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## **Arithmetic features of rational conformal field theory\***

by

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À la mémoire de Claude Itzykson

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**ABSTRACT.** – The paper differs from the Paris lecture by being shorter on the general introduction to the subject (a version of which can be found in another recent exposé [Tod94]), and by expanding and updating instead the review of the result of [ST94] concerning the Schwarz finite monodromy problem for the Knizhnik-Zamolodchikov equation. I also briefly review the first part of a joint work with Rehren and Stanev [RST94] on ratios of structure constants that characterize conformal embeddings referring for its second part to the original paper.

**RÉSUMÉ.** – Cet article diffère de la conférence donnée à Paris par une introduction au sujet (voir un exposé plus complet dans [Tod94]) et par une revue réactualisée et développée des résultats de [ST94] concernant le problème de monodromie finie de Schwarz pour l'équation de Knizhnik-Zamolodchikov. Nous réexaminons aussi brièvement la première partie d'un travail en collaboration avec Rehren et Stanev [RST94] sur les rapports des constantes de structure caractérisant les plongements conformes, tout en renvoyant le lecteur à l'article original quant à la deuxième partie.

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## INTRODUCTION

Rational conformal field theory (RCFT) gradually opened the way to an intrusion of number theoretic methods into mathematical physics. Intriguing parallels between V. Jones theory of subfactors [Jones83] and patterns of 2-dimensional ( $2D$ ) critical behaviour led to the introduction of lattice models labelled by Dynkin diagrams [Pas87] and to the A-D-E classification of  $A_1^{(1)}$  current algebra and minimal conformal models [CIZ87]. This work used number theoretic tools and triggered an interaction between the two fields (a stage of which was recorded in a subsequent Les Houches meeting – *see* [Wald92]). The trend continues in recent work on the role of Galois symmetry in the search for modular invariants ([deBG91], [RTW93], [CG94], [FG-RSS94], [FSS94], [Gan94]) and in applying the representation theory of affine superalgebras to number theoretic problems [KW94].

The present paper displays (and makes use of) some number theoretic features of braid matrices and structure constants in RCFT models (that have remained unnoticed in otherwise comprehensive studies in the past – *see*, e.g., [FFK89]).

After summarizing in Section 1 the basic ingredients of  $2D$  CFT I shall present in Sections 2 and 3 an updated account of the solution [ST94] of the Schwarz problem [Sch1873] for the Knizhnik-Zamolodchikov (KS) equation [KZ84]. It exploits the fact that the associated monodromy representation of the mapping class group has entries in a cyclotomic field. We conclude in Section 4 with an observation about the invariance under Galois automorphisms of some ratios of (squares of) structure constants in an  $A_1^{(1)}$  theory (whenever an extended chiral algebra appears at a given level  $k$  – *see* [RST94], [PZ94]). The interest in the exceptional conformal embeddings ([BN87], [SW86], [BB87], [AGO87]) involved in these computations stems from the fact that they are not of the Doplicher-Haag-Roberts [DHR69]) type: unlike the case of a  $G/Z$  theory ([Ber87], [FGK88], [SY89]) there is no gauge group (of the first kind) acting on the extended local chiral algebra, which would leave invariant the elements of the original  $A_1^{(1)}$  current algebra.

### 1. CONFORMAL CURRENT ALGEBRAS: BASIC INGREDIENTS

A CFT invariant under finite conformal transformations lives on a covering of compactified Minkowski space: the cylinder  $\widetilde{M}_{s,1} = \mathbb{S}^s \times \mathbb{R}^1$ .

In a  $2D$  CFT, for  $s = 1$ , the basic local fields are conserved chiral currents depending on a single compactified light cone coordinate,  $z = e^{i(\tau-\xi)}$ . (For reviews espousing the present point of view – see [Mack88], [FST89], [Tod94].) We shall restrict our attention to the case when the (associative) observable algebra is the tensor product of two isomorphic *chiral current algebras*

$$\mathfrak{a}_h(\mathfrak{g}) \otimes \bar{\mathfrak{a}}_h(\mathfrak{g}), \tag{1.1}$$

labelled by a simple (compact) Lie algebra  $\mathfrak{g}$  and a positive number  $h$ , the *height* (see equation (1.6) below).  $\mathfrak{a}_h(\mathfrak{g})$  is generated by currents  $J(z)$ , where  $J$  is a matrix (with a field operator entries) that transforms according to the adjoint representation of  $\mathfrak{g}$ .  $J$  satisfies the local commutation relations (CR)

$$[J^1(z_1), J^2(z_2)] = [C_{12}, J^1(z_1)] \delta(z_{12}) - k C_{12} \delta'(z_{12}) \tag{1.2}$$

where  $k (= 1, 2, \dots)$  is the (Kac-Moody) *level*,  $z_{12} = z_1 - z_2$ ,  $C_{12}$  is the *Casimir operator* in the tensor product space of two fundamental representations,

$$C_{12} = \eta^{ab} t_a^1 t_b^2 = \tau^a t_a^2 = t_a^1 \tau^a \tag{1.3a}$$

where

$$\eta_{ab} = \text{tr } t_a t_b, \quad \eta^{ab} = \text{tr } \tau^a \tau^b, \quad \eta^{as} \eta_{sb} = \delta_b^a = \text{tr } (\tau^a t_b). \tag{1.3b}$$

We are using in (1.2) and in (1.3a) the shorthand tensor product notation

$$\overset{1}{A} = A \otimes \mathbf{1}, \quad \overset{2}{B} = \mathbf{1} \otimes B. \tag{1.4}$$

$\mathfrak{a}_h(\mathfrak{g})$  contains the Sommerfield-Sugawara *stress-energy tensor* (see [GO88] or [FST89] and references therein),

$$T(z) = \frac{1}{2h} \text{tr} : J^2(z) : \tag{1.5}$$

where the height  $h$  is the shifted (“quantum”) level

$$h = k + g^\vee, \tag{1.6}$$

$g^\vee$  being the *dual Coxeter number* ( $g^\vee = n$  for  $\mathfrak{g} \simeq A_{n-1} \simeq su_n$ ,  $n \geq 2$ ;  $g^\vee = n - 2$  for  $\mathfrak{g} \simeq so_n$ ,  $n \geq 5$ ;  $g^\vee = n + 1$  for  $\mathfrak{g} \simeq sp_{2n}$ ,  $n \geq 1$ ;  $g^\vee(E_r) = \frac{\dim E_r}{r} - 1 = 12, 18, 30$  for  $r = 6, 7, 8$ ;  $g^\vee(G_2) = 4$ ,

$g^\vee(F_4) = 9$ ). The normal ordering in (1.5) is defined with respect to the current modes,

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad J_n^* = J_{-n}, \quad J_n |0\rangle = 0 \quad \text{for } n \geq 0 \quad (1.7)$$

(in particular,  $: J_m, J_n := J_n J_m := J_m J_n$  for  $m > n$ ).

The current CR (1.2), just as well as the CR for  $T(z)$  (1.5), can be deduced from the covariance relations

$$[\overset{1}{J}, \overset{2}{J}_0] = [C_{12}, \overset{1}{J}(z)] \quad \text{for } J_0 = \oint_{S^1} J(z) \frac{dz}{2\pi i} \quad (1.8)$$

$$[L_{-1}, T(z)] = \frac{d}{dz} T(z), \quad L_{-1} = \oint T(z) \frac{dz}{2\pi i}. \quad (1.9)$$

In the case of the stress energy tensor one recovers a local (world sheet) version of the Virasoro algebra,

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n-m}, \quad (1.10)$$

for

$$T(z) = \sum_n L_n z^{-n-2} \quad (L_n^* = L_{-n}) \quad (1.11)$$

$c$  being the Virasoro central charge that takes a fixed (positive) value in a (unitary) CFT. This is a consequence of Wightman axioms supplemented by the requirement of dilation invariance and by the condition (1.9) (as established by Schroer [Sch74] and Lüscher and Mack – see [Mack88] and [FST89] for a review and further references). A similar argument yields (1.2) (cf. [FST89]). The fact that  $T$  is given by (1.5) as a composite field of  $J$  [which can be deduced from the assumption that  $T$  belongs to an appropriate closure of the current algebra  $\mathfrak{a}_h(\mathfrak{g})$ ] allows to express the Virasoro charge  $c$  in terms of the level  $k$ , the height  $h$  and the dimension  $d(\mathfrak{g})$  of  $\mathfrak{g}$ :

$$c = c_k(\mathfrak{g}) = \frac{k}{h} d(\mathfrak{g}) \left( d(su_n) = n^2 - 1, d(so_n) = \frac{n(n-1)}{2} \text{ etc.} \right). \quad (1.12)$$

The splitting (1.1) of the observable algebra into chiral parts suggests, as an (important!) intermediate step in constructing a CFT, the study of chiral superselection sectors, chiral vertex operators (CVO) and conformal blocks.

The set of admissible chiral superselection sectors coincides with the set of integrable unitary highest weight modules  $\mathfrak{H}_\Lambda$  of the level  $k$  Kac-Moody algebra [Kac85]. They are labelled by weights  $\Lambda = (\lambda_1, \dots, \lambda_r)$  ( $r$  being the rank of  $\mathfrak{g}$ ) satisfying

$$(\Lambda, \theta^\vee) = \sum_{i=1}^r a_i \lambda_i \leq k \quad \left( 1 + \sum_{i=1}^r a_i = g^\vee \right) \tag{1.13}$$

where  $\theta$  is the highest root of  $\mathfrak{g}$  and  $a_i$  are positive integers such that

$$\theta^\vee = \sum_{i=1}^r a_i \alpha_i^\vee \quad \text{for} \quad \alpha_i^\vee = \frac{2\alpha_i}{\alpha_i^2}, \quad (\Lambda, \alpha_i^\vee) = \lambda_i, \tag{1.14}$$

$\alpha_i$  being the simple roots of  $\mathfrak{g}$ .  $\alpha_i^\vee$ , the corresponding coroots.

A CVO  $V_\Lambda(z)$ , that interpolates between the vacuum Hilbert space  $\mathfrak{H}_0$  and the space of weight  $\Lambda$ , has a local transformation law,

$$[V_\Lambda(z_1), J_a(z_2)] = V_\Lambda(z_1) t_a^\Lambda \delta(z_{12}), \tag{1.15}$$

but carries, in general, a non-trivial monodromy:

$$V_\Lambda(e^{2\pi i} z) = e^{-2\pi i \Delta_\Lambda} e^{2\pi i L_0} V_\Lambda(z) e^{-2\pi i L_0} \tag{1.16}$$

where the conformal dimension  $\Delta_\Lambda$  is computed from (1.5):

$$\Delta_\Lambda = \frac{C_2(\Lambda)}{2h}, \quad C_2(\Lambda) = \eta^{ab} t_a^\Lambda t_b^\Lambda. \tag{1.17}$$

Equation (1.15) together with the local reparametrization law

$$[T(z_1), V_\Lambda(z_2)] = -\Delta_\Lambda \delta'(z_{12}) V_\Lambda(z_2) + \delta(z_{12}) V_\Lambda'(z_2) \tag{1.18}$$

yields (1.17) and the operator  $KZ$  equation

$$h \frac{d}{dz} V_\Lambda(z) =: J^a(z) V_\Lambda(z) : t_a^\Lambda \quad (J^a = \eta^{ab} J_b). \tag{1.19}$$

A CVO is fully characterized by the set of its non-vanishing 3-point functions

$$w_3 \equiv w(z_1, \Lambda_1; z_2, \Lambda_2; z_3, \Lambda_3) = \frac{C_{\Lambda_1 \Lambda_2 \Lambda_3}}{z_{12}^{\Delta_{12}} z_{23}^{\Delta_{23}} z_{13}^{\Delta_{13}}} \tag{1.20}$$

where for each permutation  $(i, j, k)$  of  $(1, 2, 3)$  we have

$$\Delta_{ij} = \Delta_{\Lambda_i} + \Delta_{\Lambda_j} - \Delta_{\Lambda_k} \tag{1.21}$$

and  $C_{\Lambda_1 \Lambda_2 \Lambda_3}$  is an invariant tensor in the triple tensor product of irreducible (finite dimensional) representations of  $\mathfrak{g}$  (assuming there is one). In the case

of  $\mathfrak{g} = su_2$  in which the dimensionality of the space of 3 point invariants (for given weights  $\Lambda_i = 2I_i = 0, 1, \dots$ ) is 0 or 1 we identify  $w_3$  with the vacuum expectation value of the product of 3 CVO  $V_{\Lambda_i}$ .

The  $n$ -point blocks span a finite dimensional space  $\mathfrak{H}_n$  of solutions of the KZ equations

$$\left( h \frac{\partial}{\partial z_j} + \sum_{i \neq j} \frac{C_{ij}}{z_{ij}} \right) w(z_1, \Lambda_1; \dots; z_n, \Lambda_n) = 0, \tag{1.22}$$

where  $C_{ij} = C_{ij}(\Lambda_i, \Lambda_j)$  are the bilinear (in the generators of  $\mathfrak{g}$ ) Casimir invariants in the tensor product of irreducible representations  $\Lambda_i$  and  $\Lambda_j$ . A basis of solutions of (1.20) can be defined (in terms of integral representations – see [ZF86], [ChrF87], [TK88], [STH92]) as analytic functions in (a complex neighbourhood of) the real domain

$$z_1 > z_2 > \dots > z_{n-1} > z_n > -z_{n-1}. \tag{1.23}$$

They admit an analytic continuation to multivalued holomorphic functions in the product of projective planes  $\mathbb{P}\mathbb{C}^1$  minus the diagonal (*i.e.* for  $z_i \neq z_j$  if  $i \neq j$ ), thus giving rise to a representation of the monodromy group  $\mathcal{M}_n$ . For equal weights  $\Lambda$  (say, in the case of  $\mathfrak{g} = su_2$  and  $n$  even)  $\mathfrak{H}_n$  carries a projective representation of the mapping class group  $\mathcal{B}_n$  whose abstract definition we proceed to recall.  $\mathcal{B}_n$  is a group of  $n - 1$  generators  $B_1, \dots, B_{n-1}$  which satisfy two type of relations:

(i) the condition that they generate the braid group on  $n$  strands:

$$\begin{aligned} B_i B_{i+1} B_i &= B_{i+1} B_i B_{i+1}, & i = 1, \dots, n - 2; \\ B_i B_j &= B_j B_i & \text{for } |i - j| \geq 2; \end{aligned} \tag{1.24a}$$

(ii) three additional relations characterizing a projective representation of the mapping class group of the sphere with  $n$  punctures (*see, e.g.* [Bir74]):

$$\begin{aligned} B_1 B_2 \dots B_{n-2} B_{n-1}^2 B_{n-2} \dots B_2 B_1 \\ = B_{n-1} B_{n-2} \dots B_2 B_1^2 B_2 \dots B_{n-1} &= e^{-4\pi i \Delta_\Lambda} \end{aligned} \tag{1.24b}$$

$$(B_1 B_2 \dots B_{n-1})^n = e^{-2\pi i n \Delta_\Lambda}. \tag{1.24c}$$

$\mathcal{M}_n$  appears as an invariant subgroup of  $\mathcal{B}_n$  with quotient the permutation group of  $n$  points.

The locality of  $2D$  correlation functions (or, alternatively, the single valuedness of Euclidean Green functions) implies the existence of a  $\mathcal{B}_n$  invariant hermitean form (that is positive definite for unitary theories). The existence of different CFTs associated with the same family of representations of the underlying Kac-Moody algebra [characterized by the height  $h$  (1.6)] is reflected in the existence of different braid invariant forms (consistent with unitarity and uniqueness of the vacuum). These forms contain information about operator product expansions and associated structure constants that is more detailed than (and includes, in particular) the fusion rules of the theory. It is important to know that such an information (invariant under rescaling of conformal blocks) can be derived within the algebraic approach to local quantum physics using recent results of the theory of subfactors (*see* [RST94]). We shall demonstrate in Section 4 that invariant ratios of structure constants can be derived from our preceding study (in Section 2) of the monodromy representation of  $\mathcal{B}_4$ .

## 2. A REGULAR FORM OF THE MONODROMY REPRESENTATION $\mathcal{B}^{(h,l)}$ OF THE MAPPING CLASS GROUP $\mathcal{B}_4$

Restricting attention to 4-point functions we gain in simplicity without losing in generality: the study of  $\mathcal{B}_4$  already reduces the problem of classifying the finite mapping class groups to the study of four non-trivial cases. On the other hand, all consistency requirements for an RCFT can be formulated in terms of 4-point blocks [MS89] to that their study is a key step in constructing the full theory.

Symmetry under  $su(1, 1)$  world sheet fractional linear transformations implies that a conformal block involving 4 CVO of equal dimensions  $\Delta$  can be written in the form

$$w(z_1, z_2, z_3, z_4) = \left( \frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{14}} \right)^{2\Delta} f(\eta) \quad (2.1)$$

where  $\eta$  is the Möbius invariant cross-ratio

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} \quad (z_{ij} = z_i - z_j). \quad (2.2)$$

We shall also make a more serious restriction – to  $\mathfrak{g} = su_2$  – the study of higher rank groups being in its infancy. The general solution of the resulting KZ equation

$$\left( h \frac{d}{d\eta} - \frac{\tilde{C}_{12}}{\eta} + \frac{\tilde{C}_{23}}{1-\eta} \right) f(\eta) = 0 \quad (2.3a)$$

$\tilde{C}_{12} = C_{12} + I_1(I_1 + 1) + I_2(I_2 + 1) - I_s(I_s + 1)$ ,  $I_s = \max(|I_1 - I_2|, |I_3 - I_4|)$  (and a similar expression for  $\tilde{C}_{23}$ ) or, for equal isospins,

$$\tilde{C}_{ij} = C_{ij} + 2I(I + 1), \quad (i, j) = (1, 2) \text{ or } (2, 3) \tag{2.3b}$$

is written as a multiple ( $2I$ -fold) integral over a polynomial  $P_{hI}$  in rational powers of the integration variables  $t_i$  as well as  $t_{ij} = t_i - t_j$ ,  $1 - t_i$ ,  $\pm(\eta - t_i)$  (see [DF84], [ZF86], [ChrF87], [TK88], [STH92], [ST95]). Different solutions are obtained by different choices of contours which join (or surround) the singular points of the integrand. Most often a basis of solutions is chosen which diagonalizes one of the braid group generators  $B_1$  or  $B_2$ . This is however an unfortunate choice for ( $2I < k < 4I$ , since then  $B_i$  are not diagonalizable (and one has to introduce non-integer levels  $k$  or invent some other artificial device to deal with the arising “singularities”). We shall choose instead, following [STH92], a basis of solutions  $\{f_\lambda\}$  of (2.3) for which  $B_1$  and  $B_2$  are an upper and a lower triangular matrices, respectively. The integration contour for  $f_\lambda$  involves  $\lambda$  lines joining 0 and  $\eta$  and  $2I - \lambda$  lines joining  $\eta$  and 1. More precisely, we are dealing with  $2I$  fold integrals of the form

$$\int_0^\eta dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{\lambda-1}} dt_\lambda \int_\eta^1 dt_{\lambda+1} \int_{t_{\lambda+1}}^1 dt_{\lambda+2} \dots \times \int_{t_{2I-1}}^1 dt_{2I} P_{hI}(t_i; \eta). \tag{2.4}$$

Remarkably, the relevant properties of the corresponding braid matrices can be deduced from the very existence of such a “triangular basis” – using the general  $B_4$  relations (1.24) and our knowledge of the 3-point functions (1.20). All braid group generators are similar and hence have the same spectrum. A simple analysis of 3-point functions shows that the common (ordered) spectrum of the commuting matrices  $B_1$  and  $B_3$  is given by

$$\text{Spec } B_1 = \text{Spec } B_3 = \{(-1)^{2I-\lambda} q^{\lambda(\lambda+1)-2I(I+1)}, \lambda = 0, 1, \dots, 2I\} \tag{2.5}$$

where

$$q = e^{i\frac{\pi}{h}}. \tag{2.6}$$

It follows furthermore from (1.24) (applied to  $n = 4$ ) that the corresponding monodromy elements  $B_1^2$  and  $B_3^2$  coincide; combined with (2.5) this gives

$$B_1 = B_3 \quad (B_i \in \mathcal{B}^{(h, I)}). \tag{2.7}$$

Note that (1.24b) and (2.7) imply (1.24c) (for  $n = 4$ ) as well as the involutivity of the  $s \leftrightarrow u$  duality (also called “fusion” – see [MS89]) matrix  $F$ :

$$F := (-1)^{2I} q^{2I(I+1)} B_1 B_2 B_1 \Rightarrow F^2 = 1. \tag{2.8}$$

$F$  gives rise to a similarity transformation between the two remaining generators  $B_1$  and  $B_2$  of  $\mathcal{B}^{(h,I)}$  (which thus obey the same characteristic equation):

$$B_2 = F B_1 F. \tag{2.9}$$

The normalization of the basis (2.4) can be chosen in such a way that  $F$  becomes a simple permutation matrix:

$$F_{\lambda\mu} = \delta_{\lambda+\mu}^{2I} \left( \Rightarrow \text{tr } F = \frac{1 + (-1)^{2I}}{2} \right). \tag{2.10}$$

This still leaves us with a freedom of rescaling of a restricted type:

$$f_\lambda \rightarrow N_\lambda f_\lambda \Rightarrow B_{\lambda\mu} \rightarrow N_\lambda B_{\lambda\mu} N_\mu^{-1} \quad \text{for } N_\lambda = N_{2I-\lambda}. \tag{2.11}$$

One can adopt the following realization of  $B_1$  satisfying these conditions (see [ST94]):

$$(B_1)_{\lambda\mu} = (-1)^{2I-\mu} q^{\mu(\lambda+1)-2I(I+1)} \begin{bmatrix} 2I - \lambda \\ \mu - \lambda \end{bmatrix}. \tag{2.12}$$

Here  $\begin{bmatrix} n \\ m \end{bmatrix}$  are the (real, vanishing for  $n < m$ )  $q$ -binomial coefficients,

$$\begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix} &= \frac{[n]!}{[m]! [n-m]!}, & [n]! &= [n] [n-1]!, \\ [0]! &= 1, & [n] &= \frac{q^n - \bar{q}^{-n}}{q - \bar{q}} \end{aligned} \tag{2.13}$$

( $\bar{q} = q^{-1}$ ). We shall be, however, only interested in properties of  $B_i$  that are invariant under the above rescaling.

*Remark 2.1.* – It is worth noticing that the above basis and the resulting (regular for  $4I > k$  albeit non-unitary) realization of the braid matrices has a quantum group origin. It has in fact been realized (see [HPT91] and references therein) that the algebraic realizations of the braid group  $\mathcal{B}_n$  starting from the universal  $R$ -matrix for  $U_q(\mathfrak{sl}_2)$  ( $q^h = -1$ ) is inverse transposed to the monodromy representations of  $\mathcal{B}_n$  acting on  $n$ -point blocks of  $\mathfrak{a}_h(A_1)$  (primary) CVO. Using a realization of the  $(2I + 1)$ -dimensional

irreducible representation of  $U_q(sl_2)$  (for isospins  $I$  satisfying  $2I \leq k$ ) in a space of polynomials of degree  $2I$  in a formal variable  $u$  (that can be viewed as a coordinate in a quantum homogeneous space – see [SST94]) one can construct a canonical basis of  $n$ -point  $U_q$  invariants [FST91] given by appropriate products of “ $q$ -differences”  $q^{\mu_{ij}} u_i - \bar{q}^{\mu_{ij}} u_j$ . The basis (2.4) of solutions of the  $a_h(A_1)$  KZ equation is dual to such a monomial basis of  $U_q$  invariants for  $n = 4$ .

The representation  $\mathcal{B}^{(h,I)}$  admits at least one invariant hermitean form  $M$  such that

$$B^* MB = M (= M^*), \quad \forall B \in \mathcal{B}^{(h,I)}. \tag{2.14}$$

Noting that the inverse of the matrix  $B_1$  (2.12), and hence also that of  $B_2$  (2.9), both coincide with their complex conjugate,

$$B_i^{-1} = \bar{B}_i, \quad i = 1, 2 \tag{2.15}$$

we deduce that in the above basis the form  $M$  can be taken as real symmetric and satisfying

$${}^t B_i M = M B_i, \quad i = 1, 2, \quad [F, M] = 0, \tag{2.16}$$

where the superscript  $t$  to the left of a matrix stands for transposition. In physical terms, the form  $M$  allows to write down a (real) monodromy free euclidean Green function:

$$G_4 = \bar{w}_\lambda (\{\bar{z}_i\}) M_{\lambda\mu} w_\mu (\{z_i\}). \tag{2.17}$$

The (always existing) standard form  $M$  is degenerate for  $4I > k$  and is diagonalized in the  $s$ -channel basis [characterized by the factorization property (2.27a) below]. We have

$$M = {}^t S D S, \quad D_{\lambda\mu} = D_\lambda^{(h,I)} \delta_{\lambda\mu} \tag{2.18}$$

where  $S$  is an upper triangular matrix with elements

$$S_{\lambda\mu} = (-1)^{\mu-\lambda} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \frac{[2I - \lambda]! [2\lambda + 1]!}{[2I - \mu]! [\lambda + \mu + 1]!} \quad \text{for } 0 \leq \lambda \leq m \tag{2.19a}$$

$$S_{\lambda\mu} = \delta_{\lambda\mu} \quad \text{for } m + 1 \leq \lambda \leq 2I \quad (\text{if } 4I > k) \tag{2.19b}$$

and

$$m = \min(2I, k - 2I) = \frac{1}{2}(k - |k - 4I|). \tag{2.20}$$

The diagonal matrix  $D$  is degenerate iff  $4I > k$ , its eigenvalues being

$$D_\lambda^{(h,I)} = \left\{ \frac{[\lambda]! [2I + 1 + \lambda]!}{[2I + 1]! [2\lambda]!} \right\}^2 \frac{1}{[2\lambda + 1]} \quad \text{for } 0 \leq \lambda \leq m, \quad (2.21a)$$

$$D_\lambda^{(h,I)} = 0 \quad \text{for } \lambda \geq m + 1. \quad (2.21b)$$

The eigenvalues of  $D$  are thus positive for  $\lambda \leq m$ , hence there is, in general, an  $(m + 1)$ -dimensional unitarizable subfactor  $\mathcal{B}(h, 2I | m + 1)$  of  $\mathcal{B}^{(h,I)}$ . Let

$$B_i^{(h,I)} := S B_i S^{-1}, \quad (2.22a)$$

where  $S^{-1}$  has the following block matrix form for  $4I > k$

$$S^{-1} = \begin{pmatrix} \Sigma^{-1} & -\Sigma^{-1} \Sigma' \\ 0 & \mathbf{1} \end{pmatrix} \quad \text{for } S = \begin{pmatrix} \Sigma & \Sigma' \\ 0 & \mathbf{1} \end{pmatrix} \quad (2.22b)$$

with

$$\Sigma_{\lambda\mu}^{-1} (= S_{\lambda\mu}^{-1}) = \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \frac{[2I - \lambda]! [\lambda + \mu]!}{[2I - \mu]! [2\mu]!} \quad \text{for } (0 \leq \lambda \leq) \mu \leq m; \quad (2.22c)$$

then the matrix  $B_1(h, 2I | m + 1)$  (with the first  $m + 1$  rows and columns of  $B_1^{(h,I)}$ ) is diagonal and its eigenvalues are given by (2.5) for  $\lambda \leq m$ . The kernel of the form  $M$ , on the other hand, span (for  $4I > k$ ) a  $(4I - k)$  dimensional invariant submodule (whose factor module is unitarizable). In particular, for  $h$  even and  $I = \frac{h}{4}$  the  $2 \times 2$  bottom square of  $B_1^{(h,I)}$  is an indecomposable Jordan cell of the form

$$\begin{pmatrix} -q^{2I(I-2)} & q^{2I(I-1)} \\ 0 & q^{2I^2} \end{pmatrix} \quad \text{for } \bar{q}^{4I} = \bar{q}^h \quad (= -1).$$

The generators of  $\mathcal{B}^{(h,I)}$  and  $\mathcal{B}(h, k - 2I | 2I + 1)$  for  $2I \leq \frac{k}{2}$  differ by just a phase factor:

$$B_i(h, k - 2I | 2I + 1)_{\lambda\mu} = q^{\frac{1}{2} h (k - 4I)} B_{i\lambda\mu}^{(h,I)}, \quad i = 1, 2. \quad (2.23)$$

Hence the corresponding commutator subgroups coincide. The (common) commutator subgroup  $\mathcal{C}(h, m)$  of

$$\mathcal{B}(h, m) \equiv \mathcal{B}^{(h, \frac{m}{2})} \quad \text{and} \quad \mathcal{B}(h, k - m | m + 1)$$

for  $m \leq \frac{k}{2}$  is generated by

$$b = B_1^{-1} B_2 = B_2 B_1 B_2^{-1} B_1^{-1}, \quad \bar{b} = B_1 B_2^{-1}. \quad (2.24)$$

The factor group of either  $\mathcal{B}(h, m)$  or  $\mathcal{B}(h, k - m | m + 1)$  by  $\mathcal{C}(h, m)$  is a finite cyclic group:

$$\begin{aligned} \mathcal{B}(h, 2I | m + 1) / \mathcal{C}(h, m) &\simeq \mathbb{Z}_N, \\ N &= \begin{cases} h & \text{for } h, 2I \text{ even} \\ 2h & \text{for } h \text{ odd, } 2I \text{ even} \\ 4h & \text{for } 2I \text{ odd.} \end{cases} \end{aligned} \quad (2.25)$$

The form  $M$  is, in general, not unique. Whenever a second form  $\widetilde{M}$  exists we shall also write it as  $\widetilde{M} = {}^t S \widetilde{D} S$  with the same  $S$  (2.19) but with a different (non-diagonal)  $\widetilde{D} \neq D$ . This is motivated by the fact that the  $s$ - and  $u$ -channel bases

$$s_\lambda = S_{\lambda\mu} w_\mu, \quad u_\lambda = (SF)_{\lambda\mu} w_\mu \quad (2.26)$$

are distinguished by their simple factorization properties:

$$\begin{aligned} \lim_{z_4 \rightarrow z_3} \{s_\lambda(z_1, z_2, z_3, z_4) z_{34}^{2\Delta_I - \Delta_\lambda}\} &= w(z_1, I; z_2, I; z_3, \lambda) C_{II}^\lambda \\ (\lambda = 0, 1, \dots, m) \end{aligned} \quad (2.27a)$$

$$\lim_{z_3 \rightarrow z_2} \{u_\lambda(z_1, z_2, z_3, z_4) z_{23}^{2\Delta_I - \Delta_\lambda}\} = C_{II}^\lambda w(z_1, I; z_2, \lambda; z_4, I) \quad (2.27b)$$

where the 3-point function is given by (1.20) (with the weights  $\Lambda_i$  replaced by isospins) and indices of the structure constants  $C$  are lowered (and raised) by the 2-point ‘‘metric tensor’’  $C_{\lambda\mu 0} = C_{\lambda\lambda 0} \delta_{\lambda\mu}$  (and its inverse). We then refer to a theory with diagonal  $D_{\lambda\mu}$  [of type (2.18), (2.21)] as a ‘‘diagonal theory’’. Originally, this term has been associated with diagonal partition functions in the A-D-E classification [CIZ87], which, in fact, implies that the  $2D$  4-point function  $G_4$  has a diagonal form

$$G_4 = \sum_{\lambda=0}^m D_\lambda^{(h, I)} |s_\lambda|^2 = \sum_{\lambda=0}^m D_\lambda^{(h, I)} |u_\lambda|^2. \quad (2.28)$$

For unitary theories  $\widetilde{M}$  is admissible if it is positive semi-definite [in the  $m + 1$  dimensional space where  $M$  (2.18) is positive] and

$$\widetilde{M}_{00} (= \widetilde{D}_{00}) = D_0^{(h, I)} (= M_{00} = 1). \quad (2.29)$$

### 3. GALOIS AUTOMORPHISMS AND ALGEBRAIC SOLUTIONS OF THE KZ EQUATION

We now address the question when is the unitarizable subfactor  $\mathcal{B}(h, m)$  of the preceding section, a finite matrix group. This is equivalent to asking when does the  $su_2$  KZ equation have an algebraic solution. The classification of polynomial solutions of equation (2.3) (corresponding to the special case in which  $\mathcal{B}(h, m)$  has a trivial irreducible subrepresentation) was worked out in [MST92]. It allows to find the local extensions of the chiral current algebra. Algebraic solutions with non-trivial monodromy can be viewed as a generalization of the parafermions of [ZF85]. A classical 19 century paper by H. A. Schwarz [Sch1873] solves a similar problem for the Gauss hypergeometric equation. Much closer to our times the monodromy of higher hypergeometric functions [BH89] and the algebraic Appell-Lauricella functions [CW92] were studied. Algebraic 4-point functions for step operators in minimal conformal models have been found in [RS89].

Our study is based on the key observation that both  $\mathcal{B}(h, m)$  and  $M$  have entries in a cyclotomic field. More precisely, the matrices  $q^{2I^2} B_i$  and, as a consequence, the elements of  $\mathcal{C}(h, m)$  have entries in  $\mathbb{Q}(q)$ . In fact, they are polynomials in  $q$  of integer coefficients.  $M$  on the other hand has elements in the real subfield  $\mathbb{Q}([2])$  of  $\mathbb{Q}(q)$  ( $[2] = q + \bar{q}$ ). The point is that the algebraic properties of the group  $\mathcal{B}(h, m)$ , including the number of its elements – in fact, all properties except for the positivity of the invariant form  $M$  – do not depend on which primitive  $h$ -th root of  $-1$   $q$  is chosen to be. In other words, the algebraic properties of the mapping class group only depend on the fact that

$$q^h = -1 \text{ and, if } h = m(2n + 1) \text{ } m, n \geq 1, \text{ then } q^m \neq -1. \quad (3.1)$$

More precisely,  $q$  is any root of an irreducible polynomial equation defined as follows.  $P_h$  belongs to the ring of polynomials of integer coefficients, uniquely determined by the property of being an irreducible element of this ring such that  $P_h(e^{i\frac{\pi}{h}}) = 0$  (requiring further that the coefficient to the leading power of  $q$  is positive, we find that it is 1). Then  $q$  is any root of  $P_h(q) = 0$ .

The Galois group  $\text{Gal}_h$  of  $\mathbb{Q}(q)$  is the group of all substitutions of the form

$$\text{Gal}_h = \{q \rightarrow q^\ell, (\ell, 2h) = 1\}, \quad (3.2a)$$

$(\ell, 2h)$  standing for the greatest common divisor of  $\ell$  and  $2h$ . The substitutions  $q \rightarrow q^\ell$  and  $q \rightarrow q^{\ell+2h}$  are identical and should be counted once. Thus,  $\text{Gal}_h$  can be defined alternatively as the multiplicative group  $(\text{mod } 2h)$  of integers coprime with  $2h$ ,

$$\text{Gal}_h = \{\ell \in \mathbb{N} \pmod{2h}, (\ell, 2h) = 1\}. \quad (3.2b)$$

It permutes the roots of  $P_h$  and the degree  $n(h)$  of  $P_h$  is equal to the number of elements of  $\text{Gal}_h$ . In fact, we can write

$$P_h(q) = \prod_{\substack{\ell=1 \\ (\ell, 2h)=1}}^{2h-1} (q - e^{i \frac{\ell\pi}{h}}). \quad (3.2c)$$

We display, as an example, the multiplication table for  $\text{Gal}_6$  using the realization (3.2b).  $\text{Gal}_6$  has 4 elements which can be taken as  $\ell = 1, 5, 7, 11$  satisfying  $\ell^2 = 1 \pmod{12}$ , and  $ij = k \pmod{12}$  for any permutation  $(i, j, k)$  of  $(5, 7, 11)$ . This is, clearly, the multiplication table for  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

The eigenvalues  $D_\lambda$  (2.21) of the  $\mathcal{B}^{(h, I)}$  invariant form  $M$  (2.18) are positive for  $\lambda = 0, 1, \dots, m$ , provided this is true for the corresponding quantum dimensions:  $[2\lambda + 1] > 0$ . These ‘‘odd quantum dimensions’’ are, on the other hand, polynomials with integer coefficients in  $[3] = q^2 + 1 + \bar{q}^2$  (as a consequence of the recurrence relation  $[2n + 3] = ([3] - 1)[2n + 1] + [2n - 1]$  and of  $[1] = 1$ ). We are thus led to consider the stability subgroup  $\text{Gal}_h^{[3]}$  of  $[3]$  and the factor group

$$G_h = \text{Gal}_h / \text{Gal}_h^{[3]}; \quad (3.3)$$

$G_h$  acts effectively on the (real)  $q$ -number  $[3]$  which also obeys an irreducible polynomial equation  $R_h([3]) = 0$ . The number of elements of  $G_h$  is equal to the degree of  $R_h$  which is  $\frac{1}{2} n(h)$  for odd  $h (\geq 3)$  and  $\frac{1}{4} n(h)$  for even  $h \geq 4$ . We present for readers’ convenience a

table exhibiting  $P_h(q)$ ,  $\text{Gal}_h$ ,  $G_h$ , and  $R_h([3])$  for a few small values of  $h$ :

$h$	$P_h(q)$	$\text{Gal}_h$	$G_h$	$R_h([3])$
3	$q^2 - q + 1$	$\mathbb{Z}_2$	$\{1\}$	$[3]$
4	$q^4 + 1$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{1\}$	$[3] - 1$
5	$q^4 - q^3 + q^2 - q + 1$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$[3]^2 - [3] - 1$
6	$q^4 - q^2 + 1$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{1\}$	$[3] - 2$
10	$q^8 - q^6 + q^4 - q^2 + 1$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$[3]^2 - 3[3] + 1$
12	$q^8 - q^4 + 1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$[3]^2 - 2[3] - 2$
18	$q^{12} - q^6 + 1$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_3$	$[3]^3 - 3[3]^2 + 1$
30	$q^{16} + q^{14} - q^{10} - q^8 - q^6 + q^2 + 1$	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[3]^4 - 3[3]^3 - [3]^2 + 3[3] + 1$

(Note that the subgroup  $\text{Gal}_h^{[3]} = \mathbb{Z}_2 \times \mathbb{Z}_2$  of  $\text{Gal}_h$  for even  $h \geq 4$  involves the nontrivial substitutions  $(q \rightarrow) q^{h+1} = -q, q^{2h-1} = \bar{q}$  and their product  $q \rightarrow q^{h-1} = -\bar{q}$ . For  $h$  odd  $h \pm 1$  are even and hence not coprime with  $2h$ ; we are left in this case with  $\text{Gal}_h^{[3]} = \mathbb{Z}_2$  with nontrivial element  $q \rightarrow q^{2h-1}$ .)

A form  $M$  with entries in an algebraic field  $\mathcal{F}$  is said to be *totally positive* if it is positive for all embeddings of  $\mathcal{F}$  into  $\mathbb{C}$ . Applied to a form  $M$  with entries in  $\mathbb{Q}(q)$  where  $q$  is a particular root of  $P_h(q) = 0$  it is equivalent to the requirement that  $M$  is positive together with all its Galois transforms.

Our solution to the Schwarz problem is based on the following crucial lemma.

PROPOSITION 3.1. – *If the restriction  $M(h, m)$  of the form  $M$  (2.18) to the  $(m + 1)$ -dimensional space where it is non-degenerate,*

$$M(h, m) = {}^t \Sigma D(h, m) \Sigma$$

or

$$M(h, m)_{\lambda\mu} = \sum_{\sigma=0}^m S_{\sigma\lambda} D_{\sigma}^{(h,I)} S_{\sigma\mu}, \quad \lambda, \mu = 0, \dots, m, \quad (3.4)$$

*is totally positive then the matrix group  $\mathcal{B}(h, m)$  is finite. Conversely, if  $\mathcal{B}(h, m)$  is a finite matrix group and the  $\mathcal{B}(h, m)$  invariant hermitean form is unique, then it is totally positive.*

The *proof* of the first statement is based on the observation that any discrete compact group is finite (B. Venkov, private communication). The second uses the fact that any (finite dimensional) representation of a compact group (in particular, of a finite group) is unitarizable.

*Remark 3.1.* – Using Galois automorphisms one can also decide when a given element of  $\mathcal{C}(h, m) \subset \mathcal{B}(h, m)$  is of (in)finite order. We shall illustrate the method on the simplest example of  $2 \times 2$  braid matrices corresponding to  $su_2$  weight  $2I = 1$ , or  $2I = k - 1 = h - 3$ . In both cases the group commutators  $b$  and  $\bar{b}$  (2.24) and the product (2.24) have the following basis independent characteristic:

$$\begin{aligned} \operatorname{tr} b &= \operatorname{tr} \bar{b} = 1 - q^2 - \bar{q}^2 = 2 - [3], \\ \operatorname{tr} b^{-1} \bar{b} &= q^4 + \bar{q}^4 - q^2 - \bar{q}^2 + 2 = [3]^2 - 3[3] + 2. \end{aligned} \quad (3.5)$$

Noting that a matrix  $b$  is unitarizable if its eigenvalues lie on the unit circle and that  $\det b = 1$  ( $= \det \bar{b}$ ) we deduce that for  $h \geq 4$   $b$  is unitarizable iff

$$-2 \leq \operatorname{tr} b = 2 - [3] \leq 2. \quad (3.6)$$

The validity of (3.6) for all Galois images of  $[3]$  is also a necessary and sufficient condition for  $b$  to be of finite order (by the argument of Proposition 3.1 or, more directly, observing that the order of a cyclotomic matrix does not change by a Galois transformation). A straightforward analysis shows that this is only valid for  $h = 4, 6, 10$ ; in the first 2 cases  $[3]$  is Galois invariant (being an integer – see Table 1), in the third one  $[3]$  and its Galois transform satisfy the equation

$$[3]^2 - 3[3] + 1 = 0 \quad \text{so that} \quad [3] = \frac{3 \pm \sqrt{5}}{2} \quad \text{for } h = 10 \quad (3.7)$$

both roots obeying (3.6). According to (3.5)  $\operatorname{tr} b^{-1} \bar{b}$  takes the values 0, 0, 1 for  $h = 4, 6, 10$ , so that  $b^{-1} \bar{b}$  is also of finite order. Indeed we have

$$b^6 = \bar{b}^6 = 1 = (b^{-1} \bar{b})^4 \quad \text{for } h = 4, \quad (3.8a)$$

$$b^4 = \bar{b}^4 = 1 = (b^{-1} \bar{b})^4 \quad \text{for } h = 6, \quad (3.8b)$$

$$b^5 = \bar{b}^5 = 1 = (b^{-1} \bar{b})^6 \quad \text{for } h = 10. \quad (3.8c)$$

Thus we reproduce the result of [Jones83] about the cases of a finite matrix group whose group algebra is a Hecke (Temperley-Lieb) algebra.

**THEOREM 3.2.** – *The group  $\mathcal{B}(h, m)$  is finite only in one of the following cases:*

(i) *for all 1-dimensional representations with  $m = 0$ ; these include (besides the trivial representation  $I = 0$ ) the cases of “simple currents”,  $2I = k$ ; the resulting  $\mathcal{B}(h, k | 0)$  is the cyclic group  $\mathbb{Z}_4$  for odd  $h$ , it is*

$\mathbb{Z}_2$  for  $h = 4\rho$  and is trivial for  $h = 4\rho + 2$  ( $\rho \in \mathbb{N}$ ); there are trivial subrepresentations of  $\mathcal{B}(h, m)$  corresponding to (non-zero) integer spin fields in the  $D_{2\rho+1}$  series (for  $h = 4\rho$ ,  $\rho \geq 2$ ) and in the  $E_r$  series (for  $h = 12, 18, 30$ ) – cf. [CIZ87];

(ii) in the three cases of 2-dimensional representations corresponding to (3.8); the commutator subgroups are then

$$\mathcal{C}(4, 1) = \tilde{\mathfrak{A}}_4 \text{ (binary tetrahedral group)} \tag{3.9a}$$

$$\mathcal{C}(6, 1) = \mathbb{H}_8 \text{ (group of quaternion units)} \tag{3.9b}$$

$$\mathcal{C}(10, 1) = \tilde{\mathfrak{A}}_5 \text{ (binary icosahedral group)}, \tag{3.9c}$$

where  $\tilde{\mathfrak{A}}_n$  is the ( $n!$  element) double cover of the alternating group  $\mathfrak{A}_n$ ; the case (3.9a) is again reproduced for a subrepresentation of  $\mathcal{C}(12, 3)$  corresponding to the  $E_6$  model;

(iii) the group  $\mathcal{B}(6, 2)$  of  $3 \times 3$  matrices whose commutator subgroup  $\mathcal{C}(6, 2)$  is isomorphic to the 27 element Heisenberg-Weyl group over  $\mathbb{Z}_3$ .

We shall *prove* the theorem for the cases in which the invariant form  $M$  is unique postponing the analysis of reducible representations of  $\mathcal{B}(h, m)$  to the end of this section.

According to Proposition 3.1  $\mathcal{B}(h, m)$  is finite provided  $M$  is totally positive, or, in view of (2.21), provided the quantum dimensions  $[2\lambda + 1]$  are positive for  $\lambda = 1, \dots, m$ . The analysis of Remark 3.1 concerning  $\lambda = 1$  already selects just the 3 values of  $h$  appearing in (3.8). The corresponding commutator subgroups (3.9) are identified using their characterization in terms of generators and relations (*see* [CM57]). The 8 element group  $H_8$  is the simplest dicyclic group  $\langle 2, 2, 2 \rangle$  ([CM57] Section 1.7) with defining relations

$$b^2 = \bar{b}^2 = (b\bar{b})^2 \left( = -1 \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right); \tag{3.10}$$

it arises as a representation of the commutator subgroup for  $q^4 - q^2 + 1 = 0$ . The binary alternating groups are identified with the same conventions as follows ([CM57] Section 6.5):

$$\tilde{\mathfrak{A}}_4 = \langle 2, 3, 3 \rangle \Leftrightarrow b^3 = \bar{b}^3 = (b^{-1}\bar{b})^2 (= -1) \text{ for } q^4 = -1, \tag{3.11a}$$

$$\begin{aligned} \tilde{\mathfrak{A}}_5 = \langle 2, 3, 5 \rangle &\Leftrightarrow (b^{-1}\bar{b}^2)^2 = (b^{-1}\bar{b})^3 = \bar{b}^5 = (b^{-1}\bar{b}^2)(b^{-1}\bar{b})\bar{b} \\ &\text{for } q^8 - q^6 + q^4 - q^2 + 1 = 0. \end{aligned} \tag{3.11b}$$

Proceeding to higher spins  $I \geq 1$  which yield a  $(2I + 1)$ -dimensional representation of  $\mathcal{B}(h, 2)$  for  $h \geq 6$  we must have both  $[3] > 0$  and  $[5] > 0$  (for all Galois images of the  $q$ -numbers). This leaves us with a single new group corresponding to  $h = 6$  ( $[5] = 1$  being Galois invariant in this case). The  $SU_3$  subgroup  $\mathcal{C}(6, 2)$  can be identified with the 27 element Heisenberg-Weyl group  $H_{27}$  characterized by the relations

$$b^3 = \bar{b}^3 = \mathbf{1} = c^3, \quad b\bar{b} = c\bar{b}b \tag{3.12a}$$

where  $c$  belongs to the centre  $\mathbb{Z}_3$  of  $SU_3$ :

$$c = q^4 \mathbf{1} \left( = q^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \quad q^{12} = 1 (= q^2 - q^4). \tag{3.12b}$$

Alternatively,  $H_{27}$  can be defined as a group of triangular  $3 \times 3$  matrices with  $\mathbb{Z}_3$  entries

$$H_{27} = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}; \alpha, \beta, \gamma \in \mathbb{Z}/3\mathbb{Z} \right\} \tag{3.13}$$

(G. Pinczon, private communication).

We now proceed to summarizing the result of a study of the cases in which a second braid invariant hermitean form exists, indicating the reducibility of  $\mathcal{B}(h, m)$ . In contrast to the above simple argument this study is based on the A-D-E classification of  $su_2$  current algebra models.

The  $E_6$  and the  $E_8$  models indicate the existence of 1-dimensional invariant subspaces for the matrix groups  $\mathcal{B}(12, 4)$ ,  $\mathcal{B}(12, 6|5)$  and  $\mathcal{B}(30, 10)$ . Indeed, they correspond to conformal embeddings of the (level 10 and 28)  $su_2$  current algebra into rank 2 levels 1 algebras:

$$E_6 : \mathfrak{a}_{12}(A_1) \subset \mathfrak{a}_4(B_2) \quad (A_r \simeq su_{r+1}, B_r \simeq so_{2r+1}) \tag{3.14a}$$

$$E_8 : \mathfrak{a}_{30}(A_1) \subset \mathfrak{a}_5(G_2) \tag{3.14b}$$

and are characterized by modular invariant partition functions

$$Z(E_6) = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 \tag{3.15a}$$

$$Z(E_8) = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 \tag{3.15b}$$

(the subscript of the Virasoro characters  $\chi = \chi(\tau)$  standing for the dimension  $2I + 1$  of the associated representation of  $SU_2$ ). The  $\mathfrak{a}_{12}(A_1)$

primary field of isospin 3 appears as an  $\mathfrak{a}_4(B_2)$  current; similarly, the  $\mathfrak{a}_{30}(A_1)$  primary field of isospin 5 plays the role of an  $\mathfrak{a}_5(G_2)$  current, both having conformal dimension  $\Delta_I = 1$ . The additional currents have polynomial 4-point amplitudes ([Chr87], [MST92]) thus providing local extensions of the  $A_1^{(1)}$  current algebras of heights 12 and 30.

The level 1  $so_5$  model, corresponding to Virasoro central charge  $c_1(so_5) = c_{10}(su_2) = 5/2$ . This can be viewed as a special case of the odd orthogonal group series with

$$c_1(so_{2r+1}) = r + \frac{1}{2}. \tag{3.16}$$

In particular, it is a close analogue of the level 1  $so_3$  model,  $\mathfrak{a}_4(so_3) \simeq \mathfrak{a}_4(su_2)$ , with  $c_2(su_2) = 3/2$ , their braid groups being identical. They both involve a local Fermi field of conformal dimension 1/2 and braid group  $\mathcal{B}(h = 2I(I + 1), m = 2I) = \mathbb{Z}_2$  (the isospin  $I = 1, 2$  coinciding with the rank  $r$  of  $so_{2r+1}$ ), and a magnetization CVO of isospin  $I_r$   $\left( = \frac{1}{2}, \frac{3}{2} \right.$  for  $r = 1, 2$ ) and conformal dimension multiple of  $\frac{1}{16}$ :

$$I_r = \frac{1}{4} r(r + 1), \quad \Delta(I_r) = \frac{2r + 1}{16} \tag{3.17}$$

which intertwines the Neveu-Schwarz and the Ramond sectors of the theory. Its 4-point blocks span a 2-dimensional mapping class group isomorphic to the finite group  $\mathcal{B}(4, 1)$  (with commutator subgroup  $\tilde{\mathfrak{A}}_4$ ). This is, in fact, the only additional (compared to Theorem 3.2) case of a finite braid group of (matrix) dimension bigger than 1 within the A-D-E classification. In particular, the braid group for the 4-point blocks of  $I = 3$ ,  $\Delta_3 = \frac{2}{5}$  CVO of the  $E_8$  model coincides with  $\mathcal{B}(5, 2|2)$  and is, hence, infinite.

*Remark 3.2.* – The groups  $\mathcal{C}(h, 1)$  of equation (3.9) also appear as the commutator subgroups of the double coverings of the symmetry groups of the Platonic solids (see [Slo83]). These are: (a) the binary tetrahedral group  $\tilde{\mathfrak{A}}_4$  whose commutator subgroup is  $H_8$ ; (b) the 48-element binary octahedral group  $\tilde{\mathfrak{S}}_4$  with commutator subgroup  $\tilde{\mathfrak{A}}_4$ , and (c) the icosahedral group  $\tilde{\mathfrak{A}}_5$  which is simple and coincides with its commutator. We have the exact sequences of groups and group homomorphisms

$$1 \rightarrow H_8 \rightarrow \tilde{\mathfrak{A}}_4 \rightarrow \mathbb{Z}_3 \rightarrow 1 \tag{3.18a}$$

$$1 \rightarrow \tilde{\mathfrak{A}}_4 \rightarrow \tilde{\mathfrak{S}}_4 \rightarrow \mathbb{Z}_2 \rightarrow 1. \tag{3.18b}$$

We see, in particular, that  $\tilde{\mathfrak{A}}_4$  appears in two guises: first, in (3.9a), as the commutator subgroup of  $\mathcal{B}(4, 1)$ , and a second time, in (3.18a) as the subgroup  $\tilde{\mathcal{B}}(6, 1)$  of  $\mathcal{B}(6, 1)$  of  $2 \times 2$  matrices of determinant 1.  $\tilde{\mathcal{B}}(6, 1)$  is, in fact, generated by the elements

$$m = B_1^{-2} B_2^2 = \begin{pmatrix} q^2 + \bar{q}^2 - 1 & \bar{q} - \bar{q}^3 \\ q^3 - q & 1 \end{pmatrix}, \quad \bar{m} = B_1^2 B_2^{-2} \quad (3.19)$$

satisfying the defining relations for the group  $\langle 2, 3, 3 \rangle$ :

$$m^3 = \bar{m}^3 = (m \bar{m})^2 (= -1) \quad \text{for } q^2 + \bar{q}^2 = 1 \quad (\Rightarrow m \bar{m} \bar{b} = 1). \quad (3.20)$$

(Note that for  $q^4 = -1$ ,  $m = \bar{m} = \bar{b} b^{-2} \bar{b} = b \bar{b}^{-2} b$  so that  $\tilde{\mathfrak{S}}_4$  does not appear as a subgroup of  $\mathcal{B}(4, 1)$ . Equations (3.9) and (3.18) reflect our interpretation of [Jones83] correspondence between (finite group) Hecke algebras and symmetry groups of Platonic solids.

#### 4. CHARACTERISTIC RATIOS OF STRUCTURE CONSTANTS

We now proceed to evaluate the invariant under rescaling ratios of structure constants for the two exceptional conformal embeddings (3.14). As stated at the end of Section 1 they characterize the extended ( $E$ -even type) chiral algebra in its relation to the diagonal ( $A_{k+1}$ ) theory. Indeed, they provide a quantitative comparison between two inclusions of local field algebras: the embedding (3.14) and the inclusion of  $\mathfrak{a}_h(A_1)$  into  $2D$  (local) field algebra of the diagonal theory. Moreover, they can be computed from these data using subfactor theory [RST94]. We shall demonstrate here that they can be calculated from the two  $\mathcal{B}(h, m)$  invariant forms  $\tilde{D}$  and  $D$  (Section 2) appearing for  $h = 12$  and  $30$ . (Structure constants have also been computed by other methods – see [Pet89], [FK89], [FKS90], [PZ94].)

We shall determine the non diagonal  $s$ -channel forms  $\tilde{D}^{(I)}$  by the requirement that they satisfy

$$[B_1^{(I)}, \tilde{D}^{(I)}] = 0, \quad {}^t F^{(I)} \tilde{D}^{(I)} = \tilde{D}^{(I)} F^{(I)} \quad (4.1)$$

where  $B_i^{(I)}$  are the  $s$ -channel  $\mathcal{B}(h, m)$  generators (2.22) [a short hand for  $B_i(h, 2I | m + 1)$ ].

Commutation with the diagonal matrix  $B_1^{(I)}$  implies that a non diagonal element  $\tilde{D}_{0\lambda}^{(I)}$  ( $\lambda \neq 0$ ) can only appear if the corresponding eigenvalues of  $B_1^{(I)}$  coincide, *i.e.*, for

$$(\tilde{D}_{0\lambda}^{(I)} \neq 0 \Rightarrow) \quad (-1)^\lambda q^{\lambda(\lambda+1)} = 1. \quad (4.2)$$

This is satisfied for the “triangular number” heights

$$h = (2\rho + 1)(2\rho + 2), \quad \lambda = 2\rho + 1, \quad \rho = 1, 2. \quad (4.3)$$

The non-diagonal invariant form  $\tilde{D}^{(I)}$  is then given by

$$\tilde{D}_{\lambda\mu}^{(I)} = d_\lambda^I d_\mu^I + C_\lambda^I \delta_{\lambda\mu} \quad (C_\lambda^I d_\lambda^I = 0) \quad (4.4)$$

where  $d_\lambda^I = \tilde{D}_{\lambda 0}^{(I)}$ ,  $d_0^I = 1$ , and  $d_{2\rho+1}^I = 1$  and  $C_\lambda^I$  ( $\lambda \neq 0, 2\rho + 1$ ) are determined from  $F^{(I)}$  covariance. (The possibility to express the “structure constants”  $d_\lambda^I$  in terms of the matrix elements of  $F^{(I)}$  justifies the name fusion matrix for  $F$ .) The analysis is facilitated by the knowledge of the operator content of the  $E$  theories reflected in (3.15). The fact that some (integer) isospins  $\mu$  are not present among the primary weights implies that  $\tilde{D}_{\lambda\mu}^{(I)} = \tilde{D}_{\mu\lambda}^{(I)} = 0$ ; hence,

$$\sum_{\sigma=0}^m F_{\sigma\mu}^{(I)} \tilde{D}_{\sigma\lambda}^{(I)} = 0 \quad (\text{for } h=12, \mu=1, 4; h=30, \mu=1, 2, 4, 7, \dots). \quad (4.5)$$

Just one of these equations is sufficient to determine the non-diagonal element  $\tilde{D}_{2\rho+10}^{(I)} = d_{2\rho+1}^I$  provided it is the only non-vanishing  $d_\lambda^I$  for  $\lambda > 0$ . This is indeed the case for the  $E_6$  model; we have

$$d_3^I = -\frac{F_{01}^{(I)}}{F_{31}^{(I)}} \quad \text{for } h = 12, \quad I = \frac{3}{2}, 2, 3. \quad (4.6)$$

In the  $E_8$  model there are two  $A_1$  primary fields that mix with the identity for  $I = 5$  ( $d_\lambda^5 \neq 0$  for  $\lambda = 5, 9$ ) and one should take a system of two equations of type (4.5) (say, for  $\mu = 1, 2$ ) to determine them. Alternatively, we can view the  $E_8$  chiral algebra (that involves 4  $A_1$  primary fields corresponding to isospins 0, 5, 9, 14) as an extension of the  $D_{16}$  algebra (generated by the simple current of  $I = 14, \Delta_{14} = 7$ ). In that case the CVO  $V_I$  for  $I = 5$  and 9 ( $\Delta_I = 1$  and 3) are combined into a single  $D_{16}$  CVO, the corresponding ( $I = 5$ ) representation of  $\mathcal{B}_4$  being 9 dimensional (rather than  $2I + 1 = 11$  dimensional as it is the case for the  $A_{29}$ -theory).

Then we again have a single  $\lambda > 0$  ( $\lambda = 5$ ) for which  $\tilde{D}_{0\lambda}^{(5)} \neq 0$  ( $\tilde{D}$  denoting the second – non-diagonal – braid invariant form in the  $D_{16}$  theory).

Under a rescaling

$$s_\lambda \rightarrow N_\lambda s_\lambda \quad (\lambda = 0, \dots, m) \quad (4.7a)$$

of the  $s$ -channel blocks [equivalent to (2.11)] the entries of  $\tilde{D}^{(I)}$  also change

$$\tilde{D}_{\lambda\mu}^{(I)} \rightarrow N_\lambda^{-1} N_\mu^{-1} \tilde{D}_{\lambda\mu}^{(I)}. \quad (4.7b)$$

The diagonal form  $D$  (2.18) rescales under an identical law so that the ratios

$$\frac{\tilde{D}_{\lambda\lambda}^{(I)}}{D_{\lambda\lambda}^{(I)}} \left( = \left( \frac{F_{01}^{(I)}}{F_{\lambda 1}^{(I)}} \right)^2 \frac{F_{\lambda 0}^{(I)}}{F_{0\lambda}^{(I)}} \text{ for } h = 12 \right). \tag{4.8}$$

remain invariant. Given that  $\tilde{D}_{2\rho+1, 2\rho+1}^{(I)} = (d_{2\rho+1}^I)^2$  – see (4.4) – we identify these (positive) quotients as *ratios of squares of structure constants*. (For a study of the sign freedom in determining  $d_\lambda^I$  – see [PZ94].) To compute these characteristic ratios for the two conformal embeddings (3.14) we need [according to (4.5)] the  $s$ -channel  $F$ -matrix

$$F_{\lambda\mu}^{(I)} = \sum_{\sigma, \tau=0}^{2I} S_{\lambda\sigma} \delta_{\sigma+\tau}^{2I} S_{\tau\mu}^{-1}, \quad \lambda, \mu = 0, \dots, m \tag{4.9a}$$

– cf. (2.22) (note the difference between the range of summation and the range of indices  $\lambda, \mu$ ). Inserting  $S$  from (2.19) (2.22b) we find

$$F_{\lambda\mu}^{(I)} = \frac{[\mu]! [2\lambda + 1]! [2I - \lambda]!}{[\lambda]! [2\mu]! [2I - \mu]!} \sum_{\nu=0}^{\mu} \frac{(-1)^{2I-\lambda+\nu} [\mu + \nu]! ([2I - \nu]!)^2}{[2I - \lambda - \nu]! ([\nu]!)^2 [\mu - \nu]! [2I + \lambda - \nu + 1]!}. \tag{4.9b}$$

In evaluating (4.8) we also use  $q$ -number identities that depend on the height,

$h = 12$ :

$$[5] (= [7]) = [3] + 1 (= [2]^2), \quad [3]^2 = 2 [2]^2, \quad [6] = 2 [2]; \tag{4.10}$$

$h = 30$ :

$$[9] = [5] + [3] (= [21]), \quad [11] = [3]^2 = [9] + 1, \quad [15] = 2 [5]. \tag{4.11}$$

The outcome of a lengthy but straightforward calculation is

$h = 12$ :

$$d_3^3 = -\frac{1}{[5]}, \quad d_3^2 = -\frac{[3]}{[5]}, \quad \frac{\tilde{D}_{33}^{(I)}}{D_{33}^{(I)}} = 2 \quad \text{for } I = 2, 3, \tag{4.12a}$$

$$\frac{\tilde{D}_{33}^{(I)}}{D_{33}^{(I)}} = \frac{1}{2}, \quad \frac{\tilde{D}_{22}^{(I)}}{D_{22}^{(I)}} = 1 - \frac{F_{01}^{(I)} F_{32}^{(I)}}{F_{02}^{(I)} F_{31}^{(I)}} = \frac{3}{2} \quad \text{for } I = \frac{3}{2}; \tag{4.12b}$$

$h = 30$ :

$$\begin{aligned} \frac{\tilde{D}_{55}^{(5)}}{D_{55}^{(5)}} &= \left( \frac{F_{02}^{(5)} F_{91}^{(5)} - F_{01}^{(5)} F_{92}^{(5)}}{F_{51}^{(5)} F_{92}^{(5)} - F_{52}^{(5)} F_{91}^{(5)}} \right)^2 \frac{F_{50}^{(5)}}{F_{05}^{(5)}} = \frac{9}{4}, \\ \frac{\tilde{D}_{99}^{(5)}}{D_{99}^{(5)}} &= \left( \frac{F_{01}^{(5)} F_{52}^{(5)} - F_{02}^{(5)} F_{51}^{(5)}}{F_{51}^{(5)} F_{92}^{(5)} - F_{91}^{(5)} F_{52}^{(5)}} \right)^2 \frac{F_{90}^{(5)}}{F_{09}^{(5)}} = \frac{5}{4}. \end{aligned} \tag{4.13}$$

Remarkably, all these (independent of scale conventions) ratios are rational numbers and are, hence, invariant under Galois automorphisms. This observation (also made in [PZ94] for a wider class of examples) still awaits its explanation.

### 5. CONCLUDING REMARKS

The intuition that a local quantum field theory is determined by “the germ of its observable algebra”, consisting of local functions of the stress energy tensor and of the internal symmetry currents, requires some amendment. Depending on certain number theoretic properties of the height  $h$  a  $2D$  conformal current algebra may give rise to one or several (up to 3 for  $A_1^{(1)}$ ) RCFT. An individual  $\mathfrak{a}_h(\mathfrak{g})$  CFT is distinguished by a maximal local extension of  $\mathfrak{a}_h$  and by a (finite) series of braid invariant hermitean forms  $M$  in the spaces of 4-point blocks of the primary CVOs of the theory.

Number theoretic properties have been gradually unravelled in classifying modular invariant partition functions ([CIZ87], [CG94], [FSS94]). The work pursued (and reviewed) in this paper exhibits and exploits some number theoretic features of the forms  $M$  and of the associated monodromy representations of the mapping class group of the 2-sphere with 4 punctures. The non-unitary basis of solutions of the KZ equation introduced in [STH92] (with emphasis on its regularity for  $4I > k$ ) gives rise to a representation  $\mathcal{B}^{(h,I)}$  of  $\mathcal{B}_4$  with elements in the cyclotomic field  $\mathbb{Q}(q)$  where  $q$  is a primitive root of  $q^h = -1$ . The hermitean form  $M$  has entries in the real subfield  $\mathbb{Q}([2])$  of  $\mathbb{Q}(q)$  ( $[2] = q + \bar{q}$ ,  $\bar{q} = q^{-1}$ ). It is positive definite (in the  $m+1$  dimensional “physical subspace” of 4-point blocks) for  $[2] = 2 \cos \frac{\pi}{h}$ , but its image under a Galois automorphism  $q \rightarrow q^\ell$ ,  $(2h, l) = 1$ , in general, is not. If it is, – i.e., if for some pair  $(h, I)$   $M$  is totally positive, then the corresponding representation  $\mathcal{B}(h, 2I | m+1)$  of  $\mathcal{B}_4$  is a finite (algebraic) group. Ratios of diagonal elements of the  $s$ -channel images  $\tilde{D}$

and  $D$  of two hermitean forms  $\widetilde{M}$  and  $M$ , corresponding to the same pair  $(h, I)$ , characterize (in a normalization independent way) a conformal embedding that provides a non-trivial extension of the chiral current algebra. Surprisingly, they are found to be rational numbers (invariant under Galois automorphisms) for all local extensions of  $\mathfrak{a}_h(A_1)$  (and for a number of other examples displayed in [PZ94]).

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## REFERENCES

- [AGO87] J. R. C. ARCURI, J. F. GOMEZ and D. OLIVE, Conformal subalgebras and symmetric spaces, *Nucl. Phys.*, Vol. **B285**, 1987, pp. 327-339.
- [BB87] F. A. BAIS and P. G. BOUWKNEGT, A classification of subgroup truncations of the bosonic string, *Nucl. Phys.*, Vol. **B279**, 1987, pp. 561-570.
- [Ber87] D. BERNARD, String characters from Kac-Moody automorphisms, *Nucl. Phys.*, Vol. **B288**, 1987, pp. 628-648.
- [BH89] F. BEUKERS and G. HECKMAN, Monodromy for the hypergeometric function  ${}_nF_{n-1}$ , *Invent. Math.*, Vol. **95**, 1989, pp. 325-354; F. BEUKERS, Differential Galois theory, in [Wald92], pp. 413-439.
- [Bir74] J. S. BIRMAN, Braids, Links and Mapping Class Groups, *Ann. of Math. Studies*, Vol. **82**, Princeton Univ. Press, 1974.
- [BN87] P. BOUWKNEGT and W. NAHM, Realization of the exceptional modular invariant  $A_1^{(1)}$  partition functions, *Phys. Lett.*, Vol. **B184**, 1987, pp. 359-362.
- [CIZ87] A. CAPPELLI, C. ITZYKSON and J.-B. ZUBER, The A-D-E classification of minimal and  $A_1^{(1)}$  conformal invariant theories, *Commun. Math. Phys.*, Vol. **113**, 1987, pp. 1-26.
- [Chr87] P. CHRISTE, On exceptional operator algebras in the conformally invariant  $SU(2)$  Wess-Zumino-Witten model, *Phys. Lett.*, Vol. **189B**, 1987, pp. 215-220.
- [ChrF87] P. CHRISTE and R. FLÜME, The four point correlation functions of all primary fields of the  $d = 2$  conformally invariant  $SU(2)$   $\sigma$ -model with Wess-Zumino term, *Nucl. Phys.*, Vol. **B282**, 1987, pp. 466-494.
- [CW92] P. B. COHEN and J. WOLFART, Algebraic Appell-Lauricella functions, *Analysis*, Vol. **12**, 1992, pp. 359-376.

- [CG94] A. COSTE and T. GANNON, Galois symmetry in RCFT, *Phys. Lett.*, Vol. **B323**, 1994, pp. 316-321.
- [CM57] H. S. M. COXETER and W. O. J. MOSER, *Generators and Relations for Discrete Groups*, Springer, Berlin *et al.*, 1957.
- [deBG91] J. de BOER and J. GOEREE, Markov traces and  $II_1$  factors in conformal field theory, *Commun. Math. Phys.*, Vol. **139**, 1991, pp. 267-304.
- [DHR69] S. DOPLICHER, R. HAAG and J. E. ROBERTS, Fields, observables and gauge transformations, I and II, *Commun. Math. Phys.*, Vol. **13**, 1969, pp. 1-23 and Vol. **15**, 1969, pp.173-200; S. DOPLICHER and J. E. ROBERTS, Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics, *Commun. Math. Phys.*, Vol. **131**, 1991, pp. 51-107.
- [DF84] VI. S. DOTSENKO and V. A. FATEEV, Conformal algebra and multipoint correlation functions in 2D statistical models, *Nucl. Phys.*, Vol. **B240**, 1984, pp. 312-348.
- [FFK89] G. FELDER, J. FRÖHLICH and G. KELLER, Braid matrices and structure constants for minimal conformal models, *Commun. Math. Phys.*, Vol. **124**, 1989, pp. 647-664.
- [FGK88] G. FELDER, K. GAWEDZKI and A. KUPIAINEN, Spectra of Wess-Zumino-Witten models with arbitrary simple groups, *Commun. Math. Phys.*, Vol. **117**, 1988, pp. 127-158; *see also Nucl. Phys.*, Vol. **B299**, 1988, pp. 355-366.
- [FG-RSS94] J. FUCHS, B. GATO-RIVERA, B. SCHELLEKENS and Ch. SCHWEIGERT, Modular invariants and fusion rule automorphisms from Galois theory, *Phys. Lett.*, Vol. **B334**, 1994, pp. 113-120.
- [FK89] J. FUCHS and A. KLEMM, The computation of the operator algebra in non-diagonal conformal field theories, *Ann. Phys. (N.Y.)*, Vol. **194**, 1989, pp. 303-335.
- [FK90] J. FUCHS, A. KLEMM and C. SCHEICH, The operator algebra of the  $E_8$  type  $SU_2$  WZW theory, *Zeitschr. f. Phys.*, Vol. **C46**, 1990, pp. 71-74.
- [FSS94] J. FUCHS, B. SCHELLEKENS and Ch. SCHWEIGERT, Quasi-Galois symmetries of the modular  $S$ -matrix, Amsterdam preprint, NIKHEF-H/94-37, hep-th/94 12 009, *Commun. Math. Phys.* (to appear).
- [FST89] P. FURLAN, G. SOTKOV and I. T. TODOROV, Two-dimensional conformal quantum field theory, *Riv. Nuovo Cim.*, Vol. **12**, 6, 1989, pp. 1-202.
- [FST91] P. FURLAN, Ya. S. STANEV and I. T. TODOROV, Coherent state operators and  $n$ -point invariants for  $U_q(sl(2))$ , *Lett. Math. Phys.*, Vol. **22**, 1991, pp. 307-319.
- [Gan94] T. GANNON, The classification of  $su(m)_k$  automorphism invariants, IHES, Bures-sur-Yvette, preprint, hep-th/94 08 119; T. GANNON, C. JAKOVLEVIC and M. A. WALTON, Lie group weight multiplicities from conformal field theory, Bures-sur-Yvette, preprint, IHES/P/94/58.
- [GO88] P. GODDARD and D. OLIVE (eds), *Kac-Moody and Virasoro Algebras*, A Reprint Volume for Physicists, World Scientific, Singapore, 1988.
- [HPT91] L. K. HADJIIVANOV, R. R. PAUNOV and I. T. TODOROV, Quantum group extended chiral  $p$ -models, *Nucl. Phys.*, Vol. **B356**, 1991, pp. 387-438.
- [Jones83] V. JONES, Braid groups, Hecke algebras and type  $II_1$  factors, in: *Geometric Methods in Operator Algebras*, Kyoto, 1983, *Pitman Res. Notes in Math.*, Vol. **123**, 1986, pp. 242-273.
- [Kac85] V. G. KAC, *Infinite-Dimensional Lie Algebras*, Second edition (Cambridge Univ. Press, 1985); Third edition (*ibid.*, 1990).
- [KW94] V. G. KAC and M. WAKIMOTO, Integrable highest weight modules over affine superalgebras and number theory, in: *Lie Theory and Geometry, in Honor of Bertram Kostant, Progress in Mathematics*, Vol. **123**, Birkhäuser, Boston *et al.*, 1994, pp. 415-456.
- [KZ84] V. G. KNIZHNIK and A. B. ZAMOLODCHIKOV, Current algebra and Wess-Zumino model in two dimensions, *Nucl. Phys.*, Vol. **B247**, 1984, pp. 83-103.
- [Mack88] G. MACK, Introduction to conformal invariant quantum field theory in two and more dimensions, in: *Nonperturbative Quantum Field Theory*, G.'t Hooft *et al.*, eds. Plenum Press, N.Y., 1988, pp. 353-383.

- [MST92] L. MICHEL, Ya. S. STANEV and I. T. TODOROV, D-E classification of the local extensions of the  $su_2$  current algebras, *Teor. Mat. Fiz.*, Vol. **92**, 1992, pp. 507-521; *Theor. Math. Phys.*, Vol. **92**, 1993, p. 1063; Ya. S. STANEV and I. T. TODOROV, Local 4-point functions and the Knizhnik-Zamolodchikov equation, in: *Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups*, P. J. SALLY Jr., M. FLATO, J. LEPOWSKY, N. RESHETIKHIN and G. J. ZUCKERMAN, eds., AMS-IMS-SIAM Summer Research Conference, June 1992, Mount Holyoke College, *Contemporary Mathematics*, Vol. **175**, 1994, pp. 249-267.
- [MS89] G. MOORE and N. SEIBERG, Classical and quantum conformal field theory, *Commun. Math. Phys.*, Vol. **123**, 1989, pp. 177-254.
- [Pas87] V. PASQUIER, Operator content of the A-D-E lattice models, *J. Phys. A: Math. Gen.*, Vol. **120**, 1987, pp. 5707-5717.
- [Pet89] V. B. PETKOVA, Structure constants of the  $(A, D)$  minimal  $c < 1$  conformal models, *Phys. Lett.*, Vol. **225**, 1989, pp. 357-362.
- [PZ95] V. B. PETKOVA and J.-B. ZUBER, On structure constants of  $sl(2)$  theories, *Nucl. Phys.*, Vol. **B438**, 1995, pp. 347-372.
- [RS89] K.-H. REHREN and B. SCHROER, Einstein causality and Artin braids, *Nucl. Phys.*, Vol. **B312**, 1989, pp.715-750.
- [RST94] K.-H. REHREN, Y. S. STANEV and I. T. TODOROV, Characterizing invariants for local extensions of current algebras, Hamburg-Vienna preprint DESY 94-164, ESI 132, 1994, hep-th/94 09 165; *Commun. Math. Phys.*
- [RTW93] Ph. RUELLE, E. THIRRRAN and J. WEYERS, Implications of an arithmetical symmetry of the commutant for modular invariants, *Nucl. Phys.*, Vol. **B402**, 1993, pp. 693-708.
- [SST95] H. SAZDJIAN, Ya. S. STANEV and I. T. TODOROV,  $SU_3$  coherent state operators and invariant correlation functions and their quantum group counterparts, *J. Math. Phys.*, Vol. **36**, 1995, pp. 2030-2052.
- [SW86] A. N. SCHELLEKENS and N. P. WARNER, Conformal subalgebras of Kac-Moody algebras, *Phys. Rev.*, Vol. **D34**, 1986, pp. 3092-3101.
- [SY89] A. N. SCHELLEKENS and S. YANKIEWICZ, Extended chiral algebras and modular invariant partition functions, *Nucl. Phys.*, Vol. **B327**, 1989, pp. 673-703; Simple currents, modular invariants and fixed points, *Int. J. Mod. Phys.*, Vol. **A5**, 1990, pp. 2903-2952.
- [Sch74] B. SCHROER, A trip to scalingland, *Proceedings of the V Brazilian Symposium on Theoretical Physics*, Rio de Janeiro, Jan. 1974, ed. by E. FERREIRA (Livros Tecnicos e Cientificos Editora S.A., Rio de Janeiro, 1975).
- [Sch1873] H. A. SCHWARZ, Über diejenigen Fälle in welchen die Gaußsche hypergeometrische Reihe eine algebraische Funktion ihres vierten Elements darstellt, *Reine Angew. Math. (Crelle J.)*, Vol. **75**, 1873, pp. 292-335.
- [Slo83] P. SLODOWY, Platonic solids, Kleinian singularities and Lie groups, *Lect. Notes in Math.*, Vol. **1008**, Springer, Berlin, 1983, pp. 102-138.
- [ST94] Ya. S. STANEV and I. T. TODOROV, On Schwarz problem for the  $s\hat{u}_2$  Knizhnik-Zamolodchikov equation, Vienna preprint, ESI 121, 1994; *Lett. Math. Phys.*, Vol. **35**, 1995, pp. 123-134.
- [ST95] Ya. S. STANEV and I. T. TODOROV, Monodromy representations of the mapping class group  $B_n$  for the  $su_2$  Knizhnik-Zamolodchikov equation, 1995, Schladming lecture, Vienna preprint, ESI 233, 1995.
- [STH92] Ya. S. STANEV and I. T. TODOROV, L. K. HADJIIVANOV, Braid invariant rational conformal models with a quantum group symmetry, *Phys. Lett.*, Vol. **B276**, 1992, pp. 87-94; and in: *Quantum Symmetries*, H.-D. DOEBNER and V. K. DOBREV eds., World Scientific, Singapore, 1993, pp. 24-40.
- [Tod94] I. T. TODOROV, What are we learning from 2-dimensional conformal models? in: *Mathematical Physics Towards the 21st Century*, R. N. SEN and A. GERSTEN, eds., Ben-Gurion University of the Negev Press, 1994, pp. 160-176.

- [TK88] A. TSUCHIYA and Y. KANIE, Vertex operators in the conformal field theory on  $\mathbb{P}^1$  and monodromy representations of the braid group, in: *Conformal Field Theory and Solvable Lattice Models, Advanced Studies in Pure Mathematics*, Vol. **16**, 1988, pp. 297-372; *Lett. Math. Phys.*, Vol. **13**, 1987, pp. 303-312.
- [Wald92] M. WALDSCHMIDT *et al.*, (eds.), *From Number Theory to Physics*, Springer, Berlin, 1992.
- [ZF86] A. B. ZAMOLODCHIKOV and V. A. FATEEV, Operator algebra and correlation functions in the two-dimensional  $SU(2) \times SU(2)$  chiral Wess-Zumino model, *Yad. Fiz.*, Vol. **43**, 1986, pp. 1031-1044 (transl: *Sov. J. Nucl. Phys.*, Vol. **43**, 1986, pp. 657-664).

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