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## Wavelet transform associated to an induced representation of $SL(n+2, \mathbf{R})$

by

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ABSTRACT. — Let  $G = SL(n+2, \mathbf{R})$  and  $P = MAN$  a parabolic subgroup of  $G$  such that  $N$  is isomorphic to the Heisenberg group  $H_n$ . Let  $1 \otimes e^\lambda \otimes 1$  be a representation of  $P$  and  $\pi_\lambda = \text{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$  the induced representation of  $G$  acting on  $L^2(H_n)$ . In this paper we shall obtain a condition on  $\lambda$  and  $\psi \in \mathcal{S}'(H_n)$  for which the matrix coefficients  $\langle f, \pi_\lambda(g)\psi \rangle_{L^2(H_n)}$  are square-integrable on a subgroup  $\overline{N}A_1 \simeq H_n \times \mathbf{R}$  of  $G$  and  $\|f\|_{L^2(H_n)}^2 = c \int \int_{\overline{N}A_1} |\langle f, \pi_\lambda(\overline{n}a_1)\psi \rangle_{L^2(H_n)}|^2 d\overline{n} da_1$  for all  $f \in \mathcal{S}(H_n)$ .

RÉSUMÉ. — Soit  $G = SL(n+2, \mathbf{R})$  et  $P = MAN$  un sous-groupe parabolique de  $G$  tel que  $N$  soit isomorphe à  $H_n$  le groupe de Heisenberg. Soit  $1 \otimes e^\lambda \otimes 1$  une représentation de  $P$  et  $\pi_\lambda = \text{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$  la représentation induite de  $G$  qui opère sur  $L^2(H_n)$ . Dans cet article on obtient une condition sur  $\lambda$  et  $\psi \in \mathcal{S}'(H_n)$  pour que les coefficients de matrice  $\langle f, \pi_\lambda(g)\psi \rangle_{L^2(H_n)}$  soient de carré-intégrables sur un sous-groupe  $\overline{N}A_1 \simeq H_n \times \mathbf{R}$  de  $G$  et  $\|f\|_{L^2(H_n)}^2 = c \int \int_{\overline{N}A_1} |\langle f, \pi_\lambda(\overline{n}a_1)\psi \rangle_{L^2(H_n)}|^2 d\overline{n} da_1$  pour toute  $f \in \mathcal{S}(H_n)$ .

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## 1. INTRODUCTION

Let  $G$  be a locally compact group and  $(\pi, \mathcal{H})$  a representation of  $G$  where  $\mathcal{H}$  is a Hilbert space equipped with the norm  $\|\cdot\|_{\mathcal{H}}$  and the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ . For a subset  $S$  of  $G$  with a measure  $ds$  on  $S$  we say that  $\psi \in \mathcal{H}$  is  $S$ -admissible for  $\pi$  if there exists a positive constant  $c_{\psi}$  such that

$$(1) \quad \int_S |\langle f, \pi(s)\psi \rangle_{\mathcal{H}}|^2 ds = c_{\psi} \|f\|_{\mathcal{H}}^2 \quad \text{for all } f \in \mathcal{H}.$$

Then  $\psi \in \mathcal{H}$  is  $S$ -admissible for  $\pi$  if and only, if, as a functional on  $\mathcal{H}$ ,

$$(2) \quad f = c_{\psi}^{-1} \int_S \langle f, \pi(s)\psi \rangle_{\mathcal{H}} \pi(s)\psi ds \quad \text{for all } f \in \mathcal{H}.$$

Clearly, (2) implies (1). Conversely, we suppose (1) and define  $T_f : \mathcal{H} \rightarrow \mathbb{C}$  by  $T_f(h) = c_{\psi}^{-1} \int_S \langle f, \pi(s)\psi \rangle_{\mathcal{H}} \langle \pi(s)\psi, h \rangle_{\mathcal{H}} ds$  ( $h \in \mathcal{H}$ ). Then the Schwarz inequality yields that  $|T_f(h)| \leq \|f\|_{\mathcal{H}} \|h\|_{\mathcal{H}}$ , so  $T_f$  is a bounded linear functional on  $\mathcal{H}$ . Therefore, it follows from Riesz Representation Theorem that there exists  $f_0 \in \mathcal{H}$  such that  $T_f(h) = \langle f_0, h \rangle_{\mathcal{H}}$  and  $\|T_f\| = \|f_0\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}$ . Especially,  $T_f(f) = \langle f_0, f \rangle_{\mathcal{H}} = \|f\|_{\mathcal{H}}^2$ . Thereby,  $\|f - f_0\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - 2\Re \langle f_0, f \rangle_{\mathcal{H}} + \|f_0\|_{\mathcal{H}}^2 \leq 0$  and thus,  $f = f_0$ . This proves (2). We call  $\langle f, \pi(s)\psi \rangle$  the *wavelet transform* of  $f$  associated to  $(G, \pi, S, \psi)$  and (3) the inversion formula of the transform. Here we put  $T_f(s) = \langle f, \pi(s)\psi \rangle_{\mathcal{H}}$  ( $s \in S$ ). If  $\pi$  is unitary, it satisfies a partial covariance:  $T_{\pi(s_1)\pi(s_2)^{-1}f}(s_1) = T_f(s_2)$  ( $s_1, s_2 \in S$ ) and furthermore, if  $S$  is a subgroup of  $G$ , it is the covariance property:  $T_{\pi(s_2)f}(s_1) = T_f(s_2^{-1}s_1)$ .

We state some well-known examples of the wavelet transform in our scheme. When  $S = G$ ,  $ds$  a Haar measure of  $G$ , and  $(\pi, \mathcal{H})$  a square-integrable representation of  $G$ , Duflo and Moore [DM] find a  $G$ -admissible vector  $\psi \in \mathcal{H}$ : for example, Gabor transform and Grossmann-Morlet transform correspond to a square-integrable representation of the Weyl-Heisenberg group and the one-dimensional affine group respectively (cf. [MW, § 3]), and a reproducing formula for a weighted Bergman space on a bounded symmetric domain relates to a holomorphic discrete series of a semisimple Lie group (cf. [B], [K]). For another example we refer to [VP]. Let  $(G, H)$  be a semisimple symmetric pair and put  $S = KA = \sigma(G/H)$ ,  $\sigma : G/H \rightarrow G$  is a flat section,  $ds = dkda$ , and  $(\pi, \mathcal{H})$  a square-integrable representation of  $G \bmod H$ . Then we can find a  $H$ -invariant distribution

vector  $\psi \in \mathcal{H}_{-\infty}$  for which (1) and (2) hold. Recently this idea “square-integrability mod  $H$ ” was generalized to some other pairs  $(G, H)$  and non-flat sections  $\sigma : G/H \rightarrow G$ : Ali, Antoine, and Gazeau [AAG] obtain wavelet transforms associated to the Wigner representation of the Poincaré group  $\mathcal{P}_+^\uparrow(1, 1)$ , and Torr sani [T1, 2], Kalisa and Torr sani [KT] do to the Stone-von Neumann type representation of the affine Weyl-Heisenberg group  $G_{aWH}$ .

In this paper we shall investigate a transform associated to a principal series representation of  $SL(n + 2, \mathbf{R})$  ( $n \in \mathbf{N}$ ). To explain our goal we look at an example by taking  $G = SL(2, \mathbf{R})$ . The holomorphic discrete series  $\pi_n$  ( $n \geq 1/2$ ,  $n \in \mathbf{Z}$ ) is realized on  $H_n^2$ , here  $H_n^2$  is the Hilbert space of holomorphic functions on the upper half plane  $\mathbf{C}^+$  with inner product  $\langle \phi, \psi \rangle_{H_n^2} = \Gamma(2n - 1) \int_{\mathbf{C}^+} \phi(x + iy) \bar{\psi}(x + iy) y^{2n-2} dx dy$ . Then each  $\pi_n$  is square-integrable:

$$(3) \quad \int_G |\langle \phi, \pi_n(g) \psi \rangle_{H_n^2}|^2 ds = \frac{c}{2n - 1} \|\phi\|_{H_n^2} \|\psi\|_{H_n^2} \quad \text{for all } \phi, \psi \in H_n^2,$$

where  $c$  is independent of  $n$  (cf. [Su, § 10 and Prop. 7.18 in Chap. V]) and, as stated above, the inversion formula follows as

$$\phi(x) = c_\psi^{-1} \int_G \langle \phi, \pi_n(g) \psi \rangle_{H_n^2} \pi_n(g) \psi(x) dg \quad (x \in G),$$

where  $c_\psi = (2n - 1)^{-1} c \|\psi\|_{H_n^2}$ . Let  $\hat{H}_n^2$  be the Hilbert space of functions on  $\mathbf{R}^+$  with inner product  $\langle \Phi, \Psi \rangle_{\hat{H}_n^2} = 2^{2n-1} \int_{\mathbf{R}^+} \Phi(t) \hat{\Psi}(t) t^{-2n+1} dt$ .

Then the inverse Fourier-Laplace transform  $\mathcal{F}$  gives an isometry of  $H_n^2$  onto  $\hat{H}_n^2$  and, if we put  $\hat{\pi}_n = \mathcal{F} \circ \pi_n \circ \mathcal{F}^{-1}$ ,  $(\pi_n, H_n^2)$  and  $(\hat{\pi}_n, \hat{H}_n^2)$  are unitary equivalent (cf. [Sa, p. 20]). In particular, (3) and the inversion formula hold by replacing  $\pi_n$  and  $H_n^2$  with  $\hat{\pi}_n$  and  $\hat{H}_n^2$  respectively. Here we note that  $\|\Psi\|_{\hat{H}_{n+1/2}^2} = (2n - 1)^{-1} \|\Psi\|_{\hat{H}_n^2}$  provided that  $\Psi$  is  $K$ -invariant, and therefore, by the integral formula under the Iwasawa decomposition  $G = \bar{N}AK$  we can deduce that

$$(4) \quad \int_{\bar{N}A} |\langle \Phi, \hat{\pi}_n(\bar{n}a) \Psi \rangle_{\hat{H}_n^2}|^2 d\bar{n} da = c \|\Phi\|_{\hat{H}_n^2} \|\Psi\|_{\hat{H}_{n+1/2}^2} \quad \text{for all } \Phi \in \hat{H}_n^2.$$

Now we consider the limiting case of  $n = 1/2$ : the limit of discrete series  $(\pi_{1/2}, H_{1/2}^2) \simeq (\hat{\pi}_{1/2}, \hat{H}_{1/2}^2)$ . Obviously, (3) and (4) collapse when  $n$  goes to  $1/2$ , because  $\|\Psi\|_{\hat{H}_{n+1/2}^2} = (2n-1)^{-1} \|\Psi\|_{\hat{H}_n^2} \rightarrow \infty$  when  $n \rightarrow 1/2$ . However, if we drop the  $K$ -invariance of  $\Psi$  and assume that  $\|\Psi\|_{\hat{H}_1^2} < \infty$  for  $\Psi \in \hat{H}_{1/2}^2$ , we can deduce that

$$(5) \quad \int_{\overline{N}A} |\langle \Phi, \hat{\pi}_{1/2}(\overline{n}a)\Psi \rangle_{\hat{H}_{1/2}^2}|^2 d\overline{n} da = c \|\Phi\|_{\hat{H}_{1/2}^2} \|\Psi\|_{\hat{H}_1^2} \quad \text{for all } \Phi, \psi \in \hat{H}_{1/2}^2.$$

Observe that the wavelet transform  $\langle \Phi, \hat{\pi}_{1/2}(\overline{n}a)\Psi \rangle$  is nothing but the affine wavelet transform obtained by Grossmann and Morlet (*see* § 5). Moreover, recall that the limit of discrete series  $\pi_{1/2} \simeq \hat{\pi}_{1/2}$  is unitary equivalent to an irreducible component of a reducible unitary principal series of  $G$ , that is denoted by  $V^{1/2, 1/2}$  in [Su, p. 246]. Therefore, in this context we can say that the affine wavelet transform corresponds to the limit of discrete series or a reducible unitary principal series of  $SL(2, \mathbf{R})$ .

Our aim is to generalize this correspondance and to find a transform which associates to a principal series of  $SL(n+2, \mathbf{R})$ . Let  $P = MAN$  be a parabolic subgroup of  $G$  such that  $N \simeq H_n$ , the Heisenberg group, and and  $\pi_\lambda = \text{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$  the induced representation of  $G$ . We identify  $\overline{N}$  with  $\mathbf{R}^{2n+1}$  to define  $L^2(\overline{N})$ ,  $\mathcal{S}(\overline{N})$ , and  $\mathcal{S}'(\overline{N})$ . Then we shall find a subgroup  $\overline{N}A_1 \simeq H_n \times \mathbf{R}$  of  $G$ ,  $\psi \in \mathcal{S}'(\overline{N})$ , and a  $\lambda$  for which

$$(6) \quad \int \int_{\overline{N} \times A_1} |\langle \phi, \pi_\lambda(\overline{n}a_1)\psi \rangle_{L^2(\overline{N})}|^2 d\overline{n} da_1 = c_\psi \|\phi\|_{L^2(\overline{N})}^2$$

for all  $\phi \in \mathcal{S}(\overline{N})$ . Of course, since calculation is carried on a subgroup  $\overline{N}A$  of  $G$ , the whole results can be stated without using the  $SL(n+2, \mathbf{R})$ -scheme. However, to emphasise the correspondance of our transform and a principal series of  $G$ , we dare to use the  $SL(n+2, \mathbf{R})$ -scheme. Most of results in this paper can be generalized to the analytic continuation of discrete series, including the limiting case, and to a principal series of semisimple Lie groups. They will appear in forthcoming papers.

## 2. HEISENBERG GROUP

Before starting the representation theory of  $G = SL(n+2, \mathbf{R})$ , we recall the one of the Heisenberg group  $H_n$ , to which the subgroup  $\overline{N}$  of  $G$  is isomorphic (*see* § 3 below). We refer to the general references [F] and [G].

Let  $H_n = \{X = (p, q, t); p, q \in \mathbf{R}^n, t \in \mathbf{R}\}$  denote the polarized Heisenberg group with the group law:

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + pq'),$$

where  $xy = \sum_{i=1}^n x_i y_i$  for  $x = (x_i), y = (y_i) \in \mathbf{R}^n$ . We observe that  $\{(0, 0, t); t \in \mathbf{R}\}$  is the center of  $H_n$  and the Lebesgue measure  $dqdpdt$  is an bi-invariant measure  $dX$  on  $H_n$ . The Schrödinger representation  $(\rho_h, L^2(\mathbf{R}^n))$  of  $H_n$  with parameter  $h \in \mathbf{R}^n \setminus \{0\}$  is given by

$$(7) \quad \rho_h(p, q, t)x = e^{2\pi i h t + 2\pi i q x} f(x + hp) \quad (f \in L^2(\mathbf{R}^n)).$$

Then each  $\rho_h$  is irreducible unitary and, by Stone-von Neumann Theorem,  $\rho_h$  is, up to unitary equivalence, the only representation of  $H_n$  with the central character  $\pi(0, 0, t) = e^{2\pi i h t} I$  for  $h \in \mathbf{R}^n \setminus \{0\}$ . We define for  $\phi \in L^1(H_n)$

$$\rho_h(\phi) = \int_{H_n} \phi(X) \rho_h(X) dX.$$

Then, for  $\phi, \psi \in L^1(H_n)$

$$(8) \quad \rho_h(\phi * \psi) = \rho_h(\phi) \rho_h(\psi) \quad \text{and} \quad \rho_h(\phi)^* = \rho_h(\phi^\sim),$$

where  $\phi * \psi(X) = \int_{H_n} \phi(Y) \psi(Y^{-1}X) dY$  and  $\phi^\sim(X) = \overline{\phi}(X^{-1})$ . Similarly for  $\alpha(p, q) \in L^1(\mathbf{R}^n \times \mathbf{R}^n) = L^1(\mathbf{R}^{2n})$  we define

$$\rho_h^0(\alpha) = \int_{H_n} \alpha(p, q) \rho_h^0(p, q) dp dq,$$

where  $\rho_h^0(p, q) = \rho_h(p, q, 0)$ . It is clear that  $\rho_h^0(\alpha)$  makes sense as an operator from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  whenever  $\alpha \in \mathcal{S}'(\mathbf{R}^{2n})$  (see [F, Theorem (1.30)]). For  $\phi \in L^2(H_n)$  the Plancherel formula on  $H_n$  is given as follows.

$$(9) \quad \|\phi\|_{L^2(H_n)}^2 = \int_{\mathbf{R}} |h|^n \|\rho_h(\phi)\|_{HS}^2 dh,$$

where  $\|\cdot\|_{HS}$  denotes the Hilbert-Schmidt norm.

### 3. INDUCED REPRESENTATION OF $SL(n+2, \mathbf{R})$

Let  $G = SL(n+2, \mathbf{R})$  and  $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbf{R})$ . According to the process in [H1, § 6], we shall define a parabolic subalgebra  $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$  of  $\mathfrak{g}$ , the parabolic subgroup  $P = MAN$  of  $G$ , and an induced representation  $\pi_\lambda = \text{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$  of  $G$ .

Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_0 + \mathfrak{n}_0$  be the Iwasawa decomposition of  $\mathfrak{g}$  such that

$$\mathfrak{k} = \mathfrak{so}(n+2),$$

$$\mathfrak{a}_0 = \{\text{diagonal matrices in } \mathfrak{g}\},$$

$$\mathfrak{n}_0 = \{\text{lower triangular matrices in } \mathfrak{g} \text{ with } 0 \text{ on the diagonal}\}.$$

Let  $Z_{\mathfrak{g}}(\mathfrak{a}_0) = \mathfrak{m}_0 + \mathfrak{a}_0$ . When  $n = 0, 1$ , we put  $\mathfrak{m} = \mathfrak{m}_0$ ,  $\mathfrak{a} = \mathfrak{a}_1 = \mathfrak{a}_0$ , and  $\mathfrak{n} = \mathfrak{n}_0$ , that is,  $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$  is a minimal parabolic subalgebra of  $\mathfrak{g}$ . In the following, we may assume that  $n \geq 2$ . We define  $e_i : \mathfrak{a}_0 \rightarrow \mathbf{R}$  ( $1 \leq i \leq n+2$ ) by  $e_i(H) = h_i$  for  $H = \text{diag}(h_1, h_2, \dots, h_{n+2}) \in \mathfrak{a}_0$ , and put  $\alpha_i = e_{i+1} - e_i$  ( $1 \leq i \leq n+1$ ). The set of roots of  $(\mathfrak{g}, \mathfrak{a}_0)$  positive for  $\mathfrak{n}_0$  is given by  $\Sigma = \{e_i - e_j; i > j\}$  and the subset consisting of simple roots is  $\Sigma_0 = \{\alpha_i; 1 \leq i \leq n+1\}$ . For  $F = \{\alpha_i; 2 \leq i \leq n\}$  we set  $\mathfrak{a} = \mathfrak{a}_F = \{H \in \mathfrak{a}_0; \alpha(H) = 0 \text{ for all } \alpha \in F\}$  and  $\mathfrak{n} = \mathfrak{n}_F = \sum_{\alpha \in \Sigma \setminus \Sigma_F} \mathfrak{g}_\alpha$

where  $\Sigma_F = \{\alpha \in \Sigma; \alpha|_{\mathfrak{a}_F} \equiv 0\}$  and  $\mathfrak{g}_\alpha$  is the root space corresponding to  $\alpha$ . Explicitly, they are of the forms:

$$\mathfrak{a} = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 I_n & 0 \\ 0 & 0 & h_3 \end{pmatrix}; h_1, h_2, h_3 \in \mathbf{R}, h_1 + n h_2 + h_3 = 0 \right\},$$

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ t & q & 0 \end{pmatrix}; p, q \in \mathbf{R}^n, t \in \mathbf{R} \right\},$$

where  $I_n$  is the  $n \times n$  unit matrix. Let  $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} + \mathfrak{a}$  and put  $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ . The set of roots of  $(\mathfrak{g}, \mathfrak{a})$  positive for  $\mathfrak{n}$  is given by  $\Sigma(\mathfrak{a}) = \{\alpha^\sim; \alpha \in \Sigma\}$  where  $\alpha^\sim = \alpha|_{\mathfrak{a}}$ . We set  $\mathfrak{a}_1 = \{H \in \mathfrak{a}; \alpha_{n+1}^\sim(H) = 0\}$  and  $\rho = \sum_{\alpha \in \Sigma(\mathfrak{a})} \alpha/2$ , that is,

$$\mathfrak{a}_1 = \left\{ H_s = \begin{pmatrix} -(n+1)s & 0 \\ 0 & s I_{n+1} \end{pmatrix}; s \in \mathbf{R} \right\},$$

$$\rho = \frac{(n+1)(\alpha_1^\sim + \alpha_{n+1}^\sim)}{2}.$$

We denote by  $M_0, A, A_1$ , and  $N$  the analytic subgroups of  $G$  corresponding to  $\mathfrak{m}, \mathfrak{a}, \mathfrak{a}_1$ , and  $\mathfrak{n}$  respectively. We define  $M = Z_K(\mathfrak{a}) M_0$ , where  $Z_K(\mathfrak{a})$

is the centralizer of  $\mathfrak{a}$  in  $K = SO(n+2)$ , and put  $P = MAN$ . We denote by  $\theta$  the Cartan involution of  $G$  given by  $\theta(g) = {}^t g^{-1}$  ( $g \in G$ ) and put  $\overline{N} = \theta(H)$ . Then it is easy to see that

$$(10) \quad \overline{N} \simeq H_n \text{ and } \overline{N}P \text{ is open in } G$$

whose complement has Haar measure 0,

where the identification is given by

$$\begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \mapsto (p, q, t).$$

We put  $d\overline{n} = dX$ ,  $da = dA$ , where  $dA$  is the Lebesgue measure on  $\mathfrak{a}$  normalized in such a way that  $\int_A f(a) da = \int_{\mathfrak{a}} f(\exp A) dA$  ( $f \in C_c(A)$ ), and  $da_1 = da|_{A_1}$ . We normalize the Haar measure  $dm$  on  $M$  and  $d\overline{n} = d\theta(n)$  on  $\overline{N}$  as the following integral formula holds:

$$\int_G f(g) dg = \int \int \int \int_{\overline{N} \times M \times A \times N} f(\overline{n} man) e^{2\rho(\log a)} d\overline{n} dm da dn$$

for  $f \in C_c(G)$  (see [H2, § 19]).

Let  $\mathfrak{a}_c^*$  denote the dual space of the complexification  $\mathfrak{a}_c$  of  $\mathfrak{a}$ . For  $\lambda \in \mathfrak{a}_c^*$  we define, out of the representation  $1 \otimes e^\lambda \otimes 1$  ( $man$ ) =  $e^{\lambda(\log a)}$  of  $P = MAN$ , a representation of  $G$  denoted by

$$\pi_\lambda = \text{Ind}_P^G (1 \otimes e^\lambda \otimes 1).$$

A dense subspace of the representation space  $\mathcal{H}_\lambda$  is given by  $\mathcal{H}_\lambda^0$  consisting of continuous functions  $f$  on  $G$  such that

$$(11) \quad f(gman) = e^{-(\lambda+\rho)(\log a)} f(g) \quad (g \in G, man \in MAN)$$

with norm  $\|f\|^2 = \int_K |f(k)|^2 dk$ . Moreover,  $G$  acts on  $\mathcal{H}_\lambda$  as  $\pi_\lambda(g_0) f(g) = f(g_0^{-1} g)$  ( $g_0 \in G$ ) and  $\pi_\lambda$  is unitary whenever  $\lambda \in i\mathfrak{a}^*$ . We observe that, by restricting  $f \in \mathcal{H}_\lambda$  to  $\overline{N}$ ,  $\mathcal{H}_\lambda$  is identified with  $L^2(\overline{N}, e^{2\Re\lambda(H(\overline{n}))} d\overline{n})$ , where  $g \in G$  is decomposed under  $G = KMAN$  as  $g = kme^{H(g)}n$ , and the action of  $G$  is given by

$$(12) \quad \pi_\lambda(g) F(\overline{n}) = e^{-(\lambda+\rho) \log a(g^{-1}\overline{n})} F(\overline{n}(g^{-1}\overline{n})),$$

where  $g \in G$  is decomposed under  $G = \bar{N}MAN$  as  $g = \bar{n}(g)ma(g)n$ . We define  $\mathcal{S}(\bar{N}) = \mathcal{S}(H_n)$  by the Schwartz class  $\mathcal{S}(\mathbf{R}^{2n+1})$ . Then it follows from Lemma 8.5.23, 27, and Theorem 8.2.1 in [War] that  $\mathcal{S}(\bar{N})$  is contained in  $L^2(\bar{N}, e^{\pm 2\Re\lambda(H(\bar{n}))} d\bar{n})$  and, from (12) that  $\pi_\lambda(\bar{n}a)$  ( $\bar{n}a \in \bar{N}A$ ) is an operator on  $\mathcal{S}(\bar{N})$ . We here define  $\langle f, g \rangle_{L^2(\bar{N})} = \int_K f(k)\bar{g}(k) dk$  for  $f, g \in \mathcal{H}_\lambda$ . Since this form is nondegenerate and  $G$ -invariant on  $\mathcal{H}_{-\bar{\lambda}} \times \mathcal{H}_\lambda$  (cf. [Wal, 8.3.11]), we see that  $\pi_\lambda(\bar{n}a)$  is an operator on  $\mathcal{S}'(\bar{N})$ . Especially,  $T_f(s) = \langle f, \pi_\lambda(s)\psi \rangle_{L^2(\bar{N})}$  ( $s \in \bar{N}A$ ) satisfies the covariance property:  $T_{\pi_{-\bar{\lambda}}(s_2)f}(s_1) = T_f(s_2^{-1}s_1)$  ( $s_1, s_2 \in S$ ).

#### 4. MAIN THEOREM

Let  $P = MAN$  and  $A_1 \subset A$  be the subgroups of  $G = SL(n+2, \mathbf{R})$  introduced in paragraph 3. We suppose that  $\alpha(p, q) \in \mathcal{S}'(\mathbf{R}^{2n})$  and  $\beta(t) \in L^1(\mathbf{R})$  satisfy the following condition: there exists  $\gamma : \mathbf{R} \rightarrow \mathbf{R}$  such that

$$(13) \quad \left\{ \begin{array}{l} \text{(i)} \quad \rho_h^0(\alpha)\rho_h^0(\alpha)^* = \gamma(h)I \quad \text{a.e. } h \in \mathbf{R}, \\ \text{(ii)} \quad \int_0^\infty |\mathcal{F}^{-1}\beta(\xi)|^2 \gamma(\xi) \frac{d\xi}{|\xi|} \\ \quad \quad \quad = \int_{-\infty}^0 |\mathcal{F}^{-1}\beta(\xi)|^2 \gamma(\xi) \frac{d\xi}{|\xi|} = c_{\alpha, \beta} < \infty, \end{array} \right.$$

where  $\rho_h^0(\alpha)$  is an operator from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  (see § 2),  $I$  is the identity operator, and  $\mathcal{F}^{-1}$  is the inverse Fourier transform on  $\mathbf{R}$ . When  $G = SL(2, \mathbf{R})$ , we ignore the function  $\alpha$ . we set

$$(14) \quad \psi_{\alpha, \beta}(p, q, t) = \alpha(p, q)\beta(t).$$

**THEOREM 4.1.** – *Let  $\psi_{\alpha, \beta}$  be as above and suppose  $\lambda|_{\mathfrak{a}_1} = (n+1)\alpha_1^*/2$ . Then,  $\psi_{\alpha, \beta} \in \mathcal{S}'(\bar{N})$  is  $\bar{N}A_1$ -admissible for  $\pi_\lambda$ , that is, there exists a positive constant  $c = c_{\psi_{\alpha, \beta}}$  such that*

$$\begin{aligned} & \int \int_{\bar{N} \times A_1} |\langle f, \pi_\lambda(\bar{n}a_1)\psi_{\alpha, \beta} \rangle_{L^2(\bar{N})}|^2 d\bar{n} da_1 \\ & = c \|f\|_{L^2(\bar{N})}^2 \quad \text{for all } f \in \mathcal{S}(\bar{N}). \end{aligned}$$

*Proof.* – We first observe from (8) and (9) that for  $f \in \mathcal{S}(\overline{N})$

$$\begin{aligned} & \int \int_{\overline{N} \times A_1} |\langle f, \pi_\lambda(\overline{n} a_1) \psi_{\alpha, \beta} \rangle_{L^2(\overline{N})}|^2 d\overline{n} da_1 \\ &= \int \int_{\overline{N} \times A_1} |f * (\pi_\lambda(a_1) \psi_{\alpha, \beta})^\sim(\overline{n})|^2 d\overline{n} da_1 \\ &= \int_{A_1} \int_{\mathbf{R}} |h|^n \|\rho_h(f * (\pi_\lambda(a_1) \psi_{\alpha, \beta})^\sim)\|_{HS}^2 dh da_1 \\ &= \int_{\mathbf{R}} |h|^n \text{Tr} \left( \rho_h(f) \int_{A_1} \rho_h(\pi_\lambda(a_1) \psi_{\alpha, \beta})^* \right. \\ & \quad \left. \times \rho_h(\pi_\lambda(a_1) \psi_{\alpha, \beta}) da_1 \rho_h(f)^* \right) dh. \end{aligned}$$

Since  $A_1 = \{a_s = \exp(H_s); s \in \mathbf{R}\}$ ,  $(\lambda + \rho)|_{\mathfrak{a}_1} = (n + 1)\alpha_1^\sim$  and  $\alpha_1^\sim(\log a_s) = (n + 2)s$ , it follows from (10) and (11) that

$$\begin{aligned} \pi_\lambda(a_s) \psi_{\alpha, \beta}(\overline{n}) &= \psi_{\alpha, \beta}(a_s^{-1} \overline{n} a_s \cdot a_s^{-1}) \\ &= \psi_{\alpha, \beta}(e^{(n+2)s} p, q, e^{(n+2)s} t) e^{(n+1)(n+2)s} \end{aligned}$$

and thereby, from (7), (11), and (14) that

$$\begin{aligned} & \rho_h(\pi_\lambda(a_s) \psi_{\alpha, \beta}) \\ &= \int_{\overline{N}} \pi_\lambda(a_s) \psi_{\alpha, \beta}(\overline{n}) \rho_h(\overline{n}) d\overline{n} \\ &= \int \int \int_{\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}} \psi_{\alpha, \beta}(e^{(n+2)s} p, q, e^{(n+2)s} t) \\ & \quad \times e^{2\pi i h t} \rho_h^0(p, q) dp dq dt e^{(n+1)(n+2)s} \\ &= \mathcal{F}^{-1} \beta(e^{-(n+2)s} h) \int \int_{\mathbf{R}^n \times \mathbf{R}^n} \alpha(e^{(n+2)s} p, q) \rho_h^0(p, q) dp dq \\ & \quad \times e^{n(n+2)s} \\ &= \mathcal{F}^{-1} \beta(e^{-(n+2)s} h) \rho_{e^{-(n+2)s} h}^0(\alpha). \end{aligned}$$

Then, we can deduce from (13) (i) and (ii) that

$$\begin{aligned}
 & \int_{A_1} \rho_h (\pi_\lambda (a_1) \psi_{\alpha, \beta})^* \rho_h (\pi_\lambda (a_1) \psi_{\alpha, \beta}) da_1 \\
 &= \int_{\mathbf{R}} |\mathcal{F}^{-1} \beta (e^{-(n+2)s} h)|^2 \rho_{e^{-(n+2)s} h}^0 (\alpha)^* \rho_{e^{-(n+2)s} h}^0 (\alpha) ds \\
 &= \begin{cases} \int_0^\infty |\mathcal{F}^{-1} \beta (\xi)|^2 \gamma (\xi) I \frac{d\xi}{|\xi|} & \text{if } h > 0 \\ \int_{-\infty}^0 |\mathcal{F}^{-1} \beta (\xi)|^2 \gamma (\xi) I \frac{d\xi}{|\xi|} & \text{if } h < 0 \end{cases} \\
 &= c_n c_{\alpha, \beta} I,
 \end{aligned}$$

where  $c_n = 1/(n+2)$  and thus,

$$\begin{aligned}
 & \int_{\overline{N} \times A_1} |\langle f, \pi_\lambda (\overline{n} a_1) \psi_{\alpha, \beta} \rangle_{L^2(\overline{N})}|^2 d\overline{n} da_1 \\
 &= c_n c_{\alpha, \beta} \int_{\mathbf{R}} |h|^n \text{Tr} (\rho_h (f) (\rho_h (f))^*) dh \\
 &= c_n c_{\alpha, \beta} \|f\|_{L^2(\overline{N})}^2. \quad \square
 \end{aligned}$$

## 5. EXAMPLES

We conclude with some examples of  $\psi_{\alpha, \beta} (p, q, t) = \alpha (p, q) \beta (t)$  satisfying the condition (13) (i) and (ii). In the case of  $SL(2, \mathbf{R})$ , as said before, the function  $\alpha (p, q)$  is ignored, so the condition (13) (ii) only lives out with  $\gamma \equiv 1$ :

$$(15) \quad \int_0^\infty |\mathcal{F}^{-1} \beta (\xi)|^2 \frac{d\xi}{|\xi|} = \int_{-\infty}^0 |\mathcal{F}^{-1} \beta (\xi)|^2 \frac{d\xi}{|\xi|} < \infty.$$

This condition is noting but the admissible condition for the affine wavelet transform on  $L^2(\mathbf{R})$ . Actually, when  $G = SL(2, \mathbf{R})$ , the wavelet transform in (2) is of the form:

$$\langle f, \pi_\lambda (\overline{n}_t a_s) \beta \rangle_{L^2(H_0)} = e^{2s} \int_{\mathbf{R}} f (t') \beta (e^{2s} (t' - t)) dt'$$

and hence, if we set  $a = e^{2s}$ , it coincides with the affine wavelet transform on  $L^2(\mathbf{R})$ . This is quite natural because  $\overline{N} A_1$  is isomorphic to the affine group “ $ab + b$ ” (cf. [HW], 3.3).

We now suppose that  $n \geq 1$ , and we give some examples of  $\alpha(p, q) \in \mathcal{S}'(\mathbf{R}^{2n})$  satisfying (13) (i) and obtain the function  $\gamma : \mathbf{R} \rightarrow \mathbf{R}$ .

(a) If  $\alpha(p, q) = \delta(p - p_0) \delta(q - q_0)$  for some  $p_0, q_0 \in \mathbf{R}^n$ , it easily see that  $\rho_h^0(\alpha) f(x) = e^{2\pi i q_0 x} f(x + hp_0)$  and  $\|\rho_h^0(\alpha) f\|_{L^2(\mathbf{R}^n)}^2 = \|f\|_{L^2(\mathbf{R}^n)}^2$ . Therefore,  $\alpha(p, q)$  satisfies (13) (i) with  $\gamma \equiv 1$  and hence the condition on  $\beta$  is the same as in (15).

(b) Let  $\alpha(p, q) = \alpha_0(p) e^{\pi i p q}$  where  $|\alpha_0| \equiv 1$ . Since  $\int e^{2\pi i q x} dq = \delta(x)$ , it follows that  $\rho_h^0(\alpha) f(x) = 2^n \alpha_0(-2x) f((1 - 2h)x)$  and  $\|\rho_h^0(\alpha) f\|_{L^2(\mathbf{R}^n)}^2 = 2^{2n} |1 - 2h|^{-n} \|f\|_{L^2(\mathbf{R}^n)}^2$ . Therefore,  $\alpha(p, q)$  satisfies (13) (i) with  $\gamma(h) = 2^{2n} |1 - 2h|^{-n}$  and (13) (ii) is of the form:

$$(16) \quad \begin{aligned} & 2^n \int_0^\infty |\mathcal{F}^{-1} \beta(\xi)|^2 |1 - 2\xi|^{-n} \frac{d\xi}{|\xi|} \\ & = 2^n \int_{-\infty}^0 |\mathcal{F}^{-1} \beta(\xi)|^2 |1 - 2\xi|^{-n} \frac{d\xi}{|\xi|} < \infty. \end{aligned}$$

(c) If  $\alpha(p, q) = \alpha_1(q) e^{\pi i p q}$  where  $|\alpha_1| \equiv 1$ , then  $\rho_h^0(\alpha) f(x) = 2^n \mathcal{F}[\alpha_1' \cdot \mathcal{F}^{-1} f]((1 - 2h)x)$  where  $\alpha_1'(s) = \alpha_1(2hx)$ , so the function  $\gamma$  and the condition on  $\beta$  are the same as in (b).

(d) If  $\alpha(p, q) = 2^{-n/2} e^{in\pi/4} e^{-\pi i(p^2 + q^2)/2} e^{\pi i p q}$ , then from the formula for the distribution Fourier transform of the Gaussian functions (cf. [F, Theorem 2 in Appendix A]) it follows that  $\rho_h^0(\alpha) f(x) = h^{-n} e^{2\pi i(1-1/h)x^2} \mathcal{F}^{-1} f(x/h)$  and  $\|\rho_h^0(\alpha) f\|_{L^2(\mathbf{R}^n)}^2 = |h|^{-n} \|f\|_{L^2(\mathbf{R}^n)}^2$ . Therefore,  $\alpha(p, q)$  satisfies (13) (i) with  $\gamma(h) = |h|^{-n}$  and hence (13) (ii) is given by

$$\int_0^\infty |\mathcal{F}^{-1} \beta(\xi)|^2 \frac{d\xi}{|\xi|^{n+1}} = \int_{-\infty}^0 |\mathcal{F}^{-1} \beta(\xi)|^2 \frac{d\xi}{|\xi|^{n+1}} < \infty.$$

(e) We now consider the Gaussian functions:

$$\alpha(p, q) = e^{-\pi i(pBp - 2pAq + qCq)} e^{\pi i p q},$$

where  $A, B$ , and  $C$  denote  $n \times n$  real matrices. We set  $D = {}^t A + I/2$ . If  $C = 0$  and  $D$  is invertible, it follows as in (b) that

$$\begin{aligned} \rho_h^0(\alpha) f(x) &= \det^{-1} D \cdot e^{-\pi i x {}^t D^{-1} B D^{-1} x} f((I - h D^{-1})x), \\ \gamma(h) &= |\det D|^{-2} |\det(I - h D^{-1})|^{-1}. \end{aligned}$$

On the other hand, if  $C$  is invertible and symmetric, and  $B = {}^t DC^{-1}D$ , it follows as in (d) that

$$\begin{aligned} \rho_h^0(\alpha) f(x) &= e^{-\pi i \#(C)/4} |\det C|^{-1/2} h^{-n} \\ &\quad \times e^{2\pi i x C^{-1} \left(\frac{I}{2} - \frac{D}{h}\right) x} \mathcal{F}^{-1} f(C^{-1} Dx/h), \\ \gamma(h) &= |h|^{-n} |\det D|^{-1}, \end{aligned}$$

where  $\#(C)$  is the number of positive eigenvalues of  $C$  minus the number of negative eigenvalues.

*Remark.* – (1) We note that the process to obtain  $\rho_h^0(\alpha)$  is exactly same as the one used in the Weyl correspondence of pseudodifferential operators (cf. [F, Chap. 2]). In fact the above calculation of  $\rho_h^0(\alpha)$  also follows from Proposition (2.28) in [F] by generalizing the results for  $\rho_1$  to  $\rho_h$  and by arranging the isomorphism from the Heisenberg group to the polarized one. Especially, in the case (e) the set of operators  $\gamma(h)^{-1/2} \rho_h^0(\alpha)$  corresponds to the range of the metaplectic representation of  $Sp(n, \mathbf{R})$  (see [F, Chap. 4 and Chap. 5]).

(2) We suppose that  $B = C = 0$  and  $D$  is invertible in (e). Then  $\overline{N} A_1$ -admissible vectors  $\psi_{\alpha, \beta}$  are  $M_0 A'_1$ -invariant, where  $A'_1$  is the analytic subgroup of  $G$  corresponding to  $\mathfrak{a}'_1 = \{H \in \mathfrak{a}; (e_{n+2} - e_1)(H) = 0\}$ . In general, if  $\alpha(p, q)$  is a function of  $pq$ , then  $\overline{N} A_1$ -admissible vectors  $\psi_{\alpha, \beta}$  are  $M_0 A'_1$ -invariant, and moreover, if  $\alpha(p, q)$  is an even function of  $pq$  and  $\beta(t)$  is even, then  $\psi_{\alpha, \beta}$  are  $M_0 A'_1$ -invariant.

(3) Examples in paragraph 5 don't cover the results in [KT]. In order to obtain their transforms in our  $SL(n+2, \mathbf{R})$ -scheme we need deeper analysis on  $\psi$ . It will be done in a forthcoming paper.

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