lambda_\varphi(Y)) = \varphi(Y^\dagger X), \quad X, Y \in \mathfrak{N}_\varphi.$$

We denote by \mathcal{H}_φ the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{N}_\varphi)$. We define a $*$ -representation π_φ^0 of \mathfrak{M} by

$$\pi_\varphi^0(A) \lambda_\varphi(X) = \lambda_\varphi(AX), \quad A \in \mathfrak{M}, \quad X \in \mathfrak{N}_\varphi,$$

and denote by π_φ the closure of π_φ^0 . We call the triple $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ the *GNS construction* for φ .

From this we may now define a generalized vector. Indeed, suppose $\pi_\varphi(X) \mapsto \lambda_\varphi(X)$ ($X \in \mathfrak{N}_\varphi$) is a map and define

$$\Lambda_\varphi(\pi_\varphi(X)) = \lambda_\varphi(X), \quad X \in \mathfrak{N}_\varphi.$$

Then Λ_φ is a strongly cyclic generalized vector for $\pi_\varphi(\mathfrak{M})$ with domain $\pi_\varphi(\mathfrak{N}_\varphi)$.

Similarly, if φ is a weight on \mathfrak{M}_+ , and φ_q the quasi-weight on $\mathcal{P}(\mathfrak{M})$ generated by φ , we denote again by $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ the GNS construction for φ_q and by Λ_φ the generalized vector Λ_{φ_q} if it is defined.

If the quasi-weight φ is faithful, i.e. $\mathfrak{K}_\varphi = \{0\}$, π_φ is a *-isomorphism from \mathfrak{M} onto $\pi_\varphi(\mathfrak{M})$, so that $\pi_\varphi(X) \mapsto \lambda_\varphi(X)$ is indeed a map and the generalized vector Λ_φ is well-defined. If φ is not faithful, we have to impose the map condition explicitly.

Conversely, we proceed to show that, under some additional regularity conditions, generalized vectors for \mathfrak{M} generate quasi-weights on $\mathcal{P}(\mathfrak{M})$, thus the corresponding GNS representations.

Let \mathfrak{M} be a closed O^* -algebra on \mathcal{D} in \mathcal{H} such that $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}$ and λ a generalized vector for \mathfrak{M} for which

$$\lambda((D(\lambda) \cap D(\lambda)^\dagger)^2) \text{ is total in } \mathcal{H}. \quad (2.1)$$

We define the *commutant* λ^c of λ by

$$\begin{cases} D(\lambda^c) = \{K \in \mathfrak{M}'_w; \exists \xi_K \in \mathcal{D} \text{ s.t.} \\ \quad K\lambda(X) = X\xi_K \text{ for all } X \in D(\lambda)\}, \\ \lambda^c(K) = \xi_K, \quad K \in D(\lambda^c), \end{cases}$$

so that $K\lambda(X) = X\lambda^c(K)$. Then λ^c is a generalized vector for the von Neumann algebra \mathfrak{M}'_w . We remark that ξ_K is uniquely defined by the condition (2.1).

DEFINITION 2.7. – The generalized vector λ is said to be *regular* if it satisfies the condition (2.1) and $D(\lambda^c) \cap D(\lambda^c)^*$ is a nondegenerate *-subalgebra of the von Neumann algebra \mathfrak{M}'_w , that is, there exists a net $\{K_\alpha\}$ in $D(\lambda^c) \cap D(\lambda^c)^*$ such that $0 \leq K_\alpha \leq I$ and $K_\alpha \rightarrow I$ strongly. We denote by \mathcal{R}_λ the set of all such nets $\{K_\alpha\}$. If there exists a net $\{K_\alpha\}$ in \mathcal{R}_λ for which $K_\alpha \uparrow I$ and $K_\alpha K_\beta = K_\beta K_\alpha$ for each α, β , then λ is said to be *strongly regular*.

Let λ be a regular generalizing vector for \mathfrak{M} . We put

$$\omega_\lambda \left(\sum_{k=1}^n X_k^\dagger X_k \right) = \sum_{k=1}^n \|\lambda(X_k)\|^2, \quad \sum_{k=1}^n X_k^\dagger X_k \in \mathcal{P}(D(\lambda)).$$

Then we have the following

THEOREM 2.8. – (1) *Suppose λ is a regular generalized vector for \mathfrak{M} . Then ω_λ is a quasi-weight on $\mathcal{P}(\mathfrak{M})$. In particular, suppose $\lambda_x (x \in \mathcal{H})$ is regular. Then*

$$\omega_{\lambda_x}(X^\dagger X) = \omega_x(X^\dagger X^{\dagger*}) \equiv (X^\dagger X^{\dagger*} x | x), \quad X \in \mathcal{D}(\lambda_x).$$

(2) Suppose λ_x is strongly regular and put

$$\omega_{\lambda_x}^{\{K_\alpha\}}(X) = \begin{cases} \lim_{\alpha} (X \lambda_x^c(K_\alpha) | \lambda_x^c(K_\alpha)) & \text{if the limit exists,} \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\omega_{\lambda_x}^{\{K_\alpha\}}$ is a weight on \mathfrak{M}_+ and if \mathfrak{M} is self-adjoint, then $\mathfrak{N}_{\omega_{\lambda_x}^{\{K_\alpha\}}} = D(\lambda_x)$.

Proof. – (1) For $\{X_k\} \subset D(\lambda)$ and $\{\alpha_k\} \subset \mathbb{R}_+$ we have

$$\begin{aligned} \lim_{\alpha} \left(\sum_{k=1}^n \alpha_k X_k^\dagger X_k \lambda^c(K_\alpha) \mid \lambda^c(K_\alpha) \right) &= \lim_{\alpha} \sum_{k=1}^n \alpha_k \|K_\alpha \lambda(X_k)\|^2 \\ &= \sum_{k=1}^n \alpha_k \|\lambda(X_k)\|^2 \\ &= \omega_{\lambda} \left(\sum_{k=1}^n \alpha_k X_k^\dagger X_k \right), \end{aligned}$$

which implies that ω_{λ} is well-defined and it is a quasi-weight on $\mathcal{P}(\mathfrak{M})$. Suppose that λ_x ($x \in \mathcal{H}$) is regular. Then we have

$$\omega_{\lambda_x}(X^\dagger X) = \|\lambda_x(X)\|^2 = \|X^{*\dagger} x\|^2 = \omega_x(X^\dagger X^{*\dagger})$$

for all $X \in D(\lambda_x)$.

(2) Suppose λ_x is strongly regular. Let $\{K_\alpha\} \subset \mathcal{R}_{\lambda_x}$ be a family of mutually commuting, increasing elements. It is clear that $0 \leq \omega_{\lambda_x}^{\{K_\alpha\}}(X) \leq +\infty$ and $\omega_{\lambda_x}^{\{K_\alpha\}}(\alpha X) = \alpha \omega_{\lambda_x}^{\{K_\alpha\}}(X)$ for any $\alpha \geq 0$ and $X \in \mathfrak{M}_+$. We show that

$$\omega_{\lambda_x}^{\{K_\alpha\}}(X + Y) = \omega_{\lambda_x}^{\{K_\alpha\}}(X) + \omega_{\lambda_x}^{\{K_\alpha\}}(Y), \quad X, Y \in \mathfrak{M}_+.$$

It is sufficient to show that $X + Y \in D(\omega_{\lambda_x}^{\{K_\alpha\}})$ if and only if $X \in D(\omega_{\lambda_x}^{\{K_\alpha\}})_+$ and $Y \in D(\omega_{\lambda_x}^{\{K_\alpha\}})_+$, that is, $\lim_{\alpha} ((X + Y) K_\alpha x | K_\alpha x)$ exists if and only if both $\lim_{\alpha} (X K_\alpha x | K_\alpha x)$ and $\lim_{\alpha} (Y K_\alpha x | K_\alpha x)$ exist. Suppose that $\lim_{\alpha} ((X + Y) K_\alpha x | K_\alpha x)$ exists. Then, for $\alpha < \beta$, we have

$$(X K_\beta x | K_\beta x) - (X K_\alpha x | K_\alpha x) = (X (K_\beta^2 - K_\alpha^2) x | x). \tag{2.2}$$

Since $K_\alpha K_\beta = K_\beta K_\alpha$, we have $K_\beta^2 \geq K_\alpha^2$, and so we put

$$C_0 (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} y = (K_\beta - K_\alpha) y, \quad y \in \mathcal{H}$$

$$E = \text{Proj} (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} \mathcal{H}$$

$$C = \overline{C_0} E.$$

Then, since

$$\begin{aligned}(K_\beta - K_\alpha)y &= (K_\beta - K_\alpha)(I - F)y \\ &= \lim_{\varepsilon \downarrow 0} (K_\beta - K_\alpha)(K_\beta + K_\alpha + \varepsilon)^{-\frac{1}{2}}(K_\beta + K_\alpha)^{\frac{1}{2}}y \\ &= \lim_{\varepsilon \downarrow 0} (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}(K_\beta - K_\alpha)^{\frac{1}{2}}(K_\beta + K_\alpha + \varepsilon)^{-\frac{1}{2}}y,\end{aligned}$$

for each $y \in \mathcal{H}$, where $F \equiv \text{Proj Ker}(K_\beta - K_\alpha)^{\frac{1}{2}}$, it follows that $E(K_\beta - K_\alpha) = K_\beta - K_\alpha$, so that

$$\begin{aligned}ECy &= \lim_{n \rightarrow \infty} E\bar{C}_0(K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}y_n \\ &= \lim_{n \rightarrow \infty} E(K_\beta - K_\alpha)y_n \\ &= \lim_{n \rightarrow \infty} (K_\beta - K_\alpha)Ey_n \\ &= \lim_{n \rightarrow \infty} \bar{C}_0(K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}Ey_n \\ &= \lim_{n \rightarrow \infty} \bar{C}_0E(K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}y_n \\ &= CEy,\end{aligned}$$

for each $y \in \mathcal{H}$, where $\lim_{n \rightarrow \infty} (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}y_n = Ey$. Hence, we have

$$EC = CE. \tag{2.3}$$

Furthermore, it follows from (2.3) that

$$\begin{aligned}(Cy|z) &= \lim_{n \rightarrow \infty} (C(K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}y_n | (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}z_n) \\ &= \lim_{n \rightarrow \infty} (K_\beta - K_\alpha)y_n | (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}z_n) \\ &= \lim_{n \rightarrow \infty} ((K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}y_n | (K_\beta - K_\alpha)z_n) \\ &= (y|Cz)\end{aligned}$$

and

$$\begin{aligned}
 C A y &= \overline{C}_0 E A y = \overline{C}_0 A E y \\
 &= \lim_{n \rightarrow \infty} \overline{C}_0 A (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} y_n \\
 &= \lim_{n \rightarrow \infty} \overline{C}_0 (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} A y_n \\
 &= \lim_{n \rightarrow \infty} (K_\beta - K_\alpha) A y_n \\
 &= A \lim_{n \rightarrow \infty} (K_\beta - K_\alpha) y_n \\
 &= A \lim_{n \rightarrow \infty} C (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} y_n \\
 &= A C E y \\
 &= A C y
 \end{aligned}$$

for each $y, z \in \mathcal{H}$ and $A \in (\mathfrak{M}'_w)'$, where $\lim_{n \rightarrow \infty} (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} y_n = E y$ and $\lim_{n \rightarrow \infty} (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} z_n = E z$, which implies that

$$C^* = C \in \mathfrak{M}'_w \quad \text{and} \quad K_\beta - K_\alpha = C (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}}. \quad (2.4)$$

Hence we have

$$C (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} = (K_\beta - K_\alpha)^{\frac{1}{2}} C \quad (2.5)$$

and

$$C (K_\beta - K_\alpha) = (K_\beta - K_\alpha) C. \quad (2.6)$$

By (2.5), (2.6), we have

$$(K_\beta^2 - K_\alpha^2) \text{Ker } C \subset \text{Ker } C,$$

and so by (2.4)

$$\begin{aligned}
 G (K_\beta^2 - K_\alpha^2) &= (K_\beta^2 - K_\alpha^2) G \\
 &= (K_\beta + K_\alpha) C (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} G \\
 &= (K_\beta + K_\alpha) (K_\beta^2 - K_\alpha^2)^{\frac{1}{2}} C G \\
 &= 0,
 \end{aligned}$$

where $G = \text{Proj Ker } C$. Hence, it follows from (2.4) that

$$\begin{aligned} (X(K_\beta^2 - K_\alpha^2)x|x) &= (X(I - G)(K_\beta^2 - K_\alpha^2)x|x) \\ &= \lim_{\varepsilon \downarrow 0} ((\varepsilon + C^2)^{-1} C^2 X(K_\beta^2 - K_\alpha^2)x|x) \\ &= \lim_{\varepsilon \downarrow 0} ((\varepsilon + C^2)^{-1} X(K_\beta - K_\alpha)^2 x|x) \\ &= \lim_{\varepsilon \downarrow 0} (X(\varepsilon + C^2)^{-\frac{1}{2}}(K_\beta - K_\alpha)x| \\ &\quad \times (\varepsilon + C^2)^{-\frac{1}{2}}(K_\beta - K_\alpha)x) \\ &\geq 0, \end{aligned}$$

and similarly

$$(Y(K_\beta^2 - K_\alpha^2)x|x) \geq 0,$$

which implies by (2.2) that

$$\begin{aligned} 0 &\leq (XK_\beta x|K_\beta x) - (XK_\alpha x|K_\alpha x) \\ &\leq (XK_\beta x|K_\beta x) - (XK_\alpha x|K_\alpha x) \\ &\quad + (YK_\beta x|K_\beta x) - (YK_\alpha x|K_\alpha x) \\ &= ((X + Y)K_\beta x|K_\beta x) - ((X + Y)K_\alpha x|K_\alpha x) \xrightarrow{\alpha, \beta} 0. \end{aligned}$$

Hence, $\lim_\alpha (XK_\alpha x|K_\alpha x)$ exists, and similarly $\lim_\alpha (YK_\alpha x|K_\alpha x)$ exists. Therefore, $X, Y \in D(\omega_{\lambda_x}^{\{K_\alpha\}})$.

As for the last assertion, it is clear that $D(\lambda_x) \subset \mathfrak{M}_{\omega_{\lambda_x}^{\{K_\alpha\}}}$. When \mathfrak{M} is self-adjoint, we can prove the opposite inclusion. Take any $X \in \mathfrak{M}_{\omega_{\lambda_x}^{\{K_\alpha\}}}$. Then $\lim_\alpha \|AXK_\alpha x\|$ exists for all $A \in \mathfrak{M}$. For any $\xi \in \mathcal{D}$ and $A \in \mathfrak{M}$, we have

$$\begin{aligned} |(X^\dagger A^\dagger \xi|x)| &= \lim_\alpha ((X^\dagger A^\dagger \xi|K_\alpha x)|) \\ &= \lim_\alpha |(\xi|AXK_\alpha x)| \\ &\leq (\lim_\alpha \|AXK_\alpha x\|) \|\xi\|. \end{aligned}$$

Since $I \in \mathfrak{M}$, it follows that $x \in \mathcal{D}(X^{\dagger*})$ and $X^{\dagger*}x \in \mathcal{D}(A^{\dagger*})$ for all $A \in \mathfrak{M}$. Since \mathfrak{M} is self-adjoint, this implies that $X \in D(\lambda_x)$. This concludes the proof. \square

As asserted in the introduction, arbitrary generalized vectors are still too general for obtaining a useful Tomita-Takesaki theory. For this reason,

the authors of [7] have introduced the subclass of standard and essentially standard generalized vectors, that we now discuss.

Let \mathfrak{M} be a closed O^* -algebra on \mathcal{D} in \mathcal{H} such that $\mathfrak{M}'_{\mathfrak{w}} \mathcal{D} \subset \mathcal{D}$ and let λ be a generalized vector for \mathfrak{M} satisfying

$$(S)_1 \quad \lambda((D(\lambda) \cap D(\lambda)^\dagger)^2) \text{ is total in } \mathcal{H};$$

$$(S)_2 \quad \lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2) \text{ is total in } \mathcal{H}.$$

Since $(D(\lambda^c) \cap D(\lambda^c)^*)^2 \subset D(\lambda^c) \cap D(\lambda^c)^*$, it follows that $\lambda^c(D(\lambda^c) \cap D(\lambda^c)^*)$ is dense in \mathcal{H} , and so $D(\lambda^c) \cap D(\lambda^c)^*$ is a nondegenerate $*$ -subalgebra of $\mathfrak{M}'_{\mathfrak{w}}$, that is, λ is regular. We put

$$\begin{cases} D(\lambda^{cc}) = \{A \in (\mathfrak{M}'_{\mathfrak{w}})'; \exists \xi_A \in \mathcal{H} \text{ s.t.} \\ \quad A \lambda^c(K) = K \xi_A \text{ for all } K \in D(\lambda^c)\}, \\ \lambda^{cc}(A) = \xi_A, \quad A \in D(\lambda^{cc}). \end{cases}$$

Then λ^{cc} is a generalized vector for the von Neumann algebra $(\mathfrak{M}'_{\mathfrak{w}})'$. Furthermore, $\lambda^{cc}(D(\lambda^{cc}) \cap D(\lambda^{cc})^*)$ is an achieved left Hilbert algebra in \mathcal{H} , equipped with the following multiplication and involution:

$$\begin{aligned} \lambda^{cc}(A) \lambda^{cc}(B) &= \lambda^{cc}(AB), & \lambda^{cc}(A)^\sharp &= \lambda^{cc}(A^*), \\ A, B &\in D(\lambda^{cc}) \cap D(\lambda^{cc})^*, \end{aligned}$$

and its left von Neumann algebra equals $(\mathfrak{M}'_{\mathfrak{w}})'$. The map $\lambda^{cc}(A) \mapsto \lambda^{cc}(A^*)$, $A \in D(\lambda^{cc}) \cap D(\lambda^{cc})^*$ (resp. $\lambda(X) \mapsto \lambda(X^\dagger)$, $X \in D(\lambda) \cap D(\lambda)^\dagger$) is a closable conjugate linear operator in \mathcal{H} . We denote its closure by $S_{\lambda^{cc}}$ (resp. S_λ). Let $S_{\lambda^{cc}} = J_{\lambda^{cc}} \Delta_{\lambda^{cc}}^{\frac{1}{2}}$ and $S_\lambda = J_\lambda \Delta_\lambda^{\frac{1}{2}}$ be the polar decompositions of $S_{\lambda^{cc}}$ and S_λ , respectively. We have $S_\lambda \subset S_{\lambda^{cc}}$. Then the Tomita fundamental theorem implies the following relations:

$$J_{\lambda^{cc}} (\mathfrak{M}'_{\mathfrak{w}})' J_{\lambda^{cc}} = \mathfrak{M}'_{\mathfrak{w}};$$

$$\sigma_t^{\lambda^{cc}}(A) \equiv \Delta_{\lambda^{cc}}^{it} A \Delta_{\lambda^{cc}}^{-it} \in (\mathfrak{M}'_{\mathfrak{w}})', \quad A \in (\mathfrak{M}'_{\mathfrak{w}})', \quad t \in \mathbb{R};$$

$$\sigma_t^{\lambda^{cc}}(D(\lambda^{cc}) \cap D(\lambda^{cc})^*) = D(\lambda^{cc}) \cap D(\lambda^{cc})^*$$

$$\lambda^{cc}(\sigma_t^{\lambda^{cc}}(B)) = \Delta_{\lambda^{cc}}^{it} \lambda^{cc}(B), \quad B \in D(\lambda^{cc}) \cap D(\lambda^{cc})^*, \quad t \in \mathbb{R}.$$

Thus the unitary group $\{\Delta_{\lambda^{cc}}^{it}\}_{t \in \mathbb{R}}$ implements a one-parameter group $\{\sigma_t^{\lambda^{cc}}\}_{t \in \mathbb{R}}$ of $*$ -automorphisms of the von Neumann algebra $(\mathfrak{M}'_{\mathfrak{w}})'$. However, we do not know in general how this automorphism groups acts

on the O^* -algebra \mathfrak{M} itself. Thus we have to impose additional restrictions to the notion of generalized vector used so far.

DEFINITION 2.9. – A generalized vector λ for \mathfrak{M} is said to be *essentially standard* if it satisfies the above conditions (S)₁ and (S)₂ and the following condition (S)₃:

$$(S)_3 \quad \Delta_{\lambda^{cc}}^{it} \mathcal{D} \subset \mathcal{D} \quad \text{and} \quad \Delta_{\lambda^{cc}}^{it} \mathfrak{M} \Delta_{\lambda^{cc}}^{-it} = \mathfrak{M}, \quad t \in \mathbb{R}.$$

Furthermore, if

$$(S)_4 \quad \sigma_t^{\lambda^{cc}} (D(\lambda) \cap D(\lambda)^\dagger) = D(\lambda) \cap D(\lambda)^\dagger, \quad t \in \mathbb{R},$$

then λ is said to be *standard*.

Then, combining Theorems 5.5, 5.6 and 5.8 of [7], we have the following result, which answers the question of how to generate (generalized) KMS states.

THEOREM 2.10. – Suppose λ is a standard generalized vector for \mathfrak{M} . Then the following statements hold:

$$(1) S_\lambda = S_{\lambda^{cc}}.$$

(2) $\sigma_t^\lambda(X) \equiv \Delta_\lambda^{it} X \Delta_\lambda^{-it} = \sigma_t^{\lambda^{cc}}(X)$, $X \in \mathfrak{M}$, $t \in \mathbb{R}$ and $\{\sigma_t^\lambda\}_{t \in \mathbb{R}}$ is a one-parameter group of $*$ -automorphisms of \mathfrak{M} .

(3) The quasi-weight ω_λ on $\mathcal{P}(\mathfrak{M})$ satisfies the KMS-condition with respect to $\{\sigma_t^\lambda\}_{t \in \mathbb{R}}$, that is, for each $X, Y \in D(\lambda) \cap D(\lambda)^\dagger$ there exists an element $f_{X,Y}$ of $A(0, 1)$ such that

$$f_{X,Y}(t) = \omega_\lambda(Y^\dagger \sigma_t^\lambda(X)) = (\lambda(\sigma_t^\lambda(X)) | \lambda(Y))$$

and

$$f_{X,Y}(t+i) = \omega_\lambda(\sigma_t^\lambda(X) Y^\dagger) = (\lambda(Y^\dagger) | \lambda(\sigma_t^\lambda(X^\dagger)))$$

for all $t \in \mathbb{R}$, where $A(0, 1)$ is the set of all complex-valued functions, bounded and continuous on $0 \leq \text{Im}z \leq 1$ and analytic in the interior.

Actually the standardness condition of λ may be weakened somewhat: an essentially standard generalized vector λ will do, at the price of enlarging the domain of λ (see [7], Theorem 5.8). However, even the condition (S)₃ of Definition 2.5 is too strong for the physical applications we have in mind. We may weaken it somewhat by introducing the notion of *modular* generalized vector [7]. This will allow us to extend further the range of application of the unbounded Tomita-Takesaki theory.

DEFINITION 2.11. – A generalized vector λ for \mathfrak{M} is said to be *modular* if it satisfies the conditions (S)₁, (S)₂ and the following condition (M):

(M) There exists a dense subspace \mathcal{E} of $\mathcal{D}[t_{\mathfrak{M}}]$ such that

$$(M)_1 \quad \lambda(D(\lambda) \cap D(\lambda)^\dagger) \cap \mathcal{E},$$

$$(M)_2 \quad \{\lambda^c(K_1 K_2); K_i \in D(\lambda^c) \cap D(\lambda^c)^* \text{ s.t. } \lambda^c(K_i),$$

$$\lambda^c(K_i)^* \in \mathcal{E}, i = 1, 2\} \text{ is total in the Hilbert space } \mathcal{D}(S_{\lambda^{cc}}^*),$$

$$(M)_3 \quad \mathfrak{M} \mathcal{E} \cap \mathcal{E},$$

$$(M)_4 \quad \Delta_{\lambda^{cc}}^{it} \mathcal{E} \cap \mathcal{E} \text{ for all } t \in \mathbb{R}.$$

For a given modular generalized vector λ , there exists a largest subspace of \mathcal{D} satisfying the conditions (M)₁ ~ (M)₄ above; we denote it by \mathcal{D}_λ .

It turns out that a modular generalized vector on \mathfrak{M} defines standard generalized vectors on appropriate generalized von Neumann algebras (GW*-algebras) on the domain \mathcal{D}_λ . We first recall that concept, introduced in [5].

We denote as usual by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all linear operators X from \mathcal{D} to \mathcal{H} such that $\mathcal{D}(X^*) \supset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a \dagger -invariant vector space equipped with the usual operations: $X + Y$, λX and the involution $X \mapsto X^\dagger \equiv X^* \lceil \mathcal{D}$. We consider on $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the *strong*-topology*, that is, the locally convex topology t_s^* defined by the family $\{p_\xi^*; \xi \in \mathcal{D}\}$ of seminorms: $p_\xi^*(X) = \|X \xi\| + \|X^\dagger \xi\|$, $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[t_s^*]$ is complete and for any subset \mathfrak{N} of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ we denote by $[\mathfrak{N}]^{s*}$ the closure of \mathfrak{N} in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[t_s^*]$.

Let now λ be a modular generalized vector on \mathfrak{M} and \mathcal{D}_λ the corresponding maximal domain. It follows from the maximality of \mathcal{D}_λ that $\mathfrak{M}'_w \mathcal{D}_\lambda \subset \mathcal{D}_\lambda$, which implies by [5], Theorem 5.2, that $\mathcal{U}(\lambda) \equiv [(\mathfrak{M}'_w)' \lceil \mathcal{D}_\lambda]^{s*} \cap \mathcal{L}^\dagger(\mathcal{D}_\lambda)$ is a closed O^* -algebra on \mathcal{D}_λ such that $\mathcal{U}(\lambda)'_w = \mathfrak{M}'_w$ and

$$\begin{aligned} \mathcal{U}(\lambda) &= (\mathfrak{M}'_w)'_c \equiv \{X \in \mathcal{L}^\dagger(\mathcal{D}_\lambda); CX \subset XC, \forall C \in \mathfrak{M}'_w\} \\ &= \{X \in \mathcal{L}^\dagger(\mathcal{D}_\lambda); \bar{X} \text{ is affiliated with } (\mathfrak{M}'_w)'\}. \end{aligned}$$

In this case $\mathcal{U}(\lambda)$ is called a *generalized von Neumann algebra* (or GW^* -algebra) on \mathcal{D}_λ over $(\mathfrak{M}'_w)'$ [15]. The O^* -algebra $\mathcal{L}(\lambda)$ on \mathcal{D}_λ generated by $\{\Delta_{\lambda^{cc}}^{it} \mathfrak{M} \Delta_{\lambda^{cc}}^{-it} \lceil \mathcal{D}_\lambda; t \in \mathbb{R}\}$ is a closed O^* -subalgebra of $\mathcal{U}(\lambda)$. Furthermore, $\{\sigma_t^{\lambda^{cc}}\}_{t \in \mathbb{R}}$ is a one-parameter group of $*$ -automorphisms of $\mathcal{L}(\lambda)$ and $\mathcal{U}(\lambda)$. Then the crucial result is that the modular generalized vector λ on \mathfrak{M} generates standard generalized vectors for the O^* -algebras $\mathcal{U}(\lambda)$ and $\mathcal{L}(\lambda)$, respectively, as follows from [7], Theorem 5.11.

THEOREM 2.12. – *Suppose λ is a modular generalized vector for \mathfrak{M} and put*

$$\left\{ \begin{array}{l} D(\lambda_s) = \left\{ \sum_k Y_k X_k; Y_k \in \mathfrak{M}, X_k \in D(\bar{\lambda}) \right\}, \\ \lambda_s \left(\sum_k Y_k X_k \right) = \sum_k Y_k \bar{\lambda}(X_k), \end{array} \right.$$

where

$$\left\{ \begin{array}{l} D(\bar{\lambda}) = \{X \in \mathcal{U}(\lambda); \exists \{A_\gamma\} \subset D(\lambda^{cc}) \text{ s.t.} \\ \quad A_\gamma \xi \rightarrow X \xi, \quad \forall \xi \in \mathcal{D}_\lambda \text{ and } \lambda^{cc}(A_\gamma) \rightarrow \xi_X \in \mathcal{D}_\lambda\}, \\ \bar{\lambda}(X) = \xi_X, \quad X \in D(\bar{\lambda}). \end{array} \right.$$

Then λ_s is a standard generalized vector for $\mathcal{U}(\lambda)$ such that $D(\lambda_s^{cc}) \cap D(\lambda_s^{cc})^* = D(\lambda^{cc}) \cap D(\lambda^{cc})^*$.

Similarly, the restriction $\lambda_s|_{\mathcal{L}(\lambda)}$ of λ to the subalgebra $\mathcal{L}(\lambda)$ is a standard generalized vector for $\mathcal{L}(\lambda)$.

In conclusion, whenever we have a modular generalized vector on an O^* -algebra \mathfrak{M} , it generates quasi-weights satisfying the KMS property on appropriate GW^* -algebras. In the next section we will apply this construction to the case O^* -algebras acting in the space of Hilbert-Schmidt operators, as found, for instance, in the case of the spin systems discussed in Section 5.

3. STANDARD GENERALIZED VECTORS DETERMINED BY POSITIVE HILBERT-SCHMIDT OPERATORS

Let \mathfrak{M} be a self-adjoint O^* -algebra on the dense domain \mathcal{D} of the separable Hilbert space \mathcal{H} . Throughout this section we assume that $\mathfrak{M}'_w = \mathbb{C}I$.

We denote by $\mathcal{H} \otimes \bar{\mathcal{H}}$ the Hilbert space of all Hilbert-Schmidt operators on \mathcal{H} with the inner product $\langle S|T \rangle \equiv \text{tr}(T^* S)$, $S, T \in \mathcal{H} \otimes \bar{\mathcal{H}}$, and the norm $\|T\|_2 \equiv \langle T|T \rangle^{\frac{1}{2}}$, $T \in \mathcal{H} \otimes \bar{\mathcal{H}}$. We define some operators on $\mathcal{H} \otimes \bar{\mathcal{H}}$:

LEMMA 3.1. – *Let H and K be two closed operators in \mathcal{H} and put*

$$\left\{ \begin{array}{l} \mathcal{D}(\pi''(H)) = \{T \in \mathcal{H} \otimes \bar{\mathcal{H}}; T\mathcal{H} \subset \mathcal{D}(H) \text{ and } HT \in \mathcal{H} \otimes \bar{\mathcal{H}}\}, \\ \pi''(H)T = HT, \quad T \in \mathcal{D}(\pi''(H)), \\ \\ \mathcal{D}(\pi'(K)) = \{T \in \mathcal{H} \otimes \bar{\mathcal{H}}; TK \text{ is closable and } \overline{TK} \in \mathcal{H} \otimes \bar{\mathcal{H}}\}, \\ \pi'(K)T = \overline{TK}, \quad T \in \mathcal{D}(\pi'(K)). \end{array} \right.$$

Then the following statements hold:

(1) π'' is a $*$ -homomorphism of $\mathcal{B}(\mathcal{H})$ onto a von Neumann algebra on $\mathcal{H} \otimes \overline{\mathcal{H}}$ and π' is an anti $*$ -homomorphism of $\mathcal{B}(\mathcal{H})$ onto the commutant of the von Neumann algebra $\pi''(\mathcal{B}(\mathcal{H}))$.

(2) $\pi''(H)$ and $\pi'(K)$ are closed operators in $\mathcal{H} \otimes \overline{\mathcal{H}}$ affiliated with $\pi''(\mathcal{B}(\mathcal{H}))$ and $\pi'(\mathcal{B}(\mathcal{H}))$, respectively.

(3) Suppose that H and K are (positive) self-adjoint operators in \mathcal{H} . Then, $\pi''(H)$ and $\pi'(K)$ are (positive) self-adjoint operators in $\mathcal{H} \otimes \overline{\mathcal{H}}$, and $\pi''(H)\pi'(K)$ and $\pi'(K)\pi''(H)$ are (positive) essentially self-adjoint operators in $\mathcal{H} \otimes \overline{\mathcal{H}}$ and $\pi''(H)\pi'(K) = \pi'(K)\pi''(H)$.

(4) We put

$$\mathcal{D}(\pi) = \bigcap_{X \in \mathfrak{M}} \mathcal{D}(\pi''(\overline{X})), \quad \pi(X) = \pi''(\overline{X})|_{\mathcal{D}(\pi)}, \quad X \in \mathfrak{M}.$$

Then

$$\mathcal{D}(\pi) = \sigma_2(\mathfrak{M}) \equiv \{T \in \mathcal{H} \otimes \overline{\mathcal{H}}; \\ T\mathcal{H} \subset \mathcal{D} \text{ and } XT \in \mathcal{H} \otimes \overline{\mathcal{H}} \text{ for all } X \in \mathfrak{M}\},$$

and π a self-adjoint representation of \mathfrak{M} in $\mathcal{H} \otimes \overline{\mathcal{H}}$, that is, it is a $*$ -homomorphism of \mathfrak{M} onto the self-adjoint O^* -algebra on $\sigma_2(\mathfrak{M})$ in $\mathcal{H} \otimes \overline{\mathcal{H}}$, such that $\pi(\mathfrak{M})'_w = \pi'(\mathcal{B}(\mathcal{H}))$ and $(\pi(\mathfrak{M})'_w)' = \pi''(\mathcal{B}(\mathcal{H}))$.

Proof. – It is not difficult to show the statements (1), (2) and the first part of (3). The last part of (3) is shown by the spectral theory of the self-adjoint operators $\pi''(H)$ and $\pi'(K)$. In (4) it is easily shown that π is a self-adjoint representation of \mathfrak{M} in $\mathcal{H} \otimes \overline{\mathcal{H}}$. It follows from ([15], Lemma 2.4) that $\pi(\mathfrak{M})'_w = \pi'(\mathcal{B}(\mathcal{H}))$ and $(\pi(\mathfrak{M})'_w)' = \pi''(\mathcal{B}(\mathcal{H}))$. \square

We show now that every positive Hilbert-Schmidt operator Ω on \mathcal{H} determines a generalized vector λ_Ω for $\pi(\mathfrak{M})$. Indeed, for $\Omega \geq 0 \in \mathcal{H} \otimes \overline{\mathcal{H}}$, let us put

$$\left\{ \begin{array}{l} \mathcal{D}(\lambda_\Omega) = \{\pi(X) \in \pi(\mathfrak{M}); \Omega \in \mathcal{D}(\pi(X^\dagger)^*) \\ \text{and } \pi(X^\dagger)^* \Omega \in \sigma_2(\mathfrak{M})\}, \\ \lambda_\Omega(\pi(X)) = \pi(X^\dagger)^* \Omega, \quad \pi(X) \in \mathcal{D}(\lambda_\Omega). \end{array} \right.$$

Then λ_Ω is a generalized vector for $\pi(\mathfrak{M})$ and

$$\left\{ \begin{array}{l} \mathcal{D}(\lambda_\Omega) = \{\pi(X) \in \pi(\mathfrak{M}); \Omega \mathcal{H} \in \mathcal{D}(\pi(X^\dagger)^*) \\ \text{and } X^{\dagger*} \Omega \in \sigma_2(\mathfrak{M})\}, \\ \lambda_\Omega(\pi(X)) = X^{\dagger*} \Omega, \quad \pi(X) \in \mathcal{D}(\lambda_\Omega). \end{array} \right.$$

In the rest of this section, we search sufficient conditions for λ_Ω to be a standard or a modular generalized vector for $\pi(\mathfrak{M})$.

LEMMA 3.2. – *Suppose there exists an orthonormal basis $\{\xi_n\}$ in \mathcal{H} such that $\{\xi_n\} \subset \mathcal{D}$ and $\xi_n \otimes \bar{\xi}_m \in \mathfrak{M}$ for $n, m \in \mathbb{N}$ and $\Omega \geq 0 \in \mathcal{H} \otimes \bar{\mathcal{H}}$ such that $\Omega \mathcal{H}$ is dense in \mathcal{H} . Then λ_Ω is a cyclic generalized vector for $\pi(\mathfrak{M})$ such that $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \bar{\mathcal{H}}$. Furthermore, if $\{\xi_n\}$ is total in $\mathcal{D}[t_{\mathfrak{M}}]$, then λ_Ω is strongly cyclic.*

Proof. – We put

$$\mathcal{E} = \left\{ \sum_{k,l} \alpha_k \xi_k \otimes \overline{\beta_l \xi_l}; \alpha_k, \beta_l \in \mathbb{C} \right\}.$$

Then it is clear that $\mathcal{E} \subset (D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2$ and

$$\lambda_\Omega \left(\sum_{k,l} \alpha_k \xi_k \otimes \overline{\beta_l \xi_l} \right) = \sum_{k,l} \alpha_k \xi_k \otimes \overline{\beta_l \Omega \xi_l},$$

and since $\{\xi_n\}$ is an ONB in \mathcal{H} and $\Omega \mathcal{H}$ is dense in \mathcal{H} , it follows that $\lambda_\Omega(\mathcal{E})$ is dense in $\{x \otimes \bar{y}; x, y, \in \mathcal{H}\}$, which implies that $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \bar{\mathcal{H}}$. When $\{\xi_n\}$ is total in $\mathcal{D}[t_{\mathfrak{M}}]$, it is similarly shown that λ_Ω is strongly cyclic. \square

LEMMA 3.3. – *Let $\Omega \geq 0 \in \mathcal{H} \otimes \bar{\mathcal{H}}$. Then the following statements hold:*

(1) λ_Ω is regular if and only if there exists a net $\{K_\alpha\}$ in $\mathcal{B}(\mathcal{H})$ such that $0 \leq K_\alpha \leq I$, $K_\alpha \rightarrow I$ strongly and $\Omega K_\alpha \in \sigma_2(\mathfrak{M})$ for every α .

(2) λ_Ω is strongly regular if and only if there exists a net $\{K_\alpha\}$ in $\mathcal{B}(\mathcal{H})$ such that $0 \leq K_\alpha \leq I$, $K_\alpha \uparrow I$ strongly, $\Omega K_\alpha \in \sigma_2(\mathfrak{M})$ for every α and $K_\alpha K_\beta = K_\beta K_\alpha$ for every α, β .

(3) Suppose that $\Omega \mathcal{E} \subset \mathcal{D}$ for some dense subspace \mathcal{E} of \mathcal{H} . Then λ_Ω is strongly regular.

Proof. – (1) Suppose λ_Ω is regular. By Lemma 3.1, (4) there exists a net $\{\pi'(K_\alpha)\}$ in $\pi'(\mathcal{B}(\mathcal{H}))$ such that $0 \leq \pi'(K_\alpha) \leq I$, $\pi'(K_\alpha) \uparrow I$ strongly and $\pi'(K_\alpha) \lambda_\Omega(\pi(X)) = \pi(X^\dagger)^* \lambda_\Omega^c(\pi'(K_\alpha))$ for all $X \in D(\lambda_\Omega)$. It is clear that $0 \leq K_\alpha \leq I$ and $K_\alpha \uparrow I$ strongly. Since

$$\begin{aligned} \langle \lambda_\Omega^c(\pi'(K_\alpha)) | \lambda_\Omega(\pi(X^\dagger Y)) \rangle &= \langle \pi(X) \lambda_\Omega^c(\pi'(K_\alpha)) | \lambda_\Omega(\pi(Y)) \rangle \\ &= \langle \pi'(K_\alpha) \pi(X^\dagger)^* \Omega | \lambda_\Omega(\pi(Y)) \rangle \\ &= \langle \pi(X^\dagger)^* \Omega | \pi(Y) \lambda_\Omega^c(\pi'(K_\alpha)) \rangle \\ &= \langle \Omega | \pi(X^\dagger Y) \lambda_\Omega^c(\pi'(K_\alpha)) \rangle \\ &= \langle \Omega | \pi'(K_\alpha) \lambda_\Omega(\pi(X^\dagger Y)) \rangle \\ &= \langle \pi'(K_\alpha) \Omega | \lambda_\Omega(\pi(X^\dagger Y)) \rangle \end{aligned}$$

for all $X, Y \in D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger$ and $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$, it follows that $\Omega K_\alpha = \pi'(K_\alpha)\Omega = \lambda_\Omega^c(\pi'(K_\alpha)) \in \sigma_2(\mathfrak{M})$. The converse is trivial.

(2) This is shown in the same way as (1).

(3) Since \mathcal{H} is a separable Hilbert space and \mathcal{E} is dense in \mathcal{H} , there exists an ONB $\{\xi_n\}$ in \mathcal{H} contained in \mathcal{E} . Since $\Omega \mathcal{E} \subset \mathcal{D}$, the sequence $\left\{ \sum_{k=1}^n \xi_k \otimes \bar{\xi}_k; n \in \mathbb{N} \right\}$ satisfies the conditions in (2). Hence, λ_Ω is strongly regular. \square

Using Theorem 2.8, we can now construct the (quasi) weights on $\mathcal{P}(\mathfrak{M})$ associated to the regular generalized vector λ_Ω .

LEMMA 3.4. – *Let $\Omega \geq 0 \in \mathcal{H} \otimes \overline{\mathcal{H}}$. Then the following statements hold:*

(1) *Suppose λ_Ω is regular and put*

$$\begin{aligned} \varphi_\Omega \left(\sum_k X_k^\dagger X_k \right) &= \omega_{\lambda_\Omega} \left(\sum_k \pi(X_k^\dagger) \pi(X_k) \right), \\ \sum_k \pi(X_k^\dagger) \pi(X_k) &\in D(\lambda_\Omega). \end{aligned}$$

Then φ_Ω is a quasi-weight on $\mathcal{P}(\mathfrak{M})$ such that $\pi(\mathfrak{N}_{\varphi_\Omega}) = D(\lambda_\Omega)$.

(2) *Suppose λ_Ω is strongly regular. For each family of mutually commuting elements $\{\pi'(K_\alpha)\}$ of $\mathcal{R}_{\lambda_\Omega}$, put*

$$\varphi_\Omega^{\{K_\alpha\}}(X) = \begin{cases} \lim_\alpha \langle \pi(X) \pi'(K_\alpha)\Omega | \pi'(K_\alpha)\Omega \rangle, & \text{if the limit exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $\varphi_\Omega^{\{K_\alpha\}}$ is a weight on \mathfrak{M}_+ such that $\varphi_\Omega \subset \varphi_\Omega^{\{K_\alpha\}}$ and $\pi(\mathfrak{N}_{\varphi_\Omega^{\{K_\alpha\}}}) = \pi(\mathfrak{N}_{\varphi_\Omega}) = D(\lambda_\Omega)$. \square

LEMMA 3.5. – *Let $\Omega \geq 0 \in \mathcal{H} \otimes \overline{\mathcal{H}}$ such that Ω^{-1} is densely defined. Suppose λ_Ω is regular. Then $\lambda_\Omega^c((D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$ and $\lambda_\Omega^{cc}(D(\lambda_\Omega^{cc}) \cap D(\lambda_\Omega^{cc})^*)^2$ is an achieved left Hilbert algebra in $\mathcal{H} \otimes \overline{\mathcal{H}}$, which equals $\pi''(\mathcal{B}(\mathcal{H}))\Omega$. Its modular conjugation operator $J_{\lambda_\Omega^{cc}}$ coincides with the anti-isometry $J : T \mapsto T^*$, $T \in \mathcal{H} \otimes \overline{\mathcal{H}}$ and its modular operator $\Delta_{\lambda_\Omega^{cc}}$ coincides with the positive self-adjoint operator $\pi''(\Omega^{-2})\pi''(\Omega^2)$.*

Proof. – By Lemma 3.3 there exists a net $\{K_\alpha\}$ in $\mathcal{B}(\mathcal{H})$ such that $0 \leq K_\alpha \leq I$, $K_\alpha \rightarrow I$ strongly and $\Omega K_\alpha \in \sigma_2(\mathfrak{M})$ for each α . Then we have

$$\pi'(K_\alpha A K_\beta) \in D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*. \tag{3.1}$$

Since

$$\begin{aligned} & \|\pi'(K_\alpha AK_\beta)\Omega - \pi'(A)\Omega\|_2 \\ & \leq \|\pi'(K_\alpha)\pi'(A)\pi'(K_\beta)\Omega - \pi'(K_\alpha)\pi'(A)\Omega\|_2 \\ & \quad + \|\pi'(K_\alpha)\pi'(A)\Omega - \pi'(A)\Omega\|_2 \\ & \leq \|\pi'(A)(\pi'(K_\beta) - I)\Omega\|_2 + \|(\pi'(K_\alpha) - I)\pi'(A)\Omega\|_2 \end{aligned}$$

for all α, β and $A \in \mathcal{B}(\mathcal{H})$, we have

$$\left. \begin{aligned} \lim_{\alpha, \beta} \|\pi'(K_\alpha AK_\beta)\Omega - \pi'(A)\Omega\|_2 &= 0, \\ \lim_{\alpha, \beta} \|\pi'(K_\beta A^* K_\alpha)\Omega - \pi'(A^*)\Omega\|_2 &= 0, \end{aligned} \right\} \quad (3.2)$$

for all $A \in \mathcal{B}(\mathcal{H})$. Since $\pi'(\mathcal{B}(\mathcal{H}))\Omega$ is dense in $\mathcal{H} \otimes \overline{\mathcal{H}}$, it follows from (3.1) and (3.2) that $\lambda_\Omega^c(D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)$ is dense in $\mathcal{H} \otimes \overline{\mathcal{H}}$. Furthermore, since $\pi'(K_\gamma)\pi'(K_\alpha AK_\beta) \in (D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2$ for every α, β, γ and $A \in \mathcal{B}(\mathcal{H})$, it follows that $\lambda_\Omega^c((D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$. By (3.1) we have

$$\begin{aligned} & \{\pi'(K_\alpha AK_\beta)\Omega; A \in \mathcal{B}(\mathcal{H}); \alpha, \beta\} \\ & \subset \lambda_\Omega^c(D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*) \subset \pi'(\mathcal{B}(\mathcal{H}))\Omega, \end{aligned}$$

and so it follows from (3.2) that $\lambda_\Omega^{cc}(D(\lambda_\Omega^{cc}) \cap D(\lambda_\Omega^{cc})^*)$ is an achieved left Hilbert algebra in $\mathcal{H} \otimes \overline{\mathcal{H}}$ and it equals the achieved left Hilbert algebra $\pi''(\mathcal{B}(\mathcal{H}))\Omega$. By ([4], Lemma 5.2) we have $J_{\lambda_\Omega^{cc}} = J$ and $\Delta_{\lambda_\Omega^{cc}} = \pi'(\Omega^{-2})\pi''(\Omega^2)$. \square

THEOREM 3.6. – *Suppose $\Omega \geq 0 \in \mathcal{H} \otimes \overline{\mathcal{H}}$ satisfies the conditions:*

- (i) Ω^{-1} is densely defined.
- (ii) λ_Ω is regular.
- (iii) $\Omega^{it}\mathcal{D} \subset \mathcal{D}$ and $\Omega^{it}\mathfrak{M}\Omega^{-it} = \mathfrak{M}$, $t \in \mathbb{R}$.

*Then λ_Ω is a standard generalized vector for $\pi(\mathfrak{M})$ and the quasi-weight φ_Ω on $\mathcal{P}(\mathfrak{M})$ satisfies the KMS-condition with respect to the one-parameter group of *-automorphisms of \mathfrak{M} implemented by $\{\Omega^{it}\}_{t \in \mathbb{R}}$.*

Proof. – By (iii) we have

$$X\Omega^{it}T = \Omega^{it}(\Omega^{-it}X\Omega^{it})T \in \mathcal{H} \otimes \overline{\mathcal{H}}$$

for all $X \in \mathfrak{M}$, $T \in \sigma_2(\mathfrak{M})$ and $t \in \mathbb{R}$. Hence it follows that $\pi''(\Omega^{it})\sigma_2\mathfrak{M} \subset \sigma_2\mathfrak{M}$ for all $t \in \mathbb{R}$, which implies by Lemma 3.5 that

$$\Delta_{\lambda_\Omega^{cc}}^{it}\sigma_2(\mathfrak{M}) = \pi'(\Omega^{-2it})\pi''(\Omega^{2it})\sigma_2(\mathfrak{M}) \subset \sigma_2(\mathfrak{M})$$

and

$$\Delta_{\lambda_\Omega^{cc}}^{it}\pi(\mathfrak{M})\Delta_{\lambda_\Omega^{cc}}^{-it} = \pi(\Omega^{2it}\mathfrak{M}\Omega^{-2it}) = \pi(\mathfrak{M})$$

for all $t \in \mathbb{R}$. Therefore, λ_Ω is standard, and so the quasi-weight ω_Ω on $\mathcal{P}(\mathfrak{M})$ satisfies the KMS-condition with respect to $\{\sigma_t^{\lambda_\Omega^{cc}}\}_{t \in \mathbb{R}}$. Since $\Delta_{\lambda_\Omega^{cc}}^{it} \pi(X) \Delta_{\lambda_\Omega^{cc}}^{-it} = \pi(\Omega^{2it} X \Omega^{-2it})$ for all $X \in \mathfrak{M}$ and $t \in \mathbb{R}$, it follows from Lemma 3.4 that φ_Ω satisfies the KMS-condition with respect to the one-parameter group of *-automorphisms of \mathfrak{M} implemented by $\{\Omega^{it}\}_{t \in \mathbb{R}}$. \square

COROLLARY 3.7. – *Let H be a positive self-adjoint operator in \mathcal{H} , $\mathcal{D} = \bigcap_{n=1}^\infty \mathcal{D}(H^n)$ and $\Omega \geq 0 \in \mathcal{H} \otimes \overline{\mathcal{H}}$. Suppose Ω^{-1} is densely defined and $\Omega H \subset H \Omega$. Then λ_Ω is a standard generalized vector for $\pi(\mathcal{L}^\dagger(\mathcal{D}))$.*

Proof. – Let us take on ONB $\{\xi_n\}$ in \mathcal{H} contained in \mathcal{D} . Since $\xi_n \otimes \bar{\xi}_m \in \mathcal{L}^\dagger(\mathcal{D})$ for $n, m \in \mathbb{N}$, it follows from Lemma 3.2 that $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$. Since $\Omega H \subset H \Omega$, it follows that $\Omega, \Omega^{it} \in \mathcal{L}^\dagger(\mathcal{D})$ for all $t \in \mathbb{R}$, so that λ_Ω is standard by Lemma 3.3 and Theorem 3.6. \square

Following the general theory of Section 2, we now look for sufficient conditions for λ_Ω to be a modular generalized vector.

THEOREM 3.8. – *Let $\Omega \geq 0 \in \mathcal{H} \otimes \overline{\mathcal{H}}$. Suppose*

- (i) Ω^{-1} is densely defined;
- (ii) $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$;
- (iii) *there exists a dense subspace \mathcal{E} of $\mathcal{D}[t_{\mathfrak{M}}]$ such that*
 - (iii)₁ $\mathfrak{M} \mathcal{E} \subset \mathcal{E}$,
 - (iii)₂ $\Omega \mathcal{E} \subset \mathcal{E}$,
 - (iii)₃ $\Omega^{it} \mathcal{E} \subset \mathcal{E}, t \in \mathbb{R}$.

Then λ_Ω is a modular generalized vector for $\pi(\mathfrak{M})$.

Proof. – It follows from (ii), (iii)₂ and Lemma 3.3 that λ_Ω is regular, which implies by Lemma 3.5 that $\lambda_\Omega^{cc}(D(\lambda_\Omega^{cc}) \cap D(\lambda_\Omega^{cc})^*)$ is an achieved left Hilbert algebra in $\mathcal{H} \otimes \overline{\mathcal{H}}$, and it equals $\pi''(\mathcal{B}(\mathcal{H}))\Omega$ and $\Delta_{\lambda_\Omega^{cc}} = \pi'(\Omega^{-2})\pi''(\Omega^2)$. We denote by \mathcal{K} the linear span of $\{\xi \otimes \bar{y}; \xi \in \mathcal{E}, y \in \mathcal{H}\}$. Since \mathcal{E} is dense in $\mathcal{D}[t_{\mathfrak{M}}]$, it follows that \mathcal{K} is dense in $\sigma_2(\mathfrak{M})[t_{\pi(\mathfrak{M})}]$. We next show that

$$\begin{aligned} \{\lambda_\Omega^c(K); K \in D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^* \text{ s.t. } \lambda_\Omega^c(K), \lambda_\Omega^c(K^*) \in \mathcal{K}\}^2 \\ \text{is total in the Hilbert space } \mathcal{D}(S_{\lambda_\Omega^{cc}}^*). \end{aligned} \tag{3.3}$$

In fact, let us take an ONB $\{\eta_n\}$ in \mathcal{H} contained in \mathcal{E} and put

$$E_n = \sum_{k=1}^n \eta_k \otimes \bar{\eta}_k, \quad n \in \mathbb{N}.$$

Then we have

$$\pi'(E_n A E_m) \in (D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2,$$

$$\lambda_\Omega^c(\pi'(E_n A E_m)) = \sum_{k=1}^n \sum_{j=1}^m (A \eta_j | \eta_k) \Omega \eta_k \otimes \bar{\eta}_k \in \mathcal{K} \quad (\text{by (iii)}_2)$$

for all $A \in \mathcal{B}(\mathcal{H})$. Furthermore, we have

$$\begin{aligned} \lim_{n, m \rightarrow \infty} \|\pi'(E_n A E_m) \Omega - \pi'(A) \Omega\|_2 &= 0, \\ \lim_{n, m \rightarrow \infty} \|\pi'(E_n A E_m)^* \Omega - \pi'(A)^* \Omega\|_2 &= 0. \end{aligned}$$

Thus the statement (3.3) holds. By (iii)₂ we have $\pi(\mathfrak{M}) \mathcal{K} \subset \mathcal{K}$, and by (iii)₃, $\Delta_{\lambda_\Omega^c}^{it} \mathcal{K} \subset \mathcal{K}$ for all $t \in \mathbb{R}$. Thus, λ_Ω is modular. This completes the proof. \square

4. STANDARD GENERALIZED VECTORS DETERMINED BY POSITIVE SELF-ADJOINT OPERATORS

Let \mathfrak{M} be a self-adjoint O^* -algebra on \mathcal{D} in \mathcal{H} such that $\mathfrak{M}'_w = \mathbb{C} I$ and Ω a positive self-adjoint (possibly) unbounded operator in \mathcal{H} . We put

$$\begin{cases} D(\lambda_\Omega) = \{\pi(X); X \in \mathfrak{M} \text{ and } \overline{X^{\dagger*} \Omega} \in \sigma_2(\mathfrak{M})\}, \\ \lambda_\Omega(\pi(X)) = \overline{X^{\dagger*} \Omega}, \quad \pi(X) \in D(\lambda_\Omega). \end{cases}$$

Then λ_Ω is a generalized vector for $\pi(\mathfrak{M})$. In this section we consider when λ_Ω is standard or modular.

PROPOSITION 4.1. – *Let \mathfrak{M} be a self-adjoint O^* -algebra on \mathcal{D} in \mathcal{H} and Ω a positive self-adjoint operator in \mathcal{H} such that Ω^{-1} is densely defined. Suppose there exists a subspace \mathcal{E} of $\mathcal{D} \subset \mathcal{D}(\Omega)$ such that*

- (i) $\{\xi \otimes \bar{\eta}; \xi, \eta \in \mathcal{E}\} \subset \mathfrak{M}$.
- (ii) \mathcal{E} is a core for Ω .

Then $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$. Furthermore, if \mathcal{E} is dense in $\mathcal{D}[t_{\mathfrak{M}}]$, then λ_Ω is a strongly cyclic generalized vector for $\pi(\mathfrak{M})$.

Proof. – Since $\{\xi \otimes \bar{\eta}; \xi, \eta \in \mathcal{E}\} \subset \mathfrak{M}$ and \mathcal{E} is dense in \mathcal{H} , it follows that $\mathfrak{M}'_w = \mathbb{C}I$. It is easily shown that $\{\pi(\xi \otimes \bar{\eta}); \xi, \eta \in \mathcal{E}\} \subset (D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2$ and $\lambda_\Omega(\pi(\xi \otimes \bar{\eta})) = \xi \otimes \bar{\Omega}\eta$ for each $\xi, \eta \in \mathcal{E}$. Since $\Omega\mathcal{E}$ is total in \mathcal{H} , it follows that $\{(\xi \otimes \bar{\eta})\Omega; \xi, \eta \in \mathcal{E}\}$ is dense in $\{\xi \otimes \bar{\eta}; \xi, \eta \in \mathcal{D}\}$, and furthermore, since $\{\xi \otimes \bar{\eta}; \xi, \eta \in \mathcal{D}\}$ is total in $\sigma_2(\mathfrak{M})$, it follows that $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$. When \mathcal{E} is dense in $\mathcal{D}[t_{\mathfrak{M}}]$, we can similarly show that λ_Ω is strongly cyclic. This completes the proof. \square

THEOREM 4.2. – Let \mathfrak{M} be a self-adjoint O^* -algebra on \mathcal{D} in \mathcal{H} such that $\mathfrak{M}'_w = I$ and Ω a positive self-adjoint operator in \mathcal{H} . Suppose

- (i) Ω^{-1} is densely defined and $\mathcal{D} \cap \mathcal{D}(\Omega^{-1})$ is a core for Ω^{-1} ;
- (ii) there exists a subspace \mathfrak{N} of \mathfrak{M} such that $\pi(\mathfrak{N}) \subset D(\lambda_\Omega)$, $\mathfrak{N}^\dagger \mathcal{D} \subset \mathcal{D}(\Omega)$ and the linear span of $\mathfrak{N}^\dagger \mathcal{D}$ is a core for Ω ;
- (iii) $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$.

Then the following statements hold:

$$(1) \quad D(\lambda_\Omega^c) = \{\pi'(A); A \in \mathcal{B}(\mathcal{H}) \text{ s.t.} \\ A\mathcal{H} \subset \mathcal{D}(\Omega) \text{ and } \Omega A \in \sigma_2(\mathfrak{M})\}, \\ \lambda_\Omega^c(\pi'(A)) = \Omega A, \quad \pi'(A) \in D(\lambda_\Omega^c)$$

and $\lambda_\Omega^c((D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$.

$$(2) \quad D(\lambda_\Omega^{cc}) = \{\pi''(A); A \in \mathcal{B}(\mathcal{H}) \text{ and } \overline{A\Omega} \in \mathcal{H} \otimes \overline{\mathcal{H}}\}, \\ \lambda_\Omega^{cc}(\pi''(A)) = \overline{A\Omega}, \quad \pi''(A) \in D(\lambda_\Omega^{cc})$$

and $\lambda_\Omega^{cc}(D(\lambda_\Omega^{cc}) \cap D(\lambda_\Omega^{cc})^*)$ is an achieved left Hilbert algebra in $\mathcal{H} \otimes \overline{\mathcal{H}}$.

$$(3) \quad S_{\lambda_\Omega^{cc}} = J \overline{\pi''(\Omega) \pi'(\Omega^{-1})} = J \overline{\pi'(\Omega^{-1}) \pi''(\Omega)},$$

and so $J_{\lambda_\Omega^{cc}} = J$ and $\Delta_{\lambda_\Omega^{cc}} = \overline{\pi''(\Omega^2) \pi'(\Omega^{-2})} = \overline{\pi'(\Omega^{-2}) \pi''(\Omega^2)}$.

(4) Suppose $\Omega^{it} \mathcal{D} \subset \mathcal{D}$ for all $t \in \mathbb{R}$. Then λ_Ω is a modular generalized vector for $\pi(\mathfrak{M})$ with $\mathcal{D}_{\lambda_\Omega} = \mathcal{D}$.

(5) Suppose $\Omega^{it} \mathcal{D} \subset \mathcal{D}$ and $\Omega^{it} \mathfrak{M} \Omega^{-it} = \mathfrak{M}$ for all $t \in \mathbb{R}$. Then λ_Ω is a standard generalized vector for $\pi(\mathfrak{M})$.

Proof. – (1) Take an arbitrary $\pi'(A) \in D(\lambda_\Omega^c)$. Then there exists an element T of $\sigma_2(\mathfrak{M})$ such that

$$\pi(X)T = \pi'(A)(\lambda_\Omega)\pi(X) = \overline{X^{\dagger*}}\Omega A$$

for all $\pi(X) \in D(\lambda_\Omega)$. For each $X \in \mathfrak{N}$, $x \in \mathcal{H}$ and $\xi \in \mathcal{D}$ we have

$$(Ax|\Omega X^\dagger \xi) = (x|A^* \Omega X^\dagger \xi) = (\overline{X^{\dagger*}}\Omega Ax|\xi) = (XTx|\xi),$$

and since the linear span of $\mathfrak{N}^\dagger \mathcal{D}$ is a core for Ω , it follows that $Ax \in \mathcal{D}(\Omega)$ and $\Omega A = T \in \sigma_2(\mathfrak{M})$. Conversely, suppose that $A \in \mathcal{B}(\mathcal{H})$, $A\mathcal{H} \subset \mathcal{D}(\Omega)$ and $\Omega A \in \sigma_2(\mathfrak{M})$. Then, we have

$$\pi(X)\Omega A = X(\Omega A) = \overline{X^{\dagger*}}\Omega A = \pi'(A)\lambda_\Omega(\pi(X))$$

for all $\pi(X) \in D(\lambda_\Omega)$, and so $\pi'(A) \in D(\lambda_\Omega^c)$ and $\lambda_\Omega^c(\pi'(A)) = \Omega A$.

We show that $\lambda_\Omega^c((D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$. In fact, it follows from (1) that

$$\left. \begin{aligned} \pi'(\Omega^{-1}\xi \otimes \overline{\Omega^{-1}\eta}) &\in (D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2, \\ \lambda_\Omega^c(\pi'(\Omega^{-1}\xi \otimes \overline{\Omega^{-1}\eta})) &= \xi \otimes \overline{\Omega^{-1}\eta} \end{aligned} \right\} \quad (4.1)$$

for all $\xi, \eta \in \mathcal{D} \cap \mathcal{D}(\Omega^{-1})$. Since $\Omega^{-1}(\mathcal{D} \cap \mathcal{D}(\Omega^{-1}))$ is dense in \mathcal{H} , it follows that $\{\xi \otimes \overline{\Omega^{-1}\eta}; \xi, \eta \in \mathcal{D} \cap \mathcal{D}(\Omega^{-1})\}$ is dense in $(x \otimes \overline{y}; x, y \in \mathcal{H})$, and further $\{x \otimes \overline{y}; x, y \in \mathcal{H}\}$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$, and hence $\lambda_\Omega^c((D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$.

(2) Take an arbitrary $\pi''(A) \in D(\lambda_\Omega^{cc})$. Then there exists an element T of $\mathcal{H} \otimes \overline{\mathcal{H}}$ such that $\pi''(A)\lambda_\Omega^c(\pi'(B)) = \pi'(B)T$ for all $\pi'(B) \in D(\lambda_\Omega^c)$. By (1) we have

$$A(\Omega B) = TB \quad (4.2)$$

for all $\pi'(B) \in D(\lambda_\Omega^c)$. By (4.1) and (4.2) we have

$$A\Omega(\Omega^{-1}\xi \otimes \overline{\Omega^{-1}\eta}) = T(\Omega^{-1}\xi \otimes \overline{\Omega^{-1}\eta})$$

for all $\xi, \eta \in \mathcal{D} \cap \mathcal{D}(\Omega^{-1})$ and so $A\xi = T\Omega^{-1}\xi$ for all $\xi \in \mathcal{D} \cap \mathcal{D}(\Omega^{-1})$. Since $\mathcal{D} \cap \mathcal{D}(\Omega^{-1})$ is a core for Ω^{-1} , it follows that $A\xi = T\Omega^{-1}\xi$ for all $\xi \in \mathcal{D}(\Omega^{-1})$, and so $A\Omega\xi = T\xi$ for all $\xi \in \mathcal{D}(\Omega)$. Hence, $A\Omega$ is closable and $\overline{A\Omega} = T \in \mathcal{H} \otimes \overline{\mathcal{H}}$. Conversely, take an arbitrary $A \in \mathcal{B}(\mathcal{H})$ such that $\overline{A\Omega} \in \mathcal{H} \otimes \overline{\mathcal{H}}$. Then it follows from (1) that

$$\pi''(A)\lambda_\Omega^c(\pi'(B)) = A(\Omega B) = \overline{A\Omega}B = \pi'(B)\overline{A\Omega}$$

for all $\pi'(B) \in D(\lambda_\Omega^c)$, so that $\pi''(A) \in D(\lambda_\Omega^{cc})$ and $\lambda_\Omega^{cc}\pi''(A) = \overline{A\Omega}$.

(3) By Lemma 3.1 $\pi''(\Omega)$ and $\pi'(\Omega^{-1})$ are positive self-adjoint operators in $\mathcal{H} \otimes \overline{\mathcal{H}}$ affiliated with the von Neumann algebras $\pi''(\mathcal{B}(\mathcal{H}))$ and

$\pi'(\mathcal{B}(\mathcal{H}))$, respectively, and $\pi''(\Omega)\pi'(\Omega^{-1})$ and $\pi'(\Omega^{-1})\pi''(\Omega)$ are positive, essentially self-adjoint operators in $\mathcal{H} \otimes \overline{\mathcal{H}}$ and $\overline{\pi''(\Omega)\pi'(\Omega^{-1})} = \overline{\pi'(\Omega^{-1})\pi''(\Omega)}$. Take an arbitrary $\pi''(A) \in D(\lambda_{\Omega}^{cc}) \cap D(\lambda_{\Omega}^{cc})^*$. Let $\Omega = \int_0^{\infty} \lambda dE(\lambda)$ be the spectral decomposition of Ω and put $E_n = \int_0^n dE(\lambda)$, $n \in \mathbb{N}$. Then we have

$$\begin{aligned} & \lim_{n, m \rightarrow \infty} \pi'(E_n)\pi''(E_m)\overline{A\Omega} = \overline{A\Omega}, \\ & \lim_{n, m \rightarrow \infty} \pi''(\Omega)\pi'(\Omega^{-1})\pi'(E_n)\pi''(E_m)\overline{A\Omega} \\ & = \lim_{n, m \rightarrow \infty} \pi''(E_m)\pi'(E_n)\Omega A = \Omega A. \end{aligned}$$

Hence, it follows that $\overline{A\Omega} \in \mathcal{D}(\overline{\pi''(\Omega)\pi'(\Omega^{-1})})$ and $\overline{\pi''(\Omega)\pi'(\Omega^{-1})A\Omega} = \Omega A$, which implies by (2) that

$$\begin{aligned} J\overline{\pi''(\Omega)\pi'(\Omega^{-1})\pi''A(\Omega)} &= J\overline{\pi''(\Omega)\pi'(\Omega^{-1})A\Omega} = (\Omega A)^* \\ &= \overline{A^*\Omega} = S_{\lambda_{\Omega}^{cc}}\pi''A(\Omega). \end{aligned}$$

Hence, $S_{\lambda_{\Omega}^{cc}} \subset \overline{J\pi''(\Omega)\pi'(\Omega^{-1})}$. Conversely, take an arbitrary $T \in \mathcal{D}(\overline{\pi''(\Omega)\pi'(\Omega^{-1})})$. Then, $T = \overline{A\Omega}$ for some $A \in \mathcal{D}(\pi''(\Omega))$. Hence, we have

$$\pi''(\Omega)\pi'(\Omega^{-1})T = \pi''(\Omega)A = \Omega A \in \mathcal{H} \otimes \overline{\mathcal{H}},$$

and so $(\Omega A)^* = \overline{A^*\Omega} \in \mathcal{H} \otimes \overline{\mathcal{H}}$. Hence, $\pi''(A) \in D(\lambda_{\Omega}^{cc}) \cap D(\lambda_{\Omega}^{cc})^*$. Thus we have

$$T = \pi''(A)\Omega \in \mathcal{D}(S_{\lambda_{\Omega}^{cc}}) \quad \text{and} \quad S_{\lambda_{\Omega}^{cc}}T = J\pi''(\Omega)\pi'(\Omega^{-1})T,$$

and hence $\overline{J\pi''(\Omega)\pi'(\Omega^{-1})} \subset S_{\lambda_{\Omega}^{cc}}$. Thus, we have $S_{\lambda_{\Omega}^{cc}} = \overline{J\pi''(\Omega)\pi'(\Omega^{-1})}$. The statements (4) and (5) follow from (3). This completes the proof. \square

5. PHYSICAL APPLICATIONS

In this section we turn to physical applications and we describe some examples of the construction of Section 4, namely the BCS-Bogolubov model of superconductivity, a class of interacting boson models in Fock space and finally an application to Quantum Field Theory.

EXAMPLE 5.1. – *Dynamics of the BCS-Bogolubov model.*

Let Λ be a finite region of a lattice and $|\Lambda|$ the number of points in Λ . The local C^* -algebra \mathfrak{A}_Λ is generated by the Pauli operators $\sigma_p = (\sigma_p^x, \sigma_p^y, \sigma_p^z)$ at every point $p \in \Lambda$. The σ_p are copies of the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

\mathfrak{A}_Λ is isomorphic to the C^* -algebra of all $2^{|\Lambda|} \times 2^{|\Lambda|}$ -matrices on the $2^{|\Lambda|}$ -dimensional complex Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{p \in \Lambda} C_p^2$, where C_p^2 is the 2-dimensional complex Hilbert space at p . If $\Lambda \subset \Lambda'$ and $A_\Lambda \in \mathfrak{A}_\Lambda$, then $A_\Lambda \rightarrow A_{\Lambda'} = A_\Lambda \otimes (\bigotimes_{p \in \Lambda'} I_p)$ defines the natural imbedding of \mathfrak{A}_Λ into $\mathfrak{A}_{\Lambda'}$.

Let $n = (\ell_x, \ell_y, \ell_z)$ be a unit vector in \mathbb{R}^3 , and put

$$(\sigma n) = \ell_x \sigma^x + \ell_y \sigma^y + \ell_z \sigma^z$$

Then, denoting by $\text{Sp}(\sigma n)$ the spectrum of σn , we have

$$\text{Sp}(\sigma n) = \{1, -1\}.$$

Let $|n\rangle$ be a unit eigenvector associated with 1, and let $\{n\} = \{n_1, n_2, \dots\}$ be an infinite sequence of unit vectors in \mathbb{R}^3 . Then $|\{n\}\rangle = \bigotimes_p |n_p\rangle$ is a

unit vector in the infinite tensor product $\mathcal{H}_\infty = \bigotimes_p C_p^2$. We put

$$\mathfrak{A} = \bigcup_\Lambda \mathfrak{A}_\Lambda$$

and

$$\mathcal{D}_{\{n\}}^0 = \mathfrak{A} |\{n\}\rangle,$$

and we denote the closure of $\mathcal{D}_{\{n\}}^0$ in \mathcal{H}_∞ by $\mathcal{H}_{\{n\}}^0$. Let (n, n^1, n^2) be an orthonormal basis of \mathbb{R}^3 . We put

$$n' = \frac{1}{2} (n^1 - i n^2)$$

and

$$|m, n\rangle = (\sigma n')^m |n\rangle \quad (m = 0, 1).$$

Then we have

$$(\sigma n) |m, n\rangle = (-1)^m |m, n\rangle \quad (m = 0, 1).$$

Thus $\{ |m\rangle, |n\rangle \} = \bigotimes_p |m_p, n_p\rangle; m_p = 0, 1, \sum_p m_p < \infty$ forms

an orthonormal basis in $\mathcal{H}_{\{n\}}$. In this space we define the undounded self-adjoint operator M by

$$M |\{m\}, \{n\}\rangle = \left(\sum_p m_p \right) |\{m\}, \{n\}\rangle.$$

M counts the number of flipped spins in $|\{m\}, \{n\}\rangle$ with respect to $|\{n\}\rangle$. Now we put

$$\mathcal{D}_{\{n\}} = \bigcap_k \mathcal{D}(M^k),$$

and let $\pi_{\{n\}} : \mathfrak{A} \rightarrow \mathcal{L}^\dagger(\mathcal{D}_{\{n\}})$ be the natural realization of \mathfrak{A} on $\mathcal{D}_{\{n\}}$, i.e.

$$\begin{aligned} \pi_{\{n\}}(\sigma_p^i) |\{m\}, \{n\}\rangle &= \sigma_p^i |m_p, n_p\rangle \otimes \left(\prod_{p' \neq p} \otimes |m_{p'}, n_{p'}\rangle \right) \\ &(i = x, y, z). \end{aligned}$$

The BCS-Hamiltonian in the quasi-spin formulation is given by

$$H_\Lambda = \varepsilon \sum_{p=1}^{|\Lambda|} (1 - \sigma_p^z) - \frac{2g}{|\Lambda|} \sum_{\substack{p, q=1 \\ p \neq q}}^{|\Lambda|} \sigma_p^- \sigma_q^+,$$

where

$$\sigma^- = 2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^+ = 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that for the limit of the dynamics,

$$\alpha_t(A) = \lim_{\Lambda \rightarrow \infty} e^{iH_\Lambda t} A e^{-iH_\Lambda t} \quad (A \in \mathfrak{A})$$

fails to exist on the C^* -algebra $\overline{\mathfrak{A}}$.

Let N be a unit vector in \mathbb{R}^3 , and let $\{n\}$ be a sequence

$$\{\varepsilon_1 N, \varepsilon_2 N, \varepsilon_3 N, \dots\} \quad (\varepsilon_p = 1 \text{ or } -1 \text{ for each } p)$$

such that

$$\lim_{k \rightarrow \infty} \frac{1}{2} \sum_{p=1}^k \varepsilon_p = \eta \neq 0.$$

Then the work of Bogolubov and Haag shows that the total Hamiltonian of the BCS-model is given by a self-adjoint operator H_B in $\mathcal{H}_{\{N\}}$

$$H_B = \alpha \sum_{p=1}^k \{\varepsilon_p - (\sigma_p N)\}$$

where α is a constant.

Let \mathfrak{M} be the O^* -algebra generated by $\pi_{\{N\}} \mathfrak{A}$ and $\{f(M); f \text{ is a continuous function on } [0, \infty) \text{ and there exists a polynomial } g \text{ such that } |f(x)| \leq g(x) \text{ on } [0, \infty)\}$ with domain $\mathcal{D}_{\{N\}}$. Then \mathfrak{M} is a self-adjoint O^* -algebra such that $\mathfrak{M}'_w = \mathbb{C}1$. We put

$$\Omega = e^{-\frac{H_B}{2}}.$$

Then Ω satisfies all the conditions in Theorem 4.2. In fact, Ω is a positive self-adjoint operator in $\mathcal{H}_{\{N\}}$ such that

(i) Ω^{-1} is densely defined.

And since $\{|\{m\}, \{N\}\}; \sum_p m_p < \infty\} \subset \mathcal{D}_{\{N\}} \cap \mathcal{D}(\Omega^{-1})$, we have

(ii) $\mathcal{D}_{\{N\}} \cap \mathcal{D}(\Omega^{-1})$ is a core for Ω^{-1} .

Let \mathcal{R} be the set of all rank one projections constructed from $\{|\{m\}, \{N\}\}; \sum_p m_p < \infty\}$, and let \mathfrak{N} be the linear span of $\{R_1 \mathfrak{M} R_2; R_1, R_2 \in \mathcal{R}\}$. Since $\mathcal{R} \subset \mathfrak{M}$, we have $\mathfrak{N} \subset \mathfrak{M}$. And we have

$$\begin{aligned} \pi(\mathfrak{N}) &\subset D(\lambda_\Omega), \\ \mathfrak{N}^\dagger \mathcal{D}_{\{N\}} &\subset \mathcal{D}(\Omega), \end{aligned}$$

and the linear span of $\mathfrak{N}^\dagger \mathcal{D}_{\{N\}}$ is a core for Ω . Let \mathcal{E} be the linear span of $\{|\{m\}, \{N\}\}; \sum_p m_p < \infty\}$. Then \mathcal{E} is a subspace of $\mathcal{D}_{\{N\}} \cap \mathcal{D}(\Omega)$ such that $\{\xi \otimes \bar{\eta}; \xi, \eta \in \mathcal{E}\} \subset \mathfrak{M}$ and \mathcal{E} is a core for Ω . Thus, by Proposition 4.1, $\lambda_\Omega((D(\lambda_\Omega) \cap D(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \bar{\mathcal{H}}$. Furthermore, since $MH_B = M_B M$ on $\mathcal{D}_{\{N\}}$, we have

$$\Omega^{it} \mathcal{D}_{\{N\}} \subset \mathcal{D}_{\{N\}}$$

and

$$\Omega^{it} \mathfrak{M} \Omega^{-it} = \mathfrak{M}.$$

Therefore, by Theorem 4.2, λ_Ω is a standard generalized vector for $\pi_{\{N\}}(\mathfrak{M})$.

Applying now Theorem 2.10 to λ_Ω , we can construct the quasi-weight ω_{λ_Ω} on $\mathcal{P}(\mathfrak{M})$, which satisfies the KMS condition. As discussed in the introduction, this quasi-weight may be thought of as an equilibrium state for the BCS-Bogolubov model.

There is, however, something more: we can find quasi-weights satisfying the KMS condition for the BCS model also in the thermodynamical limit.

Let $\tilde{\mathfrak{A}}$ be the completion of the spin C^* -algebra \mathfrak{A} with respect to the topology ξ_H defined in [12]. From the very definition of ξ_H (see also [16]), one gets:

- (1) $\tilde{\mathfrak{A}}$ is still a $*$ -algebra;
- (2) each representation $\pi_{\{N\}}$ is continuous from \mathfrak{A} into $\mathcal{L}^\dagger(\mathcal{D}_{\{N\}})$ endowed with the quasi-uniform topology $\tau_*^{\mathcal{D}}$, which makes of it a complete topological $*$ -algebra [12].

Then $\pi_{\{N\}}$ can be extended by continuity to a $*$ -representation $\hat{\pi}_{\{N\}}$ defined on the whole $\tilde{\mathfrak{A}}$. Indeed, since $\mathcal{L}^\dagger(\mathcal{D}_{\{N\}})$ is complete under $\tau_*^{\mathcal{D}}$, it follows that $\pi_{\{N\}}(\tilde{\mathfrak{A}}) \subseteq \mathcal{L}^\dagger(\mathcal{D}_{\{N\}})$. This fact, and the two statements (1) and (2) above imply that $\hat{\pi}_{\{N\}}$ is a $*$ -representation and, therefore, $\hat{\pi}_{\{N\}}(\tilde{\mathfrak{A}})$ is an O^* -algebra on $\mathcal{D}_{\{N\}}$. Then we can proceed as shown above and get KMS quasi-weights for the BCS model also in the thermodynamical limit.

EXAMPLE 5.2. – *Dynamics of an Interacting Boson model.*

Let \mathcal{H} be a separable Hilbert space, and let \mathcal{H}^n be the n -fold tensor product of \mathcal{H} . We define an operator S_n on \mathcal{H}^n by

$$S_n(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi} f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n},$$

where the sum is over all permutations. We put

$$\begin{aligned} \mathcal{F}_0(\mathcal{H}) &= \mathbb{C}, \\ \mathcal{F}_N(\mathcal{H}) &= S_n \mathcal{H}^n, \end{aligned}$$

and

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{H}).$$

\mathcal{F} is called the Bose-Fock space. Let A be a self-adjoint operator in \mathcal{H} . We put

$$\begin{aligned} d\Gamma_0(A) &= 0, \\ d\Gamma_n(A) &= A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes I \otimes \cdots \otimes I \\ &\quad + \cdots + I \otimes I \otimes \cdots \otimes I \otimes A \quad (n \geq 1), \end{aligned}$$

and

$$d\Gamma(A) = \bigoplus_{n=0}^{\infty} d\Gamma_n(A).$$

Then $d\Gamma(A)$ is a self-adjoint operator in \mathcal{F} .

Let $\mathfrak{A}_{\text{CCR}}$ be the CCR algebra generated by $\{a^*(f), a(g); f, g \in \mathcal{H}\}$, where $a^*(f)$ is the creation operator in \mathcal{F} and $a(g)$ is the annihilation operator in \mathcal{F} . We also define a number operator N in \mathcal{F} by

$$\begin{cases} D(N) = \xi; \xi = \{\xi^{(n)}\}_{n=0}^\infty, \sum_{n=0}^\infty n^2 \|\xi^{(n)}\|^2 < +\infty \\ N\xi = \{n\xi^{(n)}\}_{n=0}^\infty. \end{cases}$$

Then it follows that $\mathfrak{A}_{\text{CCR}}$ leaves invariant the domain $\mathcal{D} = \bigcap_{k=0}^\infty \mathcal{D}(N^k)$, i.e. $\mathfrak{A}_{\text{CCR}} \mathcal{D} \subset \mathcal{D}$.

Let Δ be the Laplacian operator in $L^2(\mathbb{R}^3)$. We put

$$\begin{aligned} \mathcal{H} &= L^2(\mathbb{R}^3), \\ h &= -\Delta. \end{aligned}$$

A two-body potential is a real function Φ over $\mathbb{R}^3 \times \mathbb{R}^3$ whose values $\Phi(x_1, x_2)$ represent the potential energy of interaction between a particle at the point x_1 and a second particle at the point x_2 . Thus the total interaction energy of n particles at the points x_1, x_2, \dots, x_n is given by

$$U^{(n)}(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} \Phi(x_i, x_j).$$

Note that the symmetry of $U^{(n)}$ is reflected by the symmetry property

$$\Phi(x_i, x_j) = \Phi(x_j, x_i).$$

We assume that Φ is bounded, that is, there exists constant c such that

$$|\Phi(x, y)| < c \quad \text{for all } x, y \in \mathbb{R}^3.$$

The interaction operator $V^{(n)}$ is defined by

$$(V^{(n)} f)(x_1, \dots, x_n) = U^{(n)}(x_1, \dots, x_n) f(x_1, \dots, x_n) \quad (f \in \mathcal{F}_n(\mathcal{H})).$$

We put

$$H_n = d\Gamma_n(h) + V^{(n)},$$

and

$$H = \bigoplus_{n=0}^\infty H_n.$$

Then H is a self-adjoint operator in \mathcal{F} .

Let \mathfrak{M} be the O^* -algebra generated by $\mathfrak{A}_{\text{CCR}}$ and $\{f(N); f \text{ is a continuous function on } [0, \infty) \text{ and there exists a polynomial } g \text{ such that}$

$|f(x)| \leq g(x)$ on $[0, \infty)$ with domain $\mathcal{D} = \bigcap_{k=0}^{\infty} \mathcal{D}(N^k)$. Then \mathfrak{M} is a self-adjoint O^* -algebra such that $\mathfrak{M}'_{\omega} = \mathbb{C}1$. We put

$$\Omega = e^{-\frac{H}{2}}.$$

Then Ω is a positive self-adjoint operator in \mathcal{F} such that

- (i) Ω^{-1} is densely defined.

We put

$$\begin{aligned} \mathcal{F}_F(\mathcal{H}) &= \{ \xi = \{ \xi^{(n)} \}_{n=0}^{\infty} \in \mathcal{F}; \\ &\quad \text{there exists a positive integer } n_0 \text{ such that } \xi^{(n)} = 0 \\ &\quad \text{for all } n \geq n_0 \}. \end{aligned}$$

Since $\mathcal{F}_F(\mathcal{H}) \cap \mathcal{D}(\Omega^{-1})$ is included in $\mathcal{D} \cap \mathcal{D}(\Omega^{-1})$, we have

- (ii) $\mathcal{D} \cap \mathcal{D}(\Omega^{-1})$ is a core for Ω^{-1} .

We put

$$\begin{aligned} \mathcal{F}_S(\mathcal{H}) &= \{ \xi = \{ \xi^{(n)} \}_{n=0}^{\infty} \in \mathcal{F}_F(\mathcal{H}); \\ &\quad \text{each } \xi^{(n)} \text{ is a finite linear combination of simple tensors} \}. \end{aligned}$$

Let \mathcal{R} be the set of all rank one projections constructed from $\mathcal{F}_S(\mathcal{H})$, and let \mathfrak{N} be the linear span of $\{R_1 \mathfrak{M} R_2; R_1, R_2 \in \mathcal{R}\}$. Since $\mathcal{R} \subset \mathfrak{M}$, we have $\mathfrak{N} \subset \mathfrak{M}$. And we have

$$\begin{aligned} \pi(\mathfrak{N}) &\subset \mathcal{D}(\lambda_{\Omega}), \\ \mathfrak{N}^{\dagger} \mathcal{D} &\subset \mathcal{D}(\Omega), \end{aligned}$$

and the linear span of $\mathfrak{N}^{\dagger} \mathcal{D}$ is a core for Ω . Since $\mathcal{F}_S(\mathcal{H})$ is a subspace of $\mathcal{D} \cap \mathcal{D}(\Omega)$ such that $\{ \xi \otimes \bar{\eta}; \xi, \eta \in \mathcal{F}_S(\mathcal{H}) \} \subset \mathfrak{M}$ and $\mathcal{F}_S(\mathcal{H})$ is a core for Ω , by Proposition 4.1, we have $\lambda_{\Omega}((\mathcal{D}(\lambda_{\Omega}) \cap \mathcal{D}(\lambda_{\Omega})^{\dagger})^2)$ is total in $\mathcal{F} \otimes \overline{\mathcal{F}}$. Furthermore, we have

$$\Omega^{it} \mathcal{D} \subset \mathcal{D}.$$

Therefore, by Theorem 4.2, λ_{Ω} is a modular generalized vector for $\pi(\mathfrak{M})$.

Using Theorem 2.10 and Theorem 2.13, we can construct from λ_{Ω} the quasi-weight $\omega_{\lambda_{\Omega}}$ on $\mathcal{P}(\mathcal{U}(\lambda_{\Omega}))$, which satisfies the KMS condition. As in the previous case, we may interpret it as an equilibrium state of the interacting Boson model.

By the way, if the two-body potential Φ is constant, that is,

$$\Phi(x, y) = \text{const. for all } x, y \in \mathbb{R}^3,$$

then we have

$$\Omega^{it} \mathfrak{M} \Omega^{-it} = \mathfrak{M}.$$

Hence, by Theorem 4.2, λ_Ω is a standard generalized vector for $\pi(\mathfrak{M})$.

But, except for this case, the condition

$$\Omega^{it} \mathfrak{M} \Omega^{-it} = \mathfrak{M}$$

is not true in general.

EXAMPLE 5.3. – *An application to Quantum Field Theory.*

Let $f \in \mathcal{S}(\mathbb{R}^4) \mapsto A(f) \in \mathcal{L}^1(\mathcal{D})$ be a Wightman field. To guarantee a sufficient regularity of the field [17], we assume that $\mathcal{D} = \mathcal{D}^\infty(H)$, where $H \geq 0$ is the energy operator. As it is well-known, H is the generator of a unitary representation of time translations $t \in \mathbb{R} \mapsto U(t) \in \mathcal{B}(\mathcal{H})$ under which the field is covariant, *i.e.*

$$U(t) A(f) U(-t) = A(f_t) \quad (5.1)$$

where f_t denotes the time translation of f . Let $\Omega = e^{-H/2}$. Since $H \geq 0$, then Ω is a bounded operator in \mathcal{H} . Suppose that $\Omega \in \mathcal{H} \otimes \overline{\mathcal{H}}$. Let \mathfrak{M} be the O^* -algebra generated by H and $\{A(f); f \in \mathcal{S}(\mathbb{R}^4)\}$ with domain \mathcal{D} . This O^* -algebra is self-adjoint and, if we assume the absence of superselection rules, we also get $\mathfrak{M}'_{\omega} = \mathbb{C}1$. Then the results of Section 3 apply and so λ_Ω is strongly regular by Lemma 3.3(3); moreover, since $\Omega^{it} \mathcal{D} \in \mathcal{D}$ and $\Omega^{it} \mathfrak{M} \Omega^{-it} = \mathfrak{M}, \forall t \in \mathbb{R}$ (because of Eq. (5.1)), by Theorem 3.6, λ_Ω is a standard generalized vector for $\pi(\mathfrak{M})$ and the quasi-weight φ_Ω on $\mathcal{P}(\mathfrak{M})$ satisfies the KMS condition with respect to the group of $*$ -automorphisms of \mathfrak{M} implemented by $\{\Omega^{it}\}_{t \in \mathbb{R}}$.

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