ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 64, n° 4 (1996), p. 479-493

<http://www.numdam.org/item?id=AIHPA_1996__64_4_479_0>

Ann. Inst. Henri Poincaré,

Vol. 64, n° 4, 1996, p. 479-493. *Physique théorique*

Pole-factorization theorem in quantum electrodynamics

by

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ABSTRACT. – In quantum electrodynamics a classical part of the S-matrix is normally factored out in order to obtain a quantum remainder that can be treated perturbatively without the occurrence of infrared divergences. However, this separation, as usually performed, introduces spurious large-distance effects that produce an apparent breakdown of the important correspondence between stable particles and poles of the S-matrix, and, consequently, lead to apparent violations of the correspondence principle and to incorrect results for computations in the mesoscopic domain lying between the atomic and classical regimes. An improved computational technique is described that allows valid results to be obtained in this domain, and that leads, for the quantum remainder , in the cases studied, to a physical-region singularity structure that , as regards the most singular parts, is the same as the normal physical-region analytic structure in theories in which all particles have non-zero mass. The key innovations are to define the classical part in coordinate space, rather than in momentum space, and to define there a separation of the photon-electron coupling into its classical and quantum parts that has the following properties: 1) The contributions from the terms containing only classical couplings can be summed to all orders to give a unitary operator that generates the coherent state that corresponds to the appropriate classical process, and 2) The quantum remainder can be rigorously shown to exhibit, as regards its most singular parts, the normal analytic structure.

This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

RÉSUMÉ. – En électrodynamique quantique, il est d'usage de séparer par factorisation une partie dite classique de la matrice S afin d'obtenir une contribution résiduelle quantique qui soit traitable perturbativement sans production de divergences infrarouges. Cependant, telle qu'elle est réalisée habituellement, cette séparation introduit des effets pervers à grande distance se manifestant par une suppression apparente de la correspondance importante entre les particules stables et les pôles de la matrice S. Il en résulte alors des violations apparentes du principe de correspondance et des résultats incorrects dans les calculs relatifs au domaine mésoscopique, intermédiaire entre l'échelle atomique et l'échelle de la physique classique. Nous décrivons ici une méthode de calcul plus avantageuse permettant d'obtenir des résultats valables dans le domaine mentionné. Dans les cas que nous avons étudiés, la contribution résiduelle quantique obtenue par cette méthode possède une structure de singularités dans la région physique dont les parties les plus singulières sont les mêmes que celles de la structure analytique normale pour la région physique dans les théories sans particules de masse nulle. Les caractéristiques nouvelles de cette méthode consistent à introduire la partie classique dans les variables d'espace-temps plutôt que dans l'espace des impulsions et à définir ainsi une séparation de l'interaction photon-électron en ses parties classique et quantique possédant les propriétés suivantes : 1) Les contributions provenant de termes contenant seulement des couplages classiques peuvent être resommées à tous les ordres pour donner un opérateur unitaire engendrant l'état cohérent qui décrit le processus classique correspondant, et 2) La contribution résiduelle quantique fait apparaître de façon rigoureuse la structure analytique normale, quant à ses parties les plus singulières.

1. INTRODUCTION

The pole-factorization property is the analog in quantum theory of the classical concept of the stable physical particle. This property has been confirmed in a variety of rigorous contexts[1,2,3] for theories in which the vacuum is the only state of zero mass. But calculations[4,5,6] have indicated that the property fails in quantum electrodynamics, due to complications associated with infrared divergences. Specifically, the singularity associated with the propagation of a physical electron has been computed to be not a pole. Yet if the mass of the physical electron were m and the dominant singularity of a scattering function at $p^2 = m^2$ were not a pole then

physical electrons would, according to theory, not propagate over laboratory distances like stable particles, contrary to the empirical evidence.

This apparent difficulty with quantum electrodynamics has been extensively studied[7,8,9], but not fully clarified. It is shown here, at least in the context of a special case that is treated in detail, that the apparent failure in quantum electrodynamics of the classical-type spacetime behaviour of electrons and positrons in the macroscopic regime is due to approximations introduced to cope with infrared divergences. Those divergences are treated by factoring out a certain classical part, before treating the remaining part perturbatively. It can be shown, at least within the context of the case examined in detail, that if an accurate classical part of the photonic field is factored out then the required correspondence-principle and pole-factorization properties do hold. The apparent failure of these latter two properties in references 4 through 7 are artifacts of approximations that are not justified in the context of the calculation of macroscopic spacetime properties: some factors $\exp ikx$ are replaced by substitutes that introduce large errors for small k but very large x.

The need to treat the factor $\exp ikx$ approximately arises from the fact that the calculations are normally carried out in momentum space, where no variable x occurs. The present approach is based on going to a mixed representation in which both x and k appear. This is possible because the variable k refers to photonic degrees of freedom whereas the variable x refers to electronic degrees of freedom.

To have a mathematically well defined starting point we begin with processes that have no charged particles in the initial or final states: the passage to processes where charged particles are present initially or finally is to be achieved by exploiting the pole-factorization property that can be proved in the simpler case considered first. To make everything explicit we consider the case where a single charged particle runs around a spacetime closed loop: in the Feynman coordinate-space picture the loop passes through three spacetime points, x_1, x_2, and x_3, associated with, for example, an interaction with a set of three localized external disturbances. Eventually there will be an integration over these variables. The three regions are to be far apart, and situated so that a triangular electron/positron path connecting them is physically possible. To make the connection to momentum space, and to the pole-factorization theorem and correspondence principle, we must study the asymptotic behaviour of the amplitude as the three regions are moved apart.

Our procedure is based on the separation defined in reference 11 of the electromagnetic interaction operator into its "classical" and "quantum"

parts. This separation is made in the following way. Suppose we first make a conventional energy-momentum-space separation of the (real and virtual photons) into "hard" and "soft" photons, with hard and soft photons connected at "hard" and "soft" vertices, respectively. The soft photons can have small energies and momenta on the scale of the electron mass, but we shall not drop any "small" terms. Suppose a charged-particle line runs from a hard vertex x^- to a hard vertex x^+. Let soft photon j be coupled into this line at point x_j, and let the coordinate variable x_j be converted by Fourier transformation to the associated momentum variable k_j. Then the interaction operator $-ie\gamma_{\mu_j}$ is separated into its "classical" and "quantum" parts by means of the formula

$$-ie\gamma_{\mu_j} = C_{\mu_j} + Q_{\mu_j}, \tag{1.1}$$

where

$$C_{\mu_j} = -ie\frac{z_{\mu_j}}{z \cdot k_j}/k_j, \tag{1.2}$$

$z = x^+ - x^-$, and $/k_j = k_j^\nu \gamma_\nu$.

This separation of the interaction allows a corresponding separation of soft photons into "classical" and "quantum" photons: a "quantum" photon has a quantum coupling on at least one end; all other photons are called "classical" photons.

The full contribution from all classical photons is represented in an extremely neat and useful way. Specialized to our case of a single charged-particle loop $L(x_1, x_2, x_3)$ the key formula reads

$$F_{op}(L(x_1, x_2, x_3)) =: U(L(x_1, x_2, x_3))F'_{op}(L(x_1, x_2, x_3)) : . \tag{1.3}$$

Here $F_{op}(L(x_1, x_2, x_3))$ is the Feynman *operator* corresponding to the sum of contributions from *all* photons coupled into the charged-particle loop $L(x_1, x_2, x_3)$, and $F'_{op}(L(x_1, x_2, x_3))$ is the analogous operator if all contributions from classical photons are excluded. The operators F_{op} and F'_{op} are both normal ordered operators: i.e., they are operators in the asymptotic-photon Hilbert space, and the destruction operators of the incoming photons stand to the right of the creation operators of outgoing photons. On the right-hand side of (1.3) all of the contributions corresponding to classical photons are included in the unitary-operator factor $U(L)$ defined as follows:

$$U(L) = e^{<a^* \cdot J(L)>} e^{-\frac{1}{2}<J^*(L) \cdot J(L)>} e^{-<J^*(L) \cdot a>} e^{i\Phi(L)}. \tag{1.4}$$

Here, for any a and b, the symbol $< a \cdot b >$ is an abbreviation for the integral

$$< a \cdot b > \equiv \int \frac{d^4k}{(2\pi)^4} 2\pi\theta(k_0)\delta(k^2)a_\mu(k)(-g^{\mu\nu})b_\nu(k), \tag{1.5}$$

and $J(L,k)$ is formed by integrating $\exp ikx$ around the loop L:

$$J_\mu(L,k) \equiv \int_L dx_\mu e^{ikx}. \tag{1.6}$$

This classical current $J_\mu(L)$ is conserved:

$$k^\mu J_\mu(L,k) = 0. \tag{1.7}$$

The a^* and a in (1.4) are photon creation and destruction operators, respectively, and $\Phi(L)$ is the classical action associated with the motion of a charged classical particle along the loop L:

$$\Phi(L) = \frac{(-ie)^2}{8\pi} \int_L dx'_\mu g^{\mu\nu} \int_L dx''_\nu \delta((x' - x'')^2) \tag{1.8}$$

The operator $U(L)$ is *pseudo* unitary if it is written in explicitly covariant form, but it can be reduced to a strictly unitary operator using by (1.7) to eliminate all but the two transverse components of $a_\mu(k), a^*_\mu(k), J_\mu(k)$, and $J^*_\mu(k)$.

The colons in (1.3) indicate that the creation-operator parts of the normal-ordered operator F'_{op} are to be placed on the left of $U(L)$.

The unitary operator $U(L)$ has the following property:

$$U(L)|vac >= |C(L)> . \tag{1.9}$$

Here $|vac>$ is the photon vacuum, and $|C(L)>$ represents the normalized coherent state corresponding to the classical electromagnetic field radiated by a charged classical point particle moving along the closed spacetime loop L, in the Feynman sense.

The simplicity of (1.3) is worth emphasizing: it says that the complete effect of all classical photons is contained in a simple unitary operator that is independent of the quantum-photon contributions: this factor is a well-defined unitary operator that depends only on the (three) hard vertices x_1, x_2, and x_3. It is independent of the remaining details of $F'_{op}(L(x_1, x_2, c_3))$, even though the classical couplings are originally interspersed in all possibly

ways among the quantum couplings that appear in $F'_{op}(L(x_1, x_2, x_3))$. The operator $U(L)$ supplies the classical bremsstrahlung-radiation photons associated with the deflections of the charged particles that occur at the three vertices, x_1, x_2, and x_3.

Block and Nordsieck[12] have already emphasized that the infrared divergences arise from the classical aspects of the elecromagnetic field. This classical component is exactly supplied by the factor $U(L)$. One may therefore expect the remainder $F'_{op}(L(x_1, x_2, x_3))$ to be free of infrared problems: if we transform $F'_{op}(L(x_1, x_2, x_3))$ into momentum space, then it should satisfy the usual pole-factorization property. A primary goal of this work is to show that this pole-factorization property indeed holds. To recover the physics one transforms F'_{op} to coordinate space, and then incorporates the real and virtual classical photons by using 1.3 and 1.4.

The plan of the paper is as follows. In the following section 2 rules are established for writing down the functions of interest directly in momentum space. These rules are expressed in terms of operators that act on momentum–space Feynman functions and yield momentum–space functions, with classical or quantum interactions inserted into the charged-particle lines in any specified desired order.

It is advantageous always to sum together the contributions corresponding to all ways in which a photon can couple with C–type coupling into each individual side of the triangle graph G. This sum can be expressed as a sum of just two terms. In one term the photon is coupled at one endpoint, x^+, of this side of G, and in the other term the photon is coupled into the other end point, x^-, of this side of G. Thus all C–type couplings become converted into couplings at the hard–photon vertices of the original graph G.

This conversion introduces an important property. The charge–conservation (or gauge) condition $k^\mu J_\mu = 0$ normally does not hold in quantum electrodynamics for individual graphs: one must sum over all ways in which the photon can be inserted into the graph. But in the form we use, with each quantum vertex Q coupled into the interior of a line of G, but each classical vertex C placed at a hard–photon vertex of G, the charge–conservation equation (gauge invariance) holds for each vertex separately: $k^\mu J_\mu = 0$ for each vertex.

In section 3 the modification of the charged–particle propagator caused by inserting a single quantum vertex Q_μ into a charged-particle line is studied in detail. The resulting (double) propagator is re–expressed as a sum of three terms. The first two are "meromorphic" terms having poles at $p^2 = m^2$ and $p^2 = m^2 - 2pk - k^2$, respectively, in the variable p^2. Because of the special form of the quantum coupling Q_μ each residue is of

first order in k, relative to what would have been obtained with the usual coupling γ_μ. This extra power of k will lead to the infrared convergence of the residues of the pole singularities.

Our proof that this convergence property holds can be regarded as a systematization and confirmation of the argument for infrared convergence given by Grammer and Yennie[13].

The third term is a nonmeromorphic contribution. It is a difference of two logarithms. This *difference* has a power of k that renders the contribution infrared finite.

2. BASIC MOMENTUM–SPACE FORMULAS

The separation of the soft–photon interaction into its quantum and classical parts is defined in Eq. (1.1). This separation is defined in a mixed representation in which hard photons are represented in coordinate space and soft photons are represented in momentum space. In this representation one can consider a "generalized propagator". It propagates a charged particle from a hard–photon vertex y to a hard–photon vertex x with, however, the insertion of soft–photon interactions.

Suppose, for example, one inserts the interactions with two soft photons of momenta k_1 and k_2 and vector indices μ_1 and μ_2. Then the generalized propagator is

$$P_{\mu_1,\mu_2}(x,y;k_1,k_2) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx+i(p+k_1+k_2)y} \times \frac{i}{/p-m+i0}\gamma_{\mu_1}\frac{i}{/p+/k_1-m+i0}\gamma_{\mu_2}\frac{i}{/p+/k_1+/k_2-m+i0}. \tag{2.1}$$

The generalization of this formula to the case of an arbitrary number of inserted soft photons is straightforward. The soft–photon interaction γ_{μ_j} is separated into its parts Q_{μ_j} and C_{μ_j} by means of (1.1), with the x and y defined as in (1.2).

This separation of the soft–photon interaction into its quantum and classical parts can be expressed also directly in momentum space. Using (1.1) and (1.2), and the familiar identities

$$\frac{1}{/p-m}/k\frac{1}{/p+/k-m} = \frac{1}{/p-m} - \frac{1}{/p+/k-m}, \tag{2.2}$$

and

$$\left(-\frac{\partial}{\partial p^\mu}\right)\frac{1}{/p-m}=\frac{1}{/p-m}\gamma_\mu\frac{1}{/p-m}, \tag{2.3}$$

one obtains for the (generalized) propagation from y to x, with a single classical interaction inserted, the expression (with the symbol m standing henceforth for $m-i0$)

$$\begin{aligned}P_\mu(x,y;C,k)&=\int\frac{d^4p}{(2\pi)^4}\left(\frac{i}{/p-m}/k\frac{i}{/p+/k-m}\right)\frac{z_\mu}{zk+io}e^{-ipz+iky}\\&=\int\frac{d^4p}{(2\pi)^4}e^{-ipz+iky}\int_0^1 d\lambda\left(-i\frac{\partial}{\partial p^\mu}\right)\left(\frac{i}{/p+\lambda/k-m}\right)\end{aligned} \tag{2.4}$$

The derivation of this result is given in reference 14. Comparison of the result (2.4) to (2.1) shows that the result in momentum space of inserting a single classical vertex j into a propagator $i(/p-m)^{-1}$ is produced by the action of the operator

$$\widehat{C}_{\mu_j}(k_j)=\int_0^1 d\lambda_j O(p\to p+\lambda_j k_j)\left(-i\frac{\partial}{\partial p^{\mu_j}}\right) \tag{2.5}$$

upon the propagator $i(/p-m)^{-1}$ that was present *before* the insertion of the vertex j. One must, of course, also increase by k_j the momentum entering the vertex at y. The operator $O(p\to p+\lambda_j k_j)$ replaces p by $p+\lambda_j k_j$.

This result generalizes to an arbitrary number of inserted classical photons, and also to an arbitrary generalized propagator: the momentum–space result of inserting in all orders into any generalized propagator $P_{\mu_1,\cdots,\mu_n}(p;k_1,\cdots,k_n)$ a set of N classically interacting photons with $j=n+1,\cdots,n+N$ is

$$\begin{aligned}&\prod_{j=n+1}^{n+N}\widehat{C}_{\mu_j}(k_j)P_{\mu_1,\cdots,\mu_n}(p;k_1,\cdots,k_n)\\&\quad=\int_0^1\cdots\int_0^1 d\lambda_{n+1}\ldots d\lambda_{n+N}\prod_{j=1}^{N}\left(-i\frac{\partial}{\partial p^{\mu_{n+j}}}\right)\\&\quad P_{\mu_1,\cdots,\mu_n}(p+a;k_1,\cdots,k_n)\end{aligned} \tag{2.6}$$

where $a=\lambda_{n+1}k_{n+1}+\cdots+\lambda_{n+N}k_{n+N}$. The operations are commutative, and one can keep each $\lambda_j=0$ until the integration on λ_j is performed.

One may not wish to combine the results of making insertions in all orders. The result of inserting the classical interaction at just one place, identified by the subscript $j\epsilon\{1,\cdots,n\}$, into a (generalized) propagator $P_{\mu_1\cdots\mu_n}(p;k_1,\cdots,k_n)$, abbreviated now by P_{μ_j}, is produced by the action of

$$\widetilde{C}_{\mu_j}(k_j) \equiv \int_0^\infty d\lambda_j O(p_i \to p_i + \lambda_j k_j)\left(-\frac{\partial}{\partial p^{\mu_j}}\right) \tag{2.7}$$

upon $k_j^{\sigma_j} P_{\sigma_j}$.

There is a form analogous to (2.7) for the Q interaction: the momentum–space result produced by the insertion of a Q coupling into $P_{\mu_1\cdots\mu_n}(p;k_1,\cdots k_\mu) = P_{\mu_j}$ at the vertex identified by μ_j is given by the action of

$$\widetilde{Q}_{\mu_j}(k_j) \equiv (\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j})\widetilde{C}_{\rho_j}(k_j) \tag{2.8}$$

upon P_{σ_j} .

An analogous operator can be applied for each quantum interaction. Thus the generalized momentum–space propagator represented by a line L of a graph G into which n quantum interactions are inserted in a fixed order is

$$\begin{aligned}
&P_{\mu_1\cdots\mu_n}(p;Q,k_1,Q,k_2,\cdots Q,k_n) = \\
&\prod_{j=1}^{n}\left[\int_0^\infty d\lambda_j(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j})\left(-\frac{\partial}{\partial p^{\rho_j}}\right)\right] \\
&\left(\frac{i}{/p+/a-m}\gamma_{\sigma_1}\frac{i}{/p+/a+/k_1-m}\gamma_{\sigma_2}\frac{i}{/p+/a+/k_1+/k_2-m}\right. \\
&\left.\cdots\times\gamma_{\sigma_n}\frac{i}{/p+/a+/k_1+\cdots/k_n-m}\right),
\end{aligned} \tag{2.9}$$

where

$$a = \lambda_1 k_1 + \lambda_2 k_2 + \cdots \lambda_n k_n. \tag{2.10}$$

If some of the inserted interactions are classical interactions then the corresponding factors $(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j})$ are replaced by $(\delta_{\mu_j}^{\rho_j} k_j^{\sigma_j})$.

These basic momentum–space formulas provide the starting point for our examination of the analyticity properties in momentum space, and the closely related question of infrared convergence.

One point is worth mentioning here. It concerns the conservation of charge condition $k^\mu J_\mu(k) = 0$. In standard Feynman quantum electrodynamic this condition is not satisfied by the individual photon–interaction vertex, but is obtained only by summing over all the different positions where the photon interaction can be coupled into a graph. This feature is the root of many of the difficulties that arise in quantum electrodynamics.

Equation (2.9) shows that the conservation – law property holds for the individual *quantum* vertex: there is no need to sum over different positions. The classical interaction, on the other hand, has a form that allows one easily to sum over all possible locations along a generalized propagator, even before multiplication by k^μ. This summation converts the classical interaction to a sum of two interactions, one located at each end of the line associated with the generalized propagator. (See, for example, Eq. (4.1) below). We always perform this summation. Then the classical parts of the interaction are shifted to the hard–photon interaction points, at which $k^\mu J_\mu(k) = 0$ holds.

3. RESIDUES OF POLES IN GENERALIZED PROPAGATORS

Consider a generalized propagator that has only quantum–interaction insertions. Its general form is, according to (2.9),

$$\prod_{j=1}^{n}\left[\left(\delta_{\mu_j}^{\sigma_j}k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j}k_j^{\sigma_j}\right)\int_0^\infty d\lambda_j\left(-\frac{\partial}{\partial p^{\rho_j}}\right)\right]$$
$$\left(\frac{i}{/p + /a - m}\gamma_{\sigma_1}\frac{i}{/p + /a + /k_1 - m}\gamma_{\sigma_2}\frac{i}{/p + /a + /k_1 + /k_2 - m}\right.$$
$$\left.\cdots \times \gamma_{\sigma_n}\frac{i}{/p + /a + /k_1 \cdots + /k_n - m}\right) \tag{3.1}$$

where

$$a = \lambda_1 k_1 + \cdots + \lambda_n k_n. \tag{3.2}$$

The singularities of (3.1) that arise from the multiple end–point $\lambda_1 = \lambda_2 = \cdots \lambda_n = 0$ lie on the surfaces

$$p_i^2 = m^2, \tag{3.3}$$

where

$$p_i = p + k_1 + k_2 + \cdots + k_i. \tag{3.4}$$

At a point lying on only one of these surfaces the strongest of these singularities is a pole.

The Feynman function appearing in (3.1) can be decomposed into a sum of poles times residues. At the point $a = 0$ this gives

$$\frac{i(/p + m)\gamma_{\mu_1} i(/p + /k_1 + m)\gamma_{\mu_2} \cdots \gamma_{\mu_n} i(/p + \cdots + /k_n + m)}{(p^2 - m^2)((p + k_1)^2 - m^2)((p + \cdots + k_n)^2 - m^2)} = \sum_{i=0}^{n} \frac{N_{1i}}{D_{1i}} \frac{i(/p_i + m)}{p_i^2 - m^2} \frac{N_{2i}}{D_{2i}}, \tag{3.5}$$

where for each i the numerator occurring on the right–hand side of this equation is identical to the numerator occurring on the left–hand side. The denominator factors are

$$D_{1i} = \prod_{j>i} (2p_i k_{ij} + (k_{ij})^2 + i0), \tag{3.6a}$$

and

$$D_{2i} = \prod_{j>i} (2p_i k_{ij} + (k_{ij})^2 + i0), \tag{3.6b}$$

where

$$k_{ij} = \sigma_{ij}[(k_1 + \cdots + k_j) - (k_1 + \cdots + k_i)]. \tag{3.7}$$

The sign $\sigma_{ij} = \pm$ in (3.7) is specified in reference 14, where it is also shown that that the dominant singularity on $p_i^2 - m^2 = 0$ is the function obtained by simply making the replacement

$$\int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial p^{\rho_j}} \right) (O(p \to p + \lambda_j k_j)) \to p_{i\rho_j} (p_i k_j)^{-1}. \tag{3.8}$$

Each value of j can be treated in this way. Thus the dominant singularity of the generalized propagator (3.1) on $p_i^2 - m^2 = 0$ is

$$\prod_{j=1}^{n} \left[\left(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j} \right) p_{i\rho_j} (p_i k_j)^{-1} \right] \times \frac{N_{1i} i(/p_i + m) N_{2i}}{D_{1i}(p_i^2 - m^2) D_{2i}}. \tag{3.9}$$

The numerator in (3.9) has, in general, a factor

$$\begin{aligned} & i(/p_i - /k_i + m)\gamma_{\sigma_i} i(/p_i + m)\gamma_{\sigma_{i+1}} i(/p_i + /k_{i+1} + m) \\ &= i(/p_i - /k_i + m)\gamma_{\sigma_i} i((/p_i + m) i(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} /k_{i+1}) \\ &\quad + i(/p_i - /k_i + m)\gamma_{\sigma_i}\gamma_{\sigma_{i+1}}(p_i^2 - m^2) \\ &= i(2p_{i\sigma_i} - /k_i\gamma_{\sigma_i}) i(/p + m) i(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} /k_{i+1}) \\ &\quad + i(p_i^2 - m^2)\gamma_{\sigma_i}(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} /k_{i+1}) \\ &\quad + i(/p_i - /k_i + m)\gamma_{\sigma_i}\gamma_{\sigma_{i+1}}(p_i^2 - m^2) \end{aligned} \tag{3.10}$$

The last two terms in the last line of this equation have factors $p_i^2 - m^2$. Consequently, they do not contribute to the residue of the pole at $p_i^2 - m^2 = 0$. The terms in (3.10) with a factor $2p_{i\sigma_{i+1}}$, taken in conjunction with the factor in (3.9) coming from $j = i + 1$, give a dependence $2p_{i\rho_j} 2p_{i\sigma_j}$. This dependence upon the indices ρ_j and σ_j is symmetric under interchange of these two indices. But the other factor in (3.9) is antisymmetric. Thus this contribution drops out. The contribution proportional to $p_{i\sigma_i}$ drops out for similar reasons.

Omitting these terms that do not contribute to the residue of the pole at $p_i^2 - m^2$ one obtains in place of (3.10) the factor

$$(-i/k_i\gamma_{\sigma_i}) i(/p_i + m)(i\gamma_{\sigma_{i+1}} /k_{i+1}) \tag{3.11}$$

which is first–order in both $/k_i$ and $/k_{i+1}$. That these "convergence factors" actually lead to infrared convergence is shown in references 14 and 15.

4. INCLUSION OF THE CLASSICAL INTERACTIONS

The arguments of the preceeding section dealt with processes containing only Q–type interactions. In that analysis the order in which these Q–type interactions were inserted on the line L of G was held fixed: each such ordering was considered separately.

In this section the effects of adding C–type interaction are considered. Each C–type interactions introduces a coupling $k^\sigma\gamma_\sigma = /k$. Consequently, the Ward identities, illustrated in (2.2), can be used to simplify the calculation, but only if the contributions from all orders of its insertion are treated together. This we shall do. Thus for C–type interactions it is the operator $\widehat{C}$ defined in (2.5) that is to be used rather than the operator $\widetilde{C}$ defined in (2.7).

Consider, then, the generalized propagator obtained by inserting on some line L of G a set of n interactions of Q–type, placed in some definite order, and a set of N C–type interactions, inserted in all orders. The meromorphic part of the function obtained after the action of the n operators $\widetilde{Q}_j$ is given by (3.9). The action upon this of the N operators $\widehat{C}_j$ of (2.5) is obtained by arguments similar to those that gave (3.9), but differing by the fact that (2.5) acts upon the propagator present *before* the action of $\widehat{C}_j$, and the fact that now both limits of integration contribute, thus giving for each $\widehat{C}_j$ two terms on the right–hand side rather than one. Thus the action of N such $\widehat{C}_j$'s gives 2^N terms:

$$\begin{aligned}
&\left[\prod_{j=n+1}^{n+N} \widehat{C}_{\mu_j}(k_j) P_{\mu_1\cdots\mu_n}(p; Q, k_1, Q, k_2, \cdots Q, k_n)\right]_{Mero} \\
&\quad = \sum_{\Theta=1}^{2^N} Sgn(\Theta) \sum_{i=0}^{n} \prod_{j=n+1}^{n+N} \left(\frac{i p_{i\mu_j}^{\Theta}}{p_i^{\Theta} k_j}\right) \\
&\quad \times \left\{\prod_{j=1}^{n}\left[\left(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j}\right)\left(\frac{p_{i\rho_j}^{\Theta}}{p_i^{\Theta} k_j}\right)\right]\right\} \\
&\quad \times \frac{N_{1i}^{\Theta}}{D_{1i}^{\Theta}} \frac{i(\not{p}_i^{\Theta} + m)}{(p_i^{\Theta})^2 - m^2} \frac{N_{2i}^{\Theta}}{D_{2i}^{\Theta}},
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
\Theta &= (\Theta_{n+1}, \cdots, \Theta_{n+N}), \\
\Theta_j &= +1 \text{ or } 0, \\
Sgn(\Theta) &= (-1)^{\Theta_{n+1}} (-1)^{\Theta_{n+2}} \cdots (-1)^{\Theta_{n+N}} \\
p_i^{\Theta} &= p_i + \Theta_{n+1} k_{n+1} + \cdots + \Theta_{n+N} k_{n+N}, \\
p_i &= p + k_1 + \cdots + k_i,
\end{aligned} \tag{4.2}$$

and the superscript Θ on the N's and D's means that the argument p_i appearing in (3.5) and (3.6) is replaced by p_i^{Θ}. Note that even though the action of $\widehat{C}_j$ and $\widetilde{Q}_j$ involve integrations over λ and differentiations, the meromorphic parts of the resulting generalized propagators are expressed by (4.1) in relatively simple closed form. These meromorphic parts turn out to give the dominant contributions in the mesoscopic regime.

The essential simplification obtained by summing over all orders of the C–type insertions is that after this summation each C–type interaction

gives just two terms. The first term is just the function before the action of $\widehat{C}_j$ multiplied by $ip_{i\mu_j}(p_i k_j)^{-1}$; the second is minus the same thing with p_i replaced by $p_i + k_j$. Thus, apart from this simple factor, and, for one term, the overall shift in p_i, the function is just the same as it was before the action of $\widehat{C}_j$. Consequently, the power–counting arguments used for Q–type couplings go through essentially unchanged. Details can be found in references 14 and 15.

5. COMPARISON TO OTHER RECENT WORKS

The problem of formulating quantum electrodynamics in an axiomatic field-theoretic framework has been examined by Fröhlich, Morchio, and Strocchi[8] and by D. Buchholz[9], with special attention to the non-local aspects arising from Gauss' law. Their main conclusion, as it relates to the present work, is that the energy-momentum spectrum of the full system can be separated into two parts, the first being the photonic asymptotic free-field part, the second being a remainder that: 1) is tied to charged particles, 2) is nonlocal relative to the photonic part, and 3) can have a discrete part corresponding to the electron/positron mass. This separation is concordant with the structure of the QED Hamiltonian, which has a photonic free-field part and an electron/positron part that incorporates the interaction term $eA^{\mu}J_{\mu}$, but no added term corresponding to the non-free part of the electromagnetic field. It is also in line with the separation of the classical electromagnetic field, as derived from the Liénard-Wiechert potentials, into a "velocity" part that is attached (along the light cone) to the moving source particle, and an "acceleration" part that is radiated away. It is the "velocity" part, which is tied to the source particle, and which falls off only as r^{-1}, that is the origin of the "nonlocal" infraparticle structure that introduces peculiar features into quantum electrodynamics, as compared to simple local field theories.

In the present approach, the quantum analog of this entire classical structure is incorporated into the formula for the scattering operator by the unitary factor $U(L)$. It was shown in ref. 11, Appendix C, that the non-free "velocity" part of the electromagnetic field generated by $U(L)$ contributes in the correct way to the mass of the electrons and positrons. It gives also the "Coulomb" or "velocity" part of the interaction between different charged particles, which is the part of the electromagnetic field that gives the main part of Gauss' law asymptotically. Thus our formulas supply in

a computationally clean way these "velocity field" contributions that seem so strange when viewed from other points of view.

Comparisons to the works in references 17 through 22 can be found in reference 14.

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(Manuscript received on January 10th, 1996.)