ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 65, nº 4 (1996), p. 313-366
<http://www.numdam.org/item?id=AIHPA_1996__65_4_313_0>

Ann. Inst. Henri Poincaré,

Vol. 65, n° 4, 1996, p. 313-366. Physique théorique

Local existence in time of small solutions to the Davey-Stewartson systems

by

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Dedicated to Professor T. Furuta on his sixtieth birthday

ABSTRACT. – We consider the initial value problem for the Davey-Stewartson systems

$$\begin{cases} i\partial_t u + c_0\partial_x^2 u + \partial_y^2 u = c_1|u|^2u + c_2 u\partial_x\varphi & (x,y,t)\in \mathbf{R}^3, \\ \partial_x^2\varphi + c_3\partial_y^2\varphi = \partial_x|u|^2 & u(x,y,0) = u_0(x,y), \end{cases} \quad *$$

where c_0, $c_3 \in \mathbf{R}$, c_1, $c_2 \in \mathbf{C}$, u is a complex valued function and φ is a real valued funtion. Our purpose in this paper is to study the elliptic-hyperbolic $(c_0 > 0, c_3 < 0)$ and the hyperbolic-hyperbolic cases in the lower order Sobolev spaces. More precisely, we prove local existence in time of solutions to (*) under the conditions that $u_0 \in H^{1+\epsilon,0} \cap H^{0,1+\epsilon}$ and $\|u_0\|_{H^{1+\epsilon,0}} + \|u_0\|_{H^{0,1+\epsilon}}$ is sufficiently small, where $\epsilon > 0$ is sufficiently small,

$$H^{l,s} = \{f \in L^2; \|f\|_{H^{l,s}} = \|\langle\bar{x}\rangle^s\langle D\rangle^l f\|_{L^2} < \infty\},$$

$\langle\bar{x}\rangle = (1+x^2+y^2)^{1/2}$ and $\langle D\rangle = (1-\partial_x^2-\partial_y^2)^{1/2}$. Furthermore, in the case of the elliptic-hyperbolic case local existence in time of solutions is shown if $\langle\bar{x}\rangle u_0 \in H^{1/2+\epsilon,0} \cap H^{0,1/2+\epsilon}$, and $\|\langle\bar{x}\rangle u_0\|_{H^{1/2+\epsilon,0}} + \|\langle\bar{x}\rangle u_0\|_{H^{0,1/2+\epsilon}}$ is sufficiently small.

AMS Subject Classifications : 35D05, 35E15, 35Q20, 76B15.

RÉSUMÉ. – Nous considérons le problème aux valeurs initiales suivant du type de Davey-Stewartson:

$$\begin{cases} i\partial_t u + c_0\partial_x^2 u + \partial_y^2 u = c_1|u|^2 u + c_2 u\partial_x\varphi & (x,y,t) \in \mathbf{R}^3, \\ \partial_x^2\varphi + c_3\partial_y^2\varphi = \partial_x|u|^2 & u(x,y,0) = u_0(x,y), \end{cases} \quad *$$

où c_0, $c_3 \in \mathbf{R}$, c_1, $c_2 \in \mathbf{C}$, tandis que u est une fonction à valeurs complexes et φ une fonction à valeurs réelles. Notre but, dans cet article, est d'étudier les cas respectivement elliptiques-hyperboliques ($c_0 > 0, c_3 < 0$) et hyperboliques-hyperboliques dans les espaces de Sobolev de plus bas ordre. Plus précisément, nous prouvons l'existence locale en temps de solution de $*$ sous les conditions $u_0 \in H^{1+\epsilon,0} \cap H^{0,1+\epsilon}$ et $\|u_0\|_{H^{1+\epsilon,0}} + \|u_0\|_{H^{0,1+\epsilon}}$ assez petit, où $\epsilon > 0$ est assez petit, et

$$H^{l,s} = \{f \in L^2; \|f\|_{H^{l,s}} = \|\langle\bar{x}\rangle^s \langle D\rangle^l f\|_{L^2} < \infty\},$$

$\langle\bar{x}\rangle = (1 + x^2 + y^2)^{1/2}$ et $\langle D\rangle = (1 - \partial_x^2 - \partial_y^2)^{1/2}$. De plus, dans le cas elliptique-hyperbolique, l'existence locale en temps des solutions est démontrée si $\langle\bar{x}\rangle u_0 \in H^{1/2+\epsilon,0} \cap H^{0,1/2+\epsilon}$ et si $\|\langle\bar{x}\rangle u_0\|_{H^{1/2+\epsilon,0}}$ $+\|\langle\bar{x}\rangle u_0\|_{H^{0,1/2+\epsilon}}$ est suffisamment petit.

1. INTRODUCTION

In this paper we study the initial value problem for the Davey-Stewartson (DS) systems

$$\begin{cases} i\partial_t u + c_0\partial_x^2 u + \partial_y^2 u = c_1|u|^2 u + c_2 u\partial_x\varphi & (x,y,t) \in \mathbf{R}^3, \\ \partial_x^2\varphi + c_3\partial_y^2\varphi = \partial_x|u|^2 & u(x,y,0) = u_0(x,y), \end{cases} \tag{1.1}$$

where c_0, $c_3 \in \mathbf{R}$, c_1, $c_2 \in \mathbf{C}$, u is a complex valued function and φ is a real valued funtion. Our purpose in this paper is to investigate the minimal regularity assumptions necessary on the data which yield the local existence in time of small solutions to the problem (1.1). The (D-S) systems were derived by Davey-Stewartson [8] Benney-Roskes [5] and Djordjevic-Redekopp [9] and model the evolution of weakly nonlinear water waves that travel predominantly in one direction, but in which the wave amplitude is modulated slowly in horizontal directions. Independently Ablowitz and Haberman [2] and Cornille [7] obtained a particular form of (1.1) as an example of a completely integrable model which generalizes the one-dimensional Schrödinger equation. In [9] it was shown that the parameter c_3 can become negative when capillary effects are important. When $(c_0, c_1, c_2, c_3) = (1, -1, 2, -1), (-1, -2, 1, 1)$ or $(-1, 2, -1, 1)$ the system (1.1) is referred as the DSI, DSII defocusing and DS II focusing respectively in the inverse scattering literature. In these cases several results concerning the existence of solutions or lump solutions have been established ([1], [3], [4], [7], [11-13], [21]) by the inverse scattering techniques. As a matter of fact, cases where (1.1) is of inverse scattering type are exceptional. In [14], (1.1) was classified as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of $(c_0, c_3) : (+,+), (+,-), (-,+)$ and $(-,-)$. For elliptic-elliptic and hyperbolic-elliptic cases, local and global properties of solutions were studied in [14] in the usual Sobolev spaces L^2, H^1 and H^2. Their main tools were $L^p - L^q$ estimates of Stricharz type [20] and the good continuity properties of the operator $(-\Delta)^{-1}$. In this paper we consider the elliptic-hyperbolic and hyperbolic-hyperbolic cases. Without loss of generality we may take $c_3 = -1$ and $c_0 = \pm 1$. In these cases after a rotation in the xy-plane and rescaling, the systems (1.1) can be written as

$$\begin{cases} i\partial_t u + Hu = d_1|u|^2u + d_2 u\partial_x\varphi, \\ \partial_x\partial_y\varphi = d_4\partial_x|u|^2 + d_5\partial_y|u|^2, \end{cases} \tag{1.2}$$

where $d_1, \ldots, d_5$ are arbitrary constants, $H = \partial_x\partial_y$ when $c_0 = -1$ and $H = \partial_x^2 + \partial_y^2$ when $c_0 = 1$. In order to solve the system of equations, one has to assume the $\varphi(\cdot)$ satisfies the radiation condition, namely, we assume that for given functions φ_1 and φ_2

$$\lim_{y\to\infty} \varphi(x,y,t) = \varphi_1(x,t) \quad \text{and} \quad \lim_{x\to\infty} \varphi(x,y,t) = \varphi_2(y,t). \tag{1.3}$$

Under the radiation condition (1.3), the system (1.2) can be written as

$$\begin{aligned} i\partial_t u + Hu =& d_1|u|^2u + d_2 u\int_y^\infty \partial_x|u|^2(x,y',t)dy' \\ &+ d_3 u\int_x^\infty \partial_y|u|^2(x',y,t)dx' + d_4 u\partial_x\varphi_1 + d_5 u\partial_y\varphi_2 \end{aligned} \tag{1.4}$$

with the initial condition $u(x,y,0) = u_0(x,y)$.

In this paper we study the existence and uniqueness of solutions to (1.4) in the usual fractional order Sobolev space

$$H^{l,s} = \{f \in L^2; \|\langle\bar{x}\rangle^s\langle D\rangle^l f\| < \infty\}, \quad l, s \in \mathbf{R}$$

with a smallness condition on the data, where

$$\langle \bar{x} \rangle = (1+x^2+y^2)^{1/2}, \langle D \rangle = (1-\partial_x^2-\partial_y^2)^{1/2} \quad \text{and} \quad \|\cdot\|^2 = \int |\cdot|^2 dxdy.$$

Linares and Ponce [19] proved local well-posedness results for (1.4) for small data by making use of the Kato type smoothing effect obtained in [16], [17] for the group $\{e^{iHt}\}$ (see also [18]). More precisely, they obtained

THEOREM A [19]. – *We assume that* $u_0 \in H^{s,0} \cap H^{3,2} \equiv Y_s, s \geq 6$, $\varphi_1 = \varphi_2 \equiv 0$ *and* $\|u_0\|_{H^{6,0}} + \|u_0\|_{H^{3,2}}$ *is sufficiently small. Then there exists a positive constant* $T > 0$ *and a unique solution* u *of* (1.4) *with* $H = \partial_x \partial_y$ *such that* $u \in C([0,T]; Y_s)$.

THEOREM B [19]. – *We assume that* $u_0 \in H^{s,0} \cap H^{6,6} \equiv W_s, s \geq 12$, $\varphi_1 = \varphi_2 \equiv 0$ *and* $\|u_0\|_{H^{12,0}} + \|u_0\|_{H^{6,6}}$ *is sufficiently small. Then there exists a positive constant* $T > 0$ *and a unique solution* u *of* (1.4) *with* $H = \partial_x^2 + \partial_y^2$ *such that* $u \in C([0,T]; W_s)$.

In [15, Theorem 1] local existence and uniqueness of analytic solutions to (1.4) were shown without a smallness condition on the data. Furthermore global existence of small analytic solutions to (1.4) was also obtainned in [15].

Recently, Chihara [6] showed the following two results.

THEOREM 1.1 [6]. – *We assume that* $u_0 \in H^{s,0}$,*where* s *is a sufficiently large integer,* $\varphi_1 = \varphi_2 \equiv 0$ *and* $\|u_0\|_{L^2} < 1/(2\sqrt{\max_{\{d_2,d_3\}} e})$. *Then there exists a positive constant* $T > 0$ *and a unique solution* u *of* (1.4) *with* $H = \partial_x^2 + \partial_y^2$ *such that*

$$u \in C_w([0,T]; H^{s,0}) \cap C([0,T]; H^{s-1,0}).$$

THEOREM 1.2 [6]. – *We assume that* $u_0 \in \cap_{j=0}^5 H^{s-j,j}$,*where* s *is a sufficiently large integer,* $\varphi_1 = \varphi_2 \equiv 0$ *and* $\sum_{j=0}^5 \|u_0\|_{H^{s-3-j,j}}$ *is sufficiently small. Then there exists a unique global solution* u *of* (1.4) *with* $H = \partial_x^2 + \partial_y^2$ *such that*

$$u \in \cap_{j=0}^5 C_w([0,\infty); H^{s-j,j}) \cap C([0,\infty); H^{s-1-j,j}).$$

To state our result precisely we also introduce

$$H_j^{l,s} = \{f \in L^2(\mathbf{R}_j); \|\langle j \rangle^s \langle D_j \rangle^l f\|_{L^2(\mathbf{R}_j)} < \infty\}, \quad l, s \in \mathbf{R},$$

where

$$j = x, y, \langle j \rangle = (1+j^2)^{1/2}, \langle D_j \rangle = (1-\partial_j^2)^{1/2} \quad \text{and} \quad \|\cdot\|^2_{L^2(\mathbf{R}_j)} = \int |\cdot|^2 dj.$$

For simplicity we write $L^p_j = L^p(\mathbf{R}_j)$ and $L^p_x L^q_y = L^p(\mathbf{R}_x; L^q(\mathbf{R}_y))$.

We are now in a position to state our main results in this paper

THEOREM 1. – *We assume that* $u_0 \in H^{\delta,0} \cap H^{0,\delta}$, $\partial_x \varphi_1 \in C(\mathbf{R}; H^{\delta,0}_x)$, $\partial_y \varphi_2 \in C(\mathbf{R}; H^{\delta,0}_y)$*and* $\|u_0\|_{H^{\delta,0}} + \|u_0\|_{H^{0,\delta}}$ *is sufficiently small. Then there exists a positive time* T *and a unique solution* u *of (1.4) such that*

$$u \in C([0,T]; H^{\delta,0} \cap H^{0,\delta})$$

and

$$\int_0^T \|\langle \bar{x} \rangle^{-\gamma} \langle D \rangle^{\delta + \frac{1}{2}} u(t)\|^2 dt < \infty$$

where $2\gamma = \delta, \delta = 1 + \epsilon, \epsilon > 0$ *is sufficiently small.*

For $H = \partial_x^2 + \partial_y^2$ (the elliptic-hyperbolic case) we obtain

THEOREM 2. – *We assume that*

$$\langle \bar{x} \rangle u_0 \in H^{\delta - \frac{1}{2}, 0} \cap H^{0, \delta - \frac{1}{2}},$$

$$\partial_x \varphi_1 \in C(\mathbf{R}; H_x^{\delta - \frac{1}{2}, 0}), \quad \partial_y \varphi_2 \in C(\mathbf{R}; H_y^{\delta - \frac{1}{2}, 0})$$

and

$$\|\langle \bar{x} \rangle u_0\|_{H^{\delta - \frac{1}{2}, 0}} + \|\langle \bar{x} \rangle u_0\|_{H^{0, \delta - \frac{1}{2}}}$$

is sufficiently small. Then there exists a positive constant T *and a unique solution* u *of (1.4) with* $H = \partial_x^2 + \partial_y^2$ *such that*

$$u \in C([0,T]; H^{\delta - \frac{1}{2}, 0} \cap H^{0, \delta - \frac{1}{2}}) \cap C([0,T]; H_{loc}^{\delta + \frac{1}{2}, 0})$$

and

$$\int_0^T \|\langle \bar{x} \rangle^{-\gamma} \langle D \rangle^{\delta} u(t)\|^2 dt + \int_0^T t^2 \|\langle D \rangle^{1+\delta} u(t)\|^2_{L^2_{loc}} dt < \infty,$$

where δ *and* γ *are the same as those given in Theorem 1.*

In order to obatin the results we introduce

Notation and function spaces. – We introduce some pseudo-differential operators. We let $\gamma = \frac{1}{2}(1+\epsilon), \epsilon > 0$

$$K_x = \exp\left[\left(\int_0^x \langle\tau\rangle^{-2\gamma} d\tau\right)\frac{D_x}{\langle D_x\rangle}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^x \langle\tau\rangle^{-2\gamma} d\tau \frac{D_x}{\langle D_x\rangle}\right)^n$$

$$K_y = \exp\left[\left(\int_0^y \langle\tau\rangle^{-2\gamma} d\tau\right)\frac{D_y}{\langle D_y\rangle}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^y \langle\tau\rangle^{-2\gamma} d\tau \frac{D_y}{\langle D_y\rangle}\right)^n$$

and

$$\tilde{K}_x = \exp\left[\left(\int_0^y \langle\tau\rangle^{-2\gamma} d\tau\right)\frac{D_x}{\langle D_x\rangle}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^y \langle\tau\rangle^{-2\gamma} d\tau \frac{D_x}{\langle D_x\rangle}\right)^n$$

$$\tilde{K}_y = \exp\left[\left(\int_0^y \langle\tau\rangle^{-2\gamma} d\tau\right)\frac{D_y}{\langle D_y\rangle}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^x \langle\tau\rangle^{-2\gamma} d\tau \frac{D_y}{\langle D_y\rangle}\right)^n.$$

These operators were used by Doi [10] first to derive the smoothing properties of solutions to (1.4).

We easily see that the inverse and the dual operators of the above operators are defined explicitly. We denote them by $K_x^{-1}, K_y^{-1}, \tilde{K}_x^{-1}, \tilde{K}_y^{-1}$, $K_x^*, K_y^*, \tilde{K}_x^*, \tilde{K}_y^*$ respectively.

We have by the fact that the operator $\frac{D_x}{\langle D_x\rangle}$ is the bounded operator from L_x^p into itself if $1 < p < \infty$

$$\left\|\left(\int_0^x \langle\tau\rangle^{-2\gamma} d\tau\right)\frac{D_x}{\langle D_x\rangle} f\right\|_{L_x^p} \leq C_p\left(\int_0^\infty \langle\tau\rangle^{-2\gamma} d\tau\right)\|f\|_{L_x^p}.$$

Hence

$$\begin{aligned}\|K_x f\|_{L_x^p} &\leq \sum_{n=0}^{\infty}\frac{1}{n!}\left\|\left(\left(\int_0^x \langle\tau\rangle^{-2\gamma} d\tau\right)\frac{D_x}{\langle D_x\rangle}\right)^n f\right\|_{L_x^p} \\ &\leq \sum_{n=0}^{\infty}\frac{1}{n!}\left(C_p\int_0^\infty \langle\tau\rangle^{-2\gamma} d\tau\right)^n \|f\|_{L_x^p} \\ &\leq \exp\left(C_p\int_0^\infty \langle\tau\rangle^{-2\gamma} d\tau\right)\|f\|_{L_x^p} \quad (1 < p < \infty).\end{aligned} \tag{1.5}$$

Similarly, we have

$$\begin{cases} \|K_y f\|_{L_y^p} \leq M_p\|f\|_{L_y^p}, \\ \|\tilde{K}_x f\|_{L_x^p} \leq M_p\|f\|_{L_x^p}, \\ \|\tilde{K}_y f\|_{L_y^p} \leq M_p\|f\|_{L_y^p}, \\ \|K_x^{-1} f\|_{L_x^p} \leq M_p\|f\|_{L_x^p}, \\ \|K_y^{-1} f\|_{L_y^p} \leq M_p\|f\|_{L_y^p}, \\ \|\tilde{K}_x^{-1} f\|_{L_x^p} \leq M_p\|f\|_{L_x^p}, \\ \|\tilde{K}_y^{-1} f\|_{L_y^p} \leq M_p\|f\|_{L_y^p}, \end{cases} \tag{1.6}$$

where $M_p = \exp(C_p \int_0^\infty \langle \tau \rangle^{-2\gamma} d\tau)$. In order to prove our results we use the following function spaces

$$X(T) = \Big\{ f \in C([0,T]; L^2); \|f\|_{X(T)}^2 = \sup_{t \in [0,T]} \|f(t)\|_{X_1}^2 + \int_0^T \|f(t)\|_{X_2}^2 dt < \infty \Big\},$$

$$\tilde{X}(T) = \Big\{ f \in C([0,T]; L^2); \|f\|_{\tilde{X}(T)}^2 = \sup_{t \in [0,T]} \|f(t)\|_{X_1}^2 + \int_0^T \|f(t)\|_{\tilde{X}_2}^2 dt < \infty \Big\},$$

$$Y(T) = \Big\{ f \in C([0,T]; L^2); \|f\|_{Y(T)}^2 = \sup_{t \in [0,T]} \|f(t)\|_{Y_1}^2 + \int_0^T \|f(t)\|_{Y_2}^2 dt < \infty \Big\},$$

where

$$\|f\|_{X_1} = \|\langle D \rangle^\delta f\| + \|\langle \bar{x} \rangle^\delta f\|,$$

$$\|f\|_{X_2} = \|\langle x \rangle^{-\gamma} \langle D_x \rangle^{\delta + \frac{1}{2}} f\| + \|\langle y \rangle^{-\gamma} \langle D_y \rangle^{\delta + \frac{1}{2}} f\|,$$

$$\|f\|_{\tilde{X}_2} = \|\langle x \rangle^{-\gamma} \langle D_y \rangle^{\delta + \frac{1}{2}} f\| + \|\langle y \rangle^{-\gamma} \langle D_x \rangle^{\delta + \frac{1}{2}} f\|,$$

$$\|f\|_{Y_1} = \sum_{|\alpha| \le 1} (\|\langle D \rangle^{\delta - \frac{1}{2}} J^\alpha f\| + \|\langle \bar{x} \rangle^{\delta - \frac{1}{2}} J^\alpha f\|)$$

$$\|f\|_{Y_2} = \sum_{|\alpha| \le 1} (\|\langle x \rangle^{-\gamma} \langle D_x \rangle^\delta J^\alpha f\| + \|\langle y \rangle^{-\gamma} \langle D_y \rangle^\delta J^\alpha f\|),$$

$$\langle D_j \rangle = (1 - \partial_j^2)^{1/2}, \quad J = (J_x, J_y), J_x = x + 2it\partial_x, J_y = y + 2it\partial_y.$$

We use the standard notation

$$\| \cdot \|_{L_x^p L_y^q} = \left(\int \left(\int | \cdot |^q dy \right)^{p/q} dx \right)^{1/p}, \quad \| \cdot \| = \| \cdot \|_{L_x^2 L_y^2},$$

$$(f, g) = \int \int f \cdot \bar{g} dx dy, \quad (f, g)_{L_x^2} = \int f \cdot \bar{g} dx,$$

We state our strategy of the proof of Theorem 1 with $H = \partial_x^2 + \partial_y^2$ and Theorem 2. Our results are based on the soothing property of solutions to the linear equations

$$\begin{cases} i\partial_t u + Hu = f, \\ u(0, x) = u_0 \end{cases}$$

which is written as

$$\sum_{j=x,y}\left(\|\langle D_j\rangle^m u(t)\|^2+\int_0^t\|\langle j\rangle^{-\gamma}\langle D_j\rangle^{m+1/2}u(\tau)\|^2d\tau\right)$$
$$\leq C\sum_{j=x,y}\left(\|\langle D_j\rangle^m u_0\|^2+\int_0^t\|\langle j\rangle^{-\gamma}u(\tau)\|^2d\tau\right.$$
$$\left.+\int_0^t|(K_j\langle D_j\rangle^m f(\tau),K_j\langle D_j\rangle^m u(\tau))|d\tau\right).$$

obtained by using the operators K_x, K_y (see Section 3 and Appendix), and the estimates of the nonlinear terms (Section 2) derived from the commutator estimates in fractional order Sobolev spaces by C. E. Kenig, G. Ponce and L. Vega [18] (Appendix).

2. PRELIMINARY ESTIMATES

In this section we prove the estimates of nonlinear terms which are needed to obtain the result. We first state the well-known Sobolev's inequality

THE SOBOLEV INEQUALITY. – *Let* $1 < p, q, r < \infty$. *Then*

$$\|\psi\|_{L^p_x}\leq C\|D^b_x\psi\|^a_{L^r_x}\|\psi\|^{1-a}_{L^q_x},$$

where

$$0\leq a=\left(\frac{1}{q}-\frac{1}{p}\right)/\left(b-\frac{1}{r}+\frac{1}{q}\right)<1$$

if $b-1/r\in\mathbf{N}\cup\{0\}$, $0\leq a\leq 1$ *otherwise, and* $D^\alpha_x=\mathcal{F}^{-1}|\xi_x|^\alpha\mathcal{F}$. *Furthermore we have*

$$\|\psi\|_{L^\infty_x}\leq C\|\langle D_x\rangle^\gamma\psi\|_{L^2_x}\quad\textit{if}\quad\gamma>1/2.$$

LEMMA 2.1. – *We have for* $f, h\in X_1$, $g, u\in X_1\cap\tilde{X}_2$

$$\left|\left(\langle D_x\rangle^\epsilon\partial_x\left(f\int_y^\infty(\partial_x g)h\,dy'\right),\langle D_x\rangle^\epsilon\partial_x u\right)\right|$$
$$\leq C\|f\|_{X_1}\|h\|_{X_1}(\|g\|_{X_1}+\|g\|_{\tilde{X}_2})(\|u\|_{X_1}+\|u\|_{\tilde{X}_2}).\qquad(2.1)$$

Proof. – By a direct calculation we have

$$\begin{aligned}
&\left|\left(\langle D_x\rangle^\epsilon \partial_x\left(f\int_y^\infty (\partial_x g)h dy'\right), \langle D_x\rangle^\epsilon \partial_x u\right)\right| \\
&\quad\le \left|\left(\langle D_x\rangle^\epsilon \left((\partial_x f)\int_y^\infty (\partial_x g)h dy'\right), \langle D_x\rangle^\epsilon \partial_x u\right)\right| \\
&\quad + \left|\left(\langle D_x\rangle^\epsilon \left(f\int_y^\infty (\partial_x g)\partial_x h dy'\right), \langle D_x\rangle^\epsilon \partial_x u\right)\right| \\
&\quad + \left|\left(\langle D_x\rangle^\epsilon \left(f\int_y^\infty (\partial_x^2 g)h dy'\right), \langle D_x\rangle^\epsilon \partial_x u\right)\right| \equiv \sum_{j=1}^{3} F_j. \qquad (2.2)
\end{aligned}$$

We let

$$G_1 = \langle D_x\rangle^\epsilon \left((\partial_x f)\int_y^\infty (\partial_x g)h dy'\right),$$

$$G_2 = (\langle D_x\rangle^\epsilon \partial_x f)\int_y^\infty (\partial_x g)h dy',$$

$$G_3 = (\partial_x f)\int_y^\infty \langle D_x\rangle^\epsilon (\partial_x g)h dy'.$$

Then we have by the Schwarz inequality

$$F_1 \le \|G_1\|\|u\|_{X_1} \le (\|G_1 - G_2 - G_3\| + \|G_2\| + \|G_3\|)\|u\|_{X_1}. \qquad (2.3)$$

By Theorem A.1 (Appendix)

$$\begin{aligned}
&\|G_1 - G_2 - G_3\|_{L_x^2} + \|G_2\|_{L_x^2} \\
&\quad \le C\|\langle D_x\rangle^\epsilon \partial_x f\|_{L_x^2}\|\int_y^\infty (\partial_x g)h dy'\|_{L_x^\infty} \\
&\quad \le C\|\langle D_x\rangle^\epsilon \partial_x f\|_{L_x^2}\int_y^\infty \|(\partial_x g)h\|_{L_x^\infty} dy'. \qquad (2.4)
\end{aligned}$$

Applying Sobolev's inequality we see that

$$\begin{aligned}
\|(\partial_x g)h\|_{L_x^\infty} &= \|\langle y\rangle^{-\gamma}(\partial_x g)\langle y\rangle^\gamma h\|_{L_x^\infty} \\
&\le C\|\langle y\rangle^{-\gamma}\langle D_x\rangle^\gamma \partial_x g\|_{L_x^2}\|\langle y\rangle^\gamma \langle D_x\rangle^\gamma h\|_{L_x^2}. \qquad (2.5)
\end{aligned}$$

Hence by (2.4) and (2.5)

$$\|G_1 - G_2 - G_3\| + \|G_2\| \le C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}. \tag{2.6}$$

By Hölder's and Sobolev's inequalities

$$\begin{aligned}\|G_3\|_{L^2_x} &\le C\|\partial_x f\|_{L^{p_1}_x}\int_y^\infty \|\langle D_x\rangle^\epsilon((\partial_x g)h)\|_{L^{p_2}_x}dy' \\ &\le C\|\langle D_x\rangle^\epsilon \partial_x f\|_{L^2_x}\int_y^\infty \|\langle D_x\rangle^{1/2}((\partial_x g)h)\|_{L^2_x}dy',\end{aligned} \tag{2.7}$$

where $1/p_1 = 1/2 - \epsilon$, $1/p_2 = \epsilon$. From Theorem A. 1 (Appendix) it follows that

$$\begin{aligned}\|\langle D_x\rangle^{\frac{1}{2}}((\partial_x g)h)\|_{L^2_x} &= \|\langle D_x\rangle^{\frac{1}{2}}(\langle y\rangle^{-\gamma}(\partial_x g)\langle y\rangle^\gamma h)\|_{L^2_x} \\ &\le C(\|\langle D_x\rangle^{\frac{1}{2}}\langle y\rangle^{-\gamma}\partial_x g\|_{L^2_x}\|\langle y\rangle^\gamma h\|_{L^\infty_x} \\ &\quad + \|\langle y\rangle^{-\gamma}\partial_x g\|_{L^\infty_x}\|\langle D_x\rangle^{\frac{1}{2}}\langle y\rangle^\gamma h\|_{L^2_x}).\end{aligned} \tag{2.8}$$

We apply Sobolev's inequality to the right hand side of (2.8) to get

$$\begin{aligned}&\|\langle D_x\rangle^{\frac{1}{2}}((\partial_x g)h)\|_{L^2_x} \\ &\le C(\|\langle D_x\rangle^{\frac{1}{2}}\langle y\rangle^{-\gamma}\partial_x g\|_{L^2_x}\|\langle D_x\rangle^\gamma\langle y\rangle^\gamma h\|_{L^2_x} \\ &\quad + \|\langle D_x\rangle^\gamma\langle y\rangle^{-\gamma}\partial_x g\|_{L^2_x}\|\langle D_x\rangle^{\frac{1}{2}}\langle y\rangle^\gamma h\|_{L^2_x}).\end{aligned} \tag{2.9}$$

From (2.7) and (2.8) it is easy to see that

$$\|G_3\| \le C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}. \tag{2.10}$$

Thus we get by (2.3), (2.6) and (2.10)

$$F_1 \le C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}\|u\|_{X_1}. \tag{2.11}$$

We next consider the term F_2. We have by Hölder's inequality

$$\begin{aligned}F_2 &= \left|\left(f\int_y^\infty(\partial_x g)(\partial_x h)dy', \langle D_x\rangle^\epsilon\partial_x u\right)\right| \\ &= \left|\left(\langle y\rangle^\gamma f\int_y^\infty(\partial_x g)(\partial_x h)dy', \langle y\rangle^\gamma\langle D_x\rangle^\epsilon\partial_x u\right)\right| \\ &\le C\left\|\langle y\rangle^\gamma f\int_y^\infty(\partial_x g)(\partial_x h)dy'\right\|_{L^2_y L^{p_1}_x}\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L^2_y L^{p_2}_x},\end{aligned} \tag{2.12}$$

where $1/p_2 = \epsilon$ and $1/p_1 = 1 - \epsilon$. We again use Hölder's and Sobolev's inequalities to get

$$\begin{aligned}
&\left\| \langle y \rangle^\gamma f \int_y^\infty (\partial_x g)(\partial_x h) dy' \right\|_{L_x^{p_1}} \\
&\leq C \| \langle y \rangle^\gamma f \|_{L_x^\infty} \int_y^\infty \| (\partial_x g)(\partial_x h) \|_{L_x^{p_1}} dy' \\
&\leq C \| \langle D_x \rangle^\gamma \langle y \rangle^\gamma f \|_{L_x^2} \int_y^\infty \| \partial_x g \|_{L_x^{2p_1}} \| \partial_x h \|_{L_x^{2p_1}} dy' \\
&\leq C \| \langle D_x \rangle^\gamma \langle y \rangle^\gamma f \|_{L_x^2} \int_y^\infty \| \langle D_x \rangle^{\epsilon/2} \partial_x g \|_{L_x^2} \| \langle D_x \rangle^{\epsilon/2} \partial_x h \|_{L_x^2} dy' \qquad (2.13)
\end{aligned}$$

and

$$\| \langle y \rangle^{-\gamma} \langle D_x \rangle^{2\epsilon} \partial_x u \|_{L_x^{p_2}} \leq C \| \langle y \rangle^{-\gamma} \langle D_x \rangle^{\frac{1}{2}+\epsilon} \partial_x u \|_{L_x^2} \qquad (2.14)$$

Hence we have by (2.12), (2.13) and (2.14)

$$F_2 \leq C \|f\|_{X_1} \|g\|_{X_1} \|h\|_{X_1} \|u\|_{\tilde{X}_2}. \qquad (2.15)$$

We finally consider the term F_3. We let

$$G_4 = \langle D_x \rangle^{\frac{1}{2}-\epsilon} \left(f \int_y^\infty (\langle D_x \rangle^{-\frac{1}{2}+\epsilon} \partial_x^2 g) h dy' \right),$$

$$G_5 = (\langle D_x \rangle^{\frac{1}{2}-\epsilon} f) \int_y^\infty (\langle D_x \rangle^{-\frac{1}{2}+\epsilon} \partial_x^2 g) h dy',$$

$$G_6 = f \int_y^\infty \langle D_x \rangle^{\frac{1}{2}-\epsilon} ((\langle D_x \rangle^{-\frac{1}{2}+\epsilon} \partial_x^2 g) h) dy',$$

$$G_7 = f \int_y^\infty (\langle D_x \rangle^{-\frac{1}{2}+\epsilon} \partial_x^2 g)(\langle D_x \rangle^{\frac{1}{2}-\epsilon} h) dy',$$

$$G_8 = f \int_y^\infty (\partial_x^2 g) h dy'.$$

We have by (2.14)

$$\begin{aligned}
F_3 &= \left| \left(f \int_y^\infty (\partial_x^2 g) h dy', \langle D_x \rangle^{2\epsilon} \partial_x u \right) \right| \\
&\leq C \| \langle y \rangle^\gamma G_8 \|_{L_y^2 L_x^{p_1}} \|u\|_{\tilde{X}_2} \\
&\leq C (\| \langle y \rangle^\gamma (G_5 + G_6 - G_4) \|_{L_y^2 L_x^{p_1}} + \| \langle y \rangle^\gamma (-G_5) \|_{L_y^2 L_x^{p_1}} \\
&+ \| \langle y \rangle^\gamma (G_7 + G_8 - G_6) \|_{L_y^2 L_x^{p_1}} + \| \langle y \rangle^\gamma (-G_7) \|_{L_y^2 L_x^{p_1}}) \|u\|_{\tilde{X}_2} \qquad (2.16)
\end{aligned}$$

with $1/p_1 = 1-\epsilon$. By Hölder's inequality and Theorem A.1 (Appendix)

$$
\begin{aligned}
&\|\langle y\rangle^{\gamma}(G_5+G_6-G_4)\|_{L_x^{p_1}} \\
&\le C\|\langle y\rangle^{\gamma}\langle D_x\rangle^{\frac{1}{2}-\epsilon}f\|_{L_x^{p_3}}\int_y^{\infty}\|(\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2 g)h\|_{L_x^2}\,dy' \\
&\le C\|\langle y\rangle^{\gamma}\langle D_x\rangle^{\frac{1}{2}-\epsilon}f\|_{L_x^{p_3}} \\
&\quad\times\int_y^{\infty}\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2 g\|_{L_x^2}\|\langle y\rangle^{\gamma}h\|_{L_x^{\infty}}\,dy', \qquad (2.17)
\end{aligned}
$$

where $1/p_3 = 1/2-\epsilon$. We use Sobolev's inequality in the right hand side of (2.17) to get

$$
\begin{aligned}
&\|\langle y\rangle^{\gamma}(G_5+G_6-G_4)\|_{L_x^{p_1}} \\
&\le C\|\langle y\rangle^{\gamma}\langle D_x\rangle^{\frac{1}{2}}f\|_{L_x^2} \\
&\quad\times\int_y^{\infty}\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2 g\|_{L_x^2}\|\langle D_x\rangle^{\gamma}\langle y\rangle^{\gamma}h\|_{L_x^2}\,dy', \quad (2.18)
\end{aligned}
$$

Hence we have by (2.18)

$$
\begin{aligned}
&|(G_5+G_6-G_4,\langle D_x\rangle^{2\epsilon}\partial_x u)| \\
&\quad\le C\|\langle y\rangle^{\gamma}(G_5+G_6-G_4)\|_{L_y^2L_x^{p_1}}\|u\|_{\tilde{X}_2} \\
&\quad\le C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}\|u\|_{\tilde{X}_2}. \qquad (2.19)
\end{aligned}
$$

Similarly, we see that

$$
|(-G_5,\langle D_x\rangle^{2\epsilon}\partial_x u)| \le C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}\|u\|_{\tilde{X}_2}. \qquad (2.20)
$$

By Hölder's inequality and Theorem A.1 (Appendix)

$$
\begin{aligned}
&|(G_7+G_8-G_6,\langle D_x\rangle^{2\epsilon}\partial_x u)_{L_x^2}| \\
&\le \|\langle y\rangle^{\gamma}f\|_{L_x^{\infty}}\Bigg(\int_y^{\infty}\|\langle D_x\rangle^{\frac{1}{2}-\epsilon}((\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2 g)h) \\
&\qquad -(\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2 g)(\langle D_x\rangle^{\frac{1}{2}-\epsilon}h) \\
&\qquad -(\partial_x^2 g)h\|_{L_x^{p_1}}\,dy'\Bigg)\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_x^{p_2}} \\
&\le C\|\langle y\rangle^{\gamma}f\|_{L_x^{\infty}}\Bigg(\int_y^{\infty}\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2 g\|_{L_x^2} \\
&\quad\times\|\langle y\rangle^{\gamma}\langle D_x\rangle^{-\frac{1}{2}-\epsilon}h\|_{L_x^{p_3}}\,dy'\Bigg)\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_x^{p_2}}, \quad (2.21)
\end{aligned}
$$

where $1/p_2 = \epsilon, 1/p_3 = 1/2 - \epsilon, 1/p_1 = 1 - \epsilon$. We apply Sobolev's inequality to (2.21) to obatin

$$\begin{aligned} &|(G_7 + G_8 - G_6, \langle D_x\rangle^{2\epsilon}\partial_x u)_{L^2_x}| \\ &\quad \leq C\|\langle D_x\rangle^{\gamma}\langle y\rangle^{\gamma} f\|_{L^2_x} \\ &\quad \times \left(\int_y^{\infty} \|\langle y\rangle^{\gamma}\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2 g\|_{L^2_x}\|\langle y\rangle^{\gamma}\langle D_x\rangle^{\frac{1}{2}} h\|_{L^2_x} dy'\right) \\ &\quad \times \|\langle y\rangle^{-\gamma}\langle D_x\rangle^{\frac{1}{2}+\epsilon}\partial_x u\|_{L^2_x}. \end{aligned} \tag{2.22}$$

Taking the L^1_y norm in (2.22), we obtain

$$|(G_7 + G_8 - G_6, \langle D_x\rangle^{2\epsilon}\partial_x u)| \leq C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}\|u\|_{\tilde{X}_2}. \tag{2.23}$$

Sim:ilarly, we find that

$$|(-G_7, \langle D_x\rangle^{2\epsilon}\partial_x u)| \leq C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}\|u\|_{\tilde{X}_2}. \tag{2.24}$$

From (2.19), (2.20), (2.23) and (2.24) it follows that

$$F_3 \leq C\|f\|_{X_1}\|g\|_{\tilde{X}_x}\|h\|_{X_1}\|u\|_{\tilde{X}_2}. \tag{2.25}$$

Thus (2.11), (2.15) and (2.25) give the result.

LEMMA 2.2. – *We have for $f, h \in X_1$ and $g \in X_1 \cap \tilde{X}_2$*

$$\|\langle D_y\rangle^{\epsilon}\partial_y(f\int_y^{\infty}(\partial_x g)h dy')\| \leq C\|f\|_{X_1}(\|g\|_{X_1} + \|g\|_{\tilde{X}_2})\|h\|_{X_1}.$$

Proof. – By a direct calculation

$$\begin{aligned} &\left\|\langle D_y\rangle^{\epsilon}\partial_y(f\int_y^{\infty}(\partial_x g)h dy'\right)\right\| \\ &\quad \leq \left\|\langle D_y\rangle^{\epsilon}\left((\partial_y f)\int_y^{\infty}(\partial_x g)h dy'\right)\right\| + \|\langle D_y\rangle^{\epsilon}(f(\partial_x g)h)\|. \\ &\quad \leq C(\|(\partial_y f)\int_y^{\infty}(\partial_x g)h dy'\| + \|f(\partial_x g)h\| \\ &\quad + \|D_y^{\epsilon}((\partial_y f)\int_y^{\infty}(\partial_x g)h dy')\| + \|D_y^{\epsilon}(f(\partial_x g)h)\|), \end{aligned} \tag{2.26}$$

where $D_y^\epsilon = \mathcal{F}^{-1}|\xi_y|^\epsilon\mathcal{F}$. We let

$$G_9 = D_y^\epsilon((\partial_y f)\int_y^\infty(\partial_x g)hdy'),$$

$$G_{10} = (D_y^\epsilon\partial_y f)\int_y^\infty(\partial_x g)hdy',$$

$$G_{11} = (\partial_y f)D_y^\epsilon\Big(\int_y^\infty(\partial_x g)hdy'\Big).$$

Then we have

$$\begin{aligned}
\|G_9\|_{L_y^2} &\le \|G_9 - G_{10} - G_{11}\|_{L_y^2} + \|G_{10}\|_{L_y^2} + \|G_{11}\|_{L_y^2}\\
&\le C\Big(\|D_y^\epsilon\partial_y f\|_{L_y^2}\Big\|\int_y^\infty(\partial_x g)hdy'\Big\|_{L_y^\infty}\\
&\quad + \|\partial_y f\|_{L_y^{p_1}}\Big\|D_y^\epsilon\Big(\int_y^\infty(\partial_x g)hdy'\Big\|_{L_y^{p_2}}\Big)\\
&\qquad\qquad \text{(by Theorem A.1)}\\
&\le C\Big(\|D_y^\epsilon\partial_y f\|_{L_y^2}(\|(\partial_x g)h\|_{L_y^1}\\
&\quad + \Big\|\partial_y\Big(\int_y^\infty(\partial_x g)hdy'\Big)\Big\|_{L_y^1}\Big)\Big) \quad \text{(by Sobolev)}\\
&\le C\|D_y^\epsilon\partial_y f\|_{L_y^2}\|(\partial_x g)h\|_{L_y^1}\\
&\le C\|D_y^\epsilon\partial_y f\|_{L_y^2}\|\langle y\rangle^{-\gamma}\partial_x g\|_{L_y^2}\|\langle y\rangle^\gamma h\|_{L_y^2}\\
&\qquad\qquad \text{(by Schwarz)},
\end{aligned} \tag{2.27}$$

where $1/p_1 + 1/p_2 = 1/2$, $1/p_1 = 1/2 - \epsilon, 1/p_2 = \epsilon$. We again use Sobolev's inequality to get

$$\begin{aligned}
\|G_9\| &\le C\|\langle D_y\rangle^\epsilon\partial_y f\|\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{\frac{1}{2}+\epsilon}\partial_x g\|\|\langle y\rangle^\gamma\langle D_x\rangle^\gamma h\|\\
&\le C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1}.
\end{aligned} \tag{2.28}$$

We let

$$G_{12} = D_y^\epsilon(f(\partial_x g)h), G_{13} = (D_y^\epsilon\partial_x g)fh, G_{14} = (\partial_x g)(D_y^\epsilon(fh))$$

and use Theorem A.1 (Appendix). Then we have

$$\begin{aligned}\|G_{12}\|_{L^2_y} &\leq \|G_{12}-G_{13}-G_{14}\|_{L^2_y}+\|G_{13}\|_{L^2_y}+\|G_{14}\|_{L^2_y}\\ &\leq C(\|D^\epsilon_y\partial_x g\|_{L^2_y}\|fg\|_{L^\infty_y}+\|\partial_x g\|_{L^{p_1}_y}\|D^\epsilon_y(fg)\|_{L^{p_2}_y})\\ &\leq C(\|D^\epsilon_y\partial_x g\|_{L^2_y}(\|\langle D_y\rangle^\gamma f\|_{L^2_y}\|\langle D_y\rangle^\gamma g\|_{L^2_y}\\ &\qquad+\|D_y^{\frac{1}{2}}(fg)\|_{L^2_y})\quad\text{(by Sobolev).}\end{aligned}\tag{2.29}$$

Since

$$\begin{aligned}\|D_y^{\frac{1}{2}}(fg)\|_{L^2_y} &\leq \|D_y^{\frac{1}{2}}(fg)-fD_y^{\frac{1}{2}}g-gD_y^{\frac{1}{2}}f\|_{L^2_y}\\ &\quad+\|fD_y^{\frac{1}{2}}g\|_{L^2_y}+\|gD_y^{\frac{1}{2}}f\|_{L^2_y}\end{aligned}$$

we get by Theorem A.1 (Appendix) and Sobolev's inequality

$$\|G_{12}\|_{L^2_y}\leq C\|\langle D_y\rangle^\epsilon\partial_x g\|_{L^2_y}\|\langle D_y\rangle^\gamma f\|_{L^2_y}\|\langle D_y\rangle^\gamma g\|_{L^2_y}.\tag{2.30}$$

Taking the L^2_x norm in both sides of (2.30), and using Sobolev's inequality again we obtain

$$\|G_{12}\|\leq C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}.\tag{2.31}$$

From (2.28) and (2.31) it follows that

$$\begin{aligned}&\left\|D^\epsilon_y\left((\partial_y f)\int_y^\infty(\partial_x g)h\,dy'\right)\right\|+\|D^\epsilon_y(f(\partial_x g)h)\|\\ &\quad\leq C\|f\|_{X_1}(\|g\|_{X_1}+\|g\|_{\tilde{X}_2})\|h\|_{x_1}.\end{aligned}\tag{2.32}$$

In the same way as in the proof of (2.32) we see that

$$\|(\partial_y f)\int_y^\infty(\partial_x g)h\,dy'\|+\|f(\partial_x g)h\|$$

is bounded from above by the right hand side of (2.32). Hence we have the result by (2.26) and (2.32).

LEMMA 2.3. – *We have for $f,h\in X_1$ and $g\in\tilde{X}_2$ or $\in X_2$*

$$\left\|\langle\bar{x}\rangle^\delta f\int_y^\infty(\partial_x g)h\,dy'\right\|\leq C\|f\|_{X_1}\|g\|_{\tilde{X}_2}\|h\|_{X_1},$$

$$\left\| \langle \bar{x} \rangle^{\delta} f \int_y^{\infty} (\partial_x g) h dy' \right\| \leq C \|f\|_{X_1} \|g\|_{X_2} \|h\|_{X_1},$$

Proof. – By Hölder's inequality and (2.5)

$$\begin{aligned} &\left\| \langle \bar{x} \rangle^{\delta} f \int_y^{\infty} (\partial_x g) h dy' \right\|_{L_x^2} \\ &\leq C \|\langle \bar{x} \rangle^{\delta} f\|_{L_x^2} \int_y^{\infty} \|(\partial_x g) h\|_{L_x^{\infty}} dy' \\ &\leq C \|\langle \bar{x} \rangle^{\delta} f\|_{L_x^2} \int_y^{\infty} \|\langle y \rangle^{-\gamma} \langle D_x \rangle^{\gamma} \partial_x g\|_{L_x^2} \|\langle y \rangle^{\gamma} \langle D_x \rangle^{\gamma} h\|_{L_x^2} dy'. \quad (2.33) \end{aligned}$$

We take L_y^2 norm in both sides of (2.33) to obtain the first inequality. In the same way as in the proof of (2.33) we have

$$\begin{aligned} &\left\| \langle \bar{x} \rangle^{\delta} f \int_y^{\infty} (\partial_x g) h dy' \right\|_{L_x^2} \\ &\leq C \|\langle \bar{x} \rangle^{\delta} f\|_{L_x^2} \int_y^{\infty} \|\langle D_x \rangle^{\gamma} \langle x \rangle^{-\gamma} \partial_x g\|_{L_x^2} \|\langle D_x \rangle^{\gamma} \langle x \rangle^{\gamma} h\|_{L_x^2} dy'. \end{aligned}$$

We apply Lemma A.1 to the right hand side of the above inequality to get the last inequality in the lemma.

LEMMA 2.4. – *We have for $f, h \in X_1$ and $g \in X_2$*

$$\begin{aligned} &\left| \left(\langle D_x \rangle^{\epsilon} \partial_x \left(f \int_y^{\infty} (\partial_x g) h dy' \right), \langle D_x \rangle^{\epsilon} \partial_x u \right) \right| \\ &\leq C \|f\|_{X_1} \|h\|_{X_1} (\|g\|_{X_1} + \|g\|_{X_2})(\|u\|_{X_1} + \|u\|_{X_2}). \end{aligned}$$

Proof. – In the same way as in the proof of (2.6) we have

$$\begin{aligned} &\|G_1 - G_2 - G_3\| + \|G_2\| \\ &\leq C \|\langle D_x \rangle^{\epsilon} \partial_x f\| \|\langle D_x \rangle^{\gamma} \langle x \rangle^{-\gamma} \partial_x g\| \|\langle D_x \rangle^{\gamma} \langle x \rangle^{\gamma} h\|. \end{aligned}$$

By using Lemma A.1, we get

$$\|G_1 - G_2 - G_3\| + \|G_2\| \leq C \|f\|_{X_1} \|g\|_{X_2} \|h\|_{X_1}.$$

In the same way as in the proof of (2.10) we obtain

$$\|G_3\| \leq C \|\langle D_x \rangle^{\epsilon} \partial_x f\| \|\langle D_x \rangle^{\gamma} \langle x \rangle^{-\gamma} \partial_x g\| \|\langle D_x \rangle^{\gamma} \langle x \rangle^{\gamma} h\|.$$

Hence Lemma A.1 gives

$$\|G_3\| \le C\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_1}. \tag{2.35}$$

From (2.34) and (2.35) it follows that

$$F_1 \le C\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_1}\|u\|_{X_1}. \tag{2.36}$$

Similarly, we get

$$F_2 \le C\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_1}\|u\|_{X_1}. \tag{2.37}$$

We finally consider the term F_3. Since

$$\partial_x^2\langle x\rangle^{-\gamma}g = \langle x\rangle^{-\gamma}\partial_x^2 g + (\partial_x^2\langle x\rangle^{-\gamma})g + 2(\partial_x\langle x\rangle^{-\gamma})\partial_x g,$$

$$\begin{aligned} F_3 \le & \left|\left(f\int_y^\infty (\partial_x^2\langle x\rangle^{-\gamma}g)\langle x\rangle^{\gamma}h dy', \langle D_x\rangle^{2\epsilon}\partial_x u\right)\right| \\ & + \left|\left(f\int_y^\infty (\partial_x^2\langle x\rangle^{-\gamma})g\langle x\rangle^{\gamma}h dy', \langle D_x\rangle^{2\epsilon}\partial_x u\right)\right| \\ & + 2\left|\left(f\int_y^\infty (\partial_x\langle x\rangle^{-\gamma})(\partial_x g)\langle x\rangle^{\gamma}h dy', \langle D_x\rangle^{2\epsilon}\partial_x u\right)\right| \equiv \sum_{j=4}^{6} F_j. \end{aligned} \tag{2.38}$$

We have

$$\begin{aligned} F_5 + F_6 \le C\Bigg(& \left\|\left\|f\int_y^\infty g h dy'\right\|_{L_x^{p_1}} \|\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_x^{p_2}}\right\|_{L_y^1} \\ & + \Bigg\|\left\|f\int_y^\infty (\partial_x g)h dy'\right\|_{L_x^{p_1}} \\ & \times \quad \|\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_x^{p_2}}\Bigg\|_{L_y^1}\Bigg), \end{aligned} \tag{2.39}$$

where $1/p_1 + 1/p_2 = 1$, $1/p_2 = 1/2 - \epsilon$, $1/p_1 = 1/2 + \epsilon$. By Sobolev's inequality and Lemma A.1

$$\begin{aligned} F_5 + F_6 & \le C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}\|\langle D_x\rangle^{\frac{1}{2}-\epsilon}\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\| \\ & \le C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}\|u\|_{X_2}. \end{aligned} \tag{2.40}$$

We next consider F_4. We put

$$G_{15} = \langle D_x\rangle^{\frac{1}{2}-\epsilon}\Big(\langle x\rangle^{\gamma} f \int_y^{\infty} (\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2\langle x\rangle^{-\gamma} g)\langle x\rangle^{\gamma} h dy'\Big),$$

$$G_{16} = (\langle D_x\rangle^{\frac{1}{2}-\epsilon}\langle x\rangle^{\gamma} f)\int_y^{\infty} (\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2\langle x\rangle^{-\gamma} g)\langle x\rangle^{\gamma} h dy',$$

$$G_{17} = \langle x\rangle^{\gamma} f\int_y^{\infty} \langle D_x\rangle^{-\frac{1}{2}-\epsilon}((\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2\langle x\rangle^{-\gamma} g)\langle x\rangle^{\gamma} h) dy',$$

$$G_{18} = \langle x\rangle^{\gamma} f\int_y^{\infty} (\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2\langle x\rangle^{-\gamma} g)(\langle D_x\rangle^{\frac{1}{2}-\epsilon}\langle x\rangle^{\gamma} h) dy',$$

$$G_{19} = \langle x\rangle^{\gamma} f\int_y^{\infty} (\partial_x^2\langle x\rangle^{-\gamma} g)(\langle x\rangle^{\gamma} h) dy'.$$

Then

$$\begin{aligned} F_4 &= |(G_{19}, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)| \\ &\le |(-G_{15} + G_{16} + G_{17}, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)| \\ &\quad + |(-G_{16}, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)| \\ &\quad + |(-G_{17} + G_{18} + G_{19}, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)| \\ &\quad + |(-G_{18}, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)| \\ &\quad + |(-G_{15}, \langle x\rangle^{\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)|. \end{aligned} \tag{2.41}$$

By Theorem A.1 (Appendix), Hölder's and Sobolev's inequalities

$$\begin{aligned} &|(-G_{15} + G_{16} + G_{17}, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)| + |(-G_{16}, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)| \\ &\quad \le C(\|G_{15} - G_{16} - G_{17}\|_{L_x^{p_1}} + \|G_{16}\|_{L_x^{p_1}})\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_x^{p_2}} \\ &\quad \le C\|\langle D_x\rangle^{\frac{1}{2}-\epsilon}\langle x\rangle^{\gamma} f\|_{L_x^{p_3}}\Big\|\int_y^{\infty} (\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2\langle x\rangle^{-\gamma} g)\langle x\rangle^{\gamma} h dy'\Big\|_{L_x^{p_4}} \\ &\qquad \times \|\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_x^{p_2}} \\ &\quad \le C\|\langle D_x\rangle^{\frac{1}{2}}\langle x\rangle^{\gamma} f\|_{L_x^2}\Big(\int_y^{\infty} \|\langle D_x\rangle^{-\frac{1}{2}+\epsilon}\partial_x^2\langle x\rangle^{-\gamma} g\|_{L_x^2} \\ &\qquad \times \|\langle D_x\rangle^{\gamma}\langle x\rangle^{\gamma} h\|_{L_x^2} dy'\Big)\|\langle D_x\rangle^{\frac{1}{2}-\epsilon}\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_x^2}, \end{aligned}$$

where $1/p_1 = 1/2 + \epsilon, 1/p_2 = 1/2 - \epsilon$, $1/p_3 = \epsilon$ and $1/p_4 = 1/2$.

Hence we have by Lemma A.1

$$\begin{aligned}&|(-G_{15}+G_{16}+G_{17},\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)|\\&\quad+|(-G_{16},\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)|\\&\leq C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}\|u\|_{X_2}.\end{aligned}\tag{2.42}$$

Similarly,

$$\begin{aligned}&|(-G_{17}+G_{18}+G_{19},\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)_{L^2_x}|\\&\leq\|\langle x\rangle^{\gamma}f\|_{L^\infty_x}\int_y^\infty\|\langle D_x\rangle^{\frac12-\epsilon}((\langle D_x\rangle^{-\frac12+\epsilon}\partial_x^2\langle x\rangle^{-\gamma}g)\langle x\rangle^{\gamma}h)\\&\quad-(\langle D_x\rangle^{-\frac12+\epsilon}\partial_x^2\langle x\rangle^{-\gamma}g)\langle D_x\rangle^{\frac12-\epsilon}\langle x\rangle^{\gamma}h-(\partial_x^2\langle x\rangle^{-\gamma}g)(\langle x\rangle^{\gamma}h)\|_{L^{p_1}_x}dy'\\&\quad\times\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L^{p_2}_x}\quad\text{(by Hölder)}\quad 1/p_1=1-\epsilon,\quad 1/p_2=\epsilon\\&\leq C\|\langle D_x\rangle^{\gamma}\langle x\rangle^{\gamma}f\|_{L^2_x}\|\langle D_x\rangle^{-\frac12+\epsilon}\partial_x^2\langle x\rangle^{-\gamma}g\|_{L^2_x}\\&\quad\times\|\langle D_x\rangle^{\frac12}\langle x\rangle^{\gamma}h\|_{L^2_x}\|\langle D_x\rangle^{\frac12-\epsilon}\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L^2_x}\quad\text{(by Sobolev).}\end{aligned}$$

By Lemma A.1

$$\begin{aligned}&|(-G_{17}+G_{18}+G_{19},\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)|\\&\leq C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}\|u\|_{X_2}.\end{aligned}\tag{2.43}$$

Similarly

$$|(-G_{18},\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u)|\leq C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}\|u\|_{X_2}.\tag{2.44}$$

Hence by (2.41)-(2.44)

$$F_4\leq C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}\|u\|_{X_2}.\tag{2.45}$$

By (2.40) and (2.45)

$$F_3\leq C\|f\|_{X_1}\|g\|_{X_1}\|h\|_{X_1}\|u\|_{X_2}.\tag{2.46}$$

From (2.36), (2.37) and (2.46), the lemma follows.

LEMMA 2.5. – *We have for $f,h\in X_1$, $g\in X_1\cap X_2$.*

$$\left\|\langle D_y\rangle^{\epsilon}\partial_y\left(f\int_y^\infty(\partial_x g)h\,dy'\right)\right\|\leq C\|f\|_{X_1}(\|g\|_{X_1}+\|g\|_{X_2})\|h\|_{X_1}.$$

Proof. – In the same way as in the proof of (2.28)

$$\begin{aligned}&\left\|\langle D_y\rangle^\epsilon\partial_y\left(f\int_y^\infty(\partial_x g)hdy'\right)\right\|\\&\le C\|\langle D_y\rangle^\epsilon\partial_y f\|\|\langle D_x\rangle^{\frac{1}{2}+\epsilon}\langle x\rangle^{-\gamma}\partial_x g\|\|\langle D_x\rangle^\gamma\langle x\rangle^\gamma h\|\\&\le C\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_1}\quad\text{(by Lemma A.1)}.\end{aligned}\tag{2.47}$$

The inequality (2.31) implies that

$$\|\langle D_y\rangle^\epsilon(f(\partial_x g)h)\|\le C\|f\|_{X_1}\|g\|_{X_2}\|h\|_{X_1}.\tag{2.48}$$

From (2.47) and (2.48) the lemma follows.

LEMMA 2.6. – *We let $\alpha=\delta-\frac{1}{2}$, $f,\bar{h}\in Y_1$ and $g\in Y_2$. Then we have*

$$\begin{aligned}&\left|\left(\langle D_x\rangle^\alpha J_x\left(f\int_y^\infty(\partial_x g)\bar{h}dy'\right),\langle D_x\rangle^\alpha J_x u\right)\right|\\&\le Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1}+\|u\|_{Y_2}).\end{aligned}$$

Proof. – By a direct calculation

$$\begin{aligned}&\left(\langle D_x\rangle^\alpha J_x\left(f\int_y^\infty(\partial_x g)\bar{h}dy'\right),\langle D_x\rangle^\alpha J_x u\right)\\&=\left(\langle D_x\rangle^\alpha\left((J_x f)\int_y^\infty(\partial_x g)\bar{h}dy'\right),\langle D_x\rangle^\alpha J_x u\right)\\&+\left(\langle D_x\rangle^\alpha\left(f\int_y^\infty(J_x\partial_x g)\bar{h}dy'\right),\langle D_x\rangle^\alpha J_x u\right)\\&-(\langle D_x\rangle^\alpha(f\int_y^\infty(\partial_x g)(\overline{J_x h})dy'),\langle D_x\rangle^\alpha J_x u)\equiv\sum_{j=7}^{9}F_j.\end{aligned}$$

We let

$$G_{20}=\langle D_x\rangle^{\frac{1}{2}}\left((J_x f)\langle D_x\rangle^\epsilon\int_y^\infty(\partial_x g)\bar{h}dy'\right),$$

$$G_{21}=(\langle D_x\rangle^{\frac{1}{2}}J_x f)\langle D_x\rangle^\epsilon\int_y^\infty(\partial_x g)\bar{h}dy',$$

$$G_{22}=(J_x f)\langle D_x\rangle^\alpha\int_y^\infty(\partial_x g)\bar{h}dy',$$

$$G_{23} = \langle D_x \rangle^{\alpha} \Big((J_x f) \int_y^{\infty} (\partial_x g) \bar{h} dy' \Big),$$

$$G_{24} = (\langle D_x \rangle^{\alpha} J_x f) \int_y^{\infty} (\partial_x g) \bar{h} dy'.$$

Then

$$\begin{aligned} F_7 &= (G_{23}, \langle D_x \rangle^{\alpha} J_x u) \\ &= (G_{23} - G_{22} - G_{24}, \langle D_x \rangle^{\alpha} J_x u) + (G_{24}, \langle D_x \rangle^{\alpha} J_x u) \\ &\quad + (G_{22} + G_{21} - G_{20}, \langle D_x \rangle^{\alpha} J_x u) \\ &\quad - (G_{21}, \langle D_x \rangle^{\alpha} J_x u) + (G_{20}, \langle D_x \rangle^{\alpha} J_x u). \end{aligned}$$

By Theorem A.1 (Appendix)

$$\begin{aligned} |F_7| \leq C\Bigg(&\|\langle D_x \rangle^{\alpha} J_x f\| \Big\| \int_y^{\infty} (\partial_x g) \bar{h} dy' \Big\|_{L^{\infty}} \\ &+ \|\langle D_x \rangle^{\frac{1}{2}} J_x f\|_{L_y^2 L_x^{p_1}} \Big\| \langle D_x \rangle^{\epsilon} \int_y^{\infty} (\partial_x g) \bar{h} dy' \Big\|_{L_y^{\infty} L_x^{p_2}} \Bigg) \|\langle D_x \rangle^{\alpha} J_x u\| \\ &+ \Big| \Big((J_x f) \langle D_x \rangle^{\epsilon} \int_y^{\infty} (\partial_x g) \bar{h} dy', \langle D_x \rangle^{\frac{1}{2}+\alpha} J_x u \Big) \Big| \\ \leq\ & \|f\|_{Y_1} \|u\|_{Y_1} (\|(\partial_x g) \bar{h}\|_{L_y^1 L_x^{\infty}} + \|\langle D_x \rangle^{\epsilon} (\partial_x g \cdot \bar{h})\|_{L_y^1 L_x^{p_2}}) \\ &+ \|\langle x \rangle^{\gamma} J_x f\|_{L_y^2 L_x^{p_3}} \|\langle D_x \rangle^{\epsilon} (\partial_x g \cdot \bar{h})\|_{L_y^1 L_x^{p_4}} \|u\|_{Y_2}. \end{aligned} \tag{2.49}$$

where $1/p_1 = 1/2 - \epsilon$, $1/p_2 = \epsilon$, $1/p_3 = 1/2 - \epsilon/2$ and $p_4 = \epsilon/2$.

By Sobolev's inequality

$$\begin{aligned} &\|(\partial_x g) \bar{h}\|_{L_x^{\infty}} + \|\langle D_x \rangle^{\epsilon} ((\partial_x g) \bar{h})\|_{L_x^{p_2}} \\ &\quad \leq C(\|(\partial_x g) \bar{h}\|_{L_x^2}^{1/2} + \|\partial_x ((\partial_x g) \bar{h})\|_{L_x^2}^{1/2}) \|(\partial_x g) \bar{h}\|_{L_x^2}^{1/2} \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} &\|\langle D_x \rangle^{\epsilon} ((\partial_x g) \bar{h})\|_{L_x^{p_4}} \\ &\quad \leq C(\|(\partial_x g) \bar{h}\|_{L_x^2}^{1/2+\epsilon/2} + \|\partial_x ((\partial_x g) \bar{h})\|_{L_x^2}^{1/2+\epsilon/2}) \|(\partial_x g) \bar{h}\|_{L_x^2}^{1/2-\epsilon/2}. \end{aligned} \tag{2.51}$$

We again use Sobolev's inequality to get

$$\begin{aligned}
&\|\partial_x((\partial_x g)\bar{h})\|_{L^2_x} \\
&\quad \le Ct^{-1}(\|(J_x\partial_x g)\bar{h})\|_{L^2_x} + \|(\partial_x g)\overline{J_x h}\|_{L^2_x}) \\
&\quad \le Ct^{-1}(\|\langle x\rangle^{-\gamma} J_x\partial_x g\|_{L^{p_1}_x}\|\langle x\rangle^{\gamma} h\|_{L^{p_2}_x} \\
&\qquad\qquad + \|\langle x\rangle^{-\gamma}\partial_x g\|_{L^{p_2}_x}\|\langle x\rangle^{\gamma} J_x h\|_{L^{p_1}_x}) \\
&\quad \le Ct^{-1}(\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma} J_x\partial_x g\|_{L^2_x}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^{p_3}_x} \\
&\qquad + \|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma}\partial_x g\|_{L^{p_3}_x}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x h\|_{L^2_x}),
\end{aligned} \tag{2.52}$$

where $1/p_1 = 1/2 - \epsilon/2$, $1/p_2 = \epsilon/2$ and $1/p_3 = \epsilon$. Since

$$\|f\|_{L^p_x} \le Ct^{-\frac{1}{2}(1-\frac{2}{p})}\|J_x f\|_{L^2_x}^{\frac{1}{2}(1-\frac{2}{p})}\|f\|_{L^2_x}^{\frac{1}{2}(1+\frac{2}{p})} \quad \text{for} \quad p \ge 2 \tag{2.53}$$

we have by (2.52)

$$\begin{aligned}
&\|\partial_x((\partial_x g)\bar{h})\|_{L^2_x} \\
&\quad \le Ct^{-\frac{3}{2}+\epsilon}(\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma} J_x\partial_x g\|_{L^2_x}\sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^2_x} \\
&\qquad + \sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma}\partial_x g\|_{L^2_x}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x h\|_{L^2_x}).
\end{aligned} \tag{2.54}$$

Similarly,

$$\begin{aligned}
&\|(\partial_x g)\bar{h}\|_{L^2_x} \\
&\quad \le Ct^{-\frac{1}{2}+\epsilon}(\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma}\partial_x g\|_{L^2_x}\sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^2_x} \\
&\qquad + \sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma}\partial_x g\|_{L^2_x}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^2_x}).
\end{aligned} \tag{2.55}$$

We apply (2.54) and (2.55) to (2.50), (2.51) to get

$$\begin{aligned}
&\|(\partial_x g)\bar{h}\|_{L^1_y L^\infty_x} + \|\langle D_x\rangle^{\epsilon}((\partial_x g)\bar{h})\|_{L^1_y L^{p_2}_x} \\
&\quad \le Ct^{-1+\epsilon}(\sum_{\beta|\le 1}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma} J_x^{\beta}\partial_x g\|\sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\| \\
&\qquad + \sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{-\gamma}\partial_x g\|_{L^2_x}\sum_{|\beta|\le 1}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x^{\beta} h\|_{L^2_x}) \\
&\quad \le Ct^{-1+\epsilon}\|g\|_{Y_2}\|h\|_{Y_1} \quad \text{(by Lemma A.1)}
\end{aligned} \tag{2.56}$$

and

$$\|\langle D_x\rangle^{\epsilon}((\partial_x g)\bar{h})\|_{L_x^{p_4}} \le Ct^{-1+\frac{\epsilon}{2}}\|g\|_{Y_2}\|h\|_{Y_1}. \tag{2.57}$$

From (2.49), (2.56) and (2.57) it follows that

$$|F_7| \le Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1}+\|u\|_{Y_2}). \tag{2.58}$$

In the same way as in the proof of (2.58) we have

$$|F_9| \le Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1}+\|u\|_{Y_2}). \tag{2.59}$$

We now consider F_8. We have

$$\begin{aligned}
|F_8| &= \left|\left(\langle D_x\rangle^{\alpha}\left(f\int_y^{\infty} J_x\partial_x g\right)\bar{h}dy'\right), \langle D_x\rangle^{\alpha}J_x u\right)\right| \\
&= \left|\left(\langle x\rangle^{\gamma}\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty} J_x\partial_x g\right)\bar{h}dy'\right), \langle x\rangle^{-\gamma}\langle D_x\rangle^{1+\epsilon}J_x u\right)\right| \\
&\le \left\|\langle x\rangle^{\gamma}\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\|u\|_{Y_2} \\
&\le \left\|\langle D_x\rangle^{\epsilon}\left(\langle x\rangle^{\gamma}f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\|u\|_{Y_2}.
\end{aligned} \tag{2.60}$$

We consider the term

$$\|\langle D_x\rangle^{\epsilon}(\langle x\rangle^{\gamma}f\int_y^{\infty} J_x\partial_x g)\bar{h}dy')\|_{L_x^2}. \tag{2.61}$$

We have

$$\begin{aligned}
&\|\langle D_x\rangle^{\epsilon}(\langle x\rangle^{2\gamma}f\bar{h}\cdot\langle x\rangle^{-\gamma}J_x\partial_x g)\|_{L_x^2} \\
&\quad \le C(\|\langle D_x\rangle^{\epsilon}(\langle x\rangle^{2\gamma}f\bar{h})\|_{L_x^{p_1}}\|\langle x\rangle^{-\gamma}J_x\partial_x g\|_{L_x^{p_2}} \\
&\qquad + \|\langle x\rangle^{2\gamma}f\bar{h}\|_{L_x^{\infty}}\|\langle D_x\rangle^{\epsilon}\langle x\rangle^{-\gamma}J_x\partial_x g\|_{L_x^2}) \quad \text{(by Theorem A.1)} \\
&\quad \le C(\|\langle D_x\rangle(\langle x\rangle^{2\gamma}f\bar{h})\|_{L_x^2}^{1/2}\|\langle x\rangle^{2\gamma}f\bar{h}\|_{L_x^2}^{1/2}\|\langle D_x\rangle^{\epsilon}\langle x\rangle^{-\gamma}J_x\partial_x g\|_{L_x^2}) \\
&\qquad\qquad \text{(by Sobolev)} \\
&\quad \le C(\|\langle x\rangle^{2\gamma}f\bar{h}\|_{L_x^2} + \|\partial_x(\langle x\rangle^{2\gamma}f\bar{h})\|_{L_x^2}^{1/2}\|\langle x\rangle^{2\gamma}f\bar{h}\|_{L_x^2}^{1/2}) \\
&\qquad \times \|\langle D_x\rangle^{\epsilon}\langle x\rangle^{-\gamma}J_x\partial_x g\|_{L_x^2},
\end{aligned} \tag{2.62}$$

where $1/p_1 = \epsilon, 1/p_2 = 1/2 - \epsilon$.

By a direct calculation

$$\|\partial_x(\langle x\rangle^{2\gamma} f\bar{h})\|_{L^2_x} \le Ct^{-1}(\|\langle x\rangle^{2\gamma} J_x f\cdot \bar{h}\|_{L^2_x} + \|\langle x\rangle^{2\gamma} f\cdot \overline{J_x h}\|_{L^2_x})$$

and by Sobolev's inequality and (2.53)

$$\begin{aligned}&\|\langle x\rangle^{\gamma} J_x f\|_{L^{p_1}_x}\|\langle x\rangle^{\gamma} h\|_{L^{p_2}_x}\\ &\le C\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x f\|_{L^2_x}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^{p_3}_x}\\ &\le Ct^{-\frac{1}{2}+\epsilon}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x f\|_{L^2_x}\sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^2_x},\end{aligned}$$

with $1/p_3 = \epsilon$. Hence

$$\begin{aligned}&\|\partial_x(\langle x\rangle^{2\gamma} f\bar{h})\|_{L^2_x}\\ &\le Ct^{-\frac{3}{2}+\epsilon}\sum_{|\beta|\le 1}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x^{\beta} f\|_{L^2_x}\sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^2_x}\end{aligned}\quad (2.63)$$

and by (2.53)

$$\begin{aligned}&\|\langle x\rangle^{2\gamma} f\bar{h}\|_{L^2_x}\\ &\le Ct^{-\frac{1}{2}+\epsilon}\sum_{|\beta|\le 1}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x^{\beta} f\|_{L^2_x}\sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^2_x}\end{aligned}\quad (2.64)$$

From (2.62)-(2.64) it follows that

$$\begin{aligned}&\|\langle D_x\rangle^{\epsilon}(\langle x\rangle^{2\gamma} f\bar{h}\cdot\langle x\rangle^{-\gamma} J_x\partial_x g)\|_{L^2_x}\\ &\le Ct^{-1+\epsilon}\sum_{|\beta|\le 1}\|\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} J_x^{\beta} f\|_{L^2_x}\\ &\quad\times\sum_{|\beta|\le 1}\|J_x^{\beta}\langle D_x\rangle^{\frac{\epsilon}{2}}\langle x\rangle^{\gamma} h\|_{L^2_x}\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{\epsilon}\partial_x J_x g\|_{L^2_x}.\end{aligned}\quad (2.65)$$

Therefore we have by (2.60), (2.65) and Lemma A.1

$$|F_8| \le Ct^{-1+\epsilon}\|f\|_{Y_1}\|h\|_{Y_1}\|g\|_{Y_2}\|u\|_{y_2}. \quad (2.66)$$

From (2.58), (2.59) and (2.66) the result follows. In the same way as in the proof of Lemma 2.6 we have

LEMMA 2.7. – *We let $\alpha = \delta - \frac{1}{2}$, $f, \bar{h} \in Y_1$ and $g \in Y_2$. Then we have*

$$\begin{aligned}&\left|\left(\langle D_x\rangle^{\alpha} J_y\left(f\int_y^{\infty}(\partial_x g)\bar{h}dy'\right), \langle D_x\rangle^{\alpha} J_y u\right)\right|\\ &\le Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1} + \|u\|_{Y_2}).\end{aligned}$$

We next prove

LEMMA 2.8. – *We let* $\alpha = \delta - \frac{1}{2}$, $f, \bar{h} \in Y_1$ *and* $g \in Y_2$. *Then we have*

$$\left| \left(D_y^\alpha J_x \left(f \int_y^\infty (\partial_x g) \bar{h} dy' \right), D_y^\alpha J_x u \right) \right|$$
$$\leq C t^{-1+\frac{\epsilon^2}{2}} \|f\|_{Y_1} \|g\|_{Y_2} \|h\|_{Y_1} (\|u\|_{Y_1} + \|u\|_{Y_2}).$$

Proof. – By a direct calculation

$$\begin{aligned} &\left(D_y^\alpha J_x \left(f \int_y^\infty (\partial_x g) \bar{h} dy' \right), D_y^\alpha J_x u \right) \\ &= \left(D_y^\alpha \left((J_x f) \int_y^\infty (\partial_x g) \bar{h} dy' \right), D_y^\alpha J_x u \right) \\ &\quad + \left(D_y^\alpha \left(f \int_y^\infty (J_x \partial_x g) \bar{h} dy' \right), D_y^\alpha J_x u \right) \\ &\quad - \left(D_y^\alpha \left(f \int_y^\infty (\partial_x g) (\overline{J_x h}) dy' \right), D_y^\alpha J_x u \right) \equiv \sum_{j=10}^{12} F_j. \quad (2.67) \end{aligned}$$

We let

$$G_{25} = D_y^{\frac{1}{2}} \left((J_x f) D_y^\epsilon \int_y^\infty (\partial_x g) \bar{h} dy' \right),$$

$$G_{26} = (D_y^{\frac{1}{2}} J_x f) D_y^\epsilon \int_y^\infty (\partial_x g) \bar{h} dy',$$

$$G_{27} = (J_x f) D_y^\alpha \int_y^\infty (\partial_x g) \bar{h} dy',$$

$$G_{28} = D_y^\alpha \left((J_x f) \int_y^\infty (\partial_x g) \bar{h} dy' \right),$$

$$G_{29} = (D_y^\alpha J_x f) \int_y^\infty (\partial_x g) \bar{h} dy'.$$

Then

$$\begin{aligned} F_{10} &= (G_{28}, D_y^\alpha J_x u) \\ &= (G_{28} - G_{27} - G_{29}, \langle D_x \rangle^\alpha J_x u) + (G_{29}, \langle D_x \rangle^\alpha J_x u) \\ &\quad + (G_{27} + G_{26} - G_{25}, \langle D_x \rangle^\alpha J_x u) \\ &\quad - (G_{26}, \langle D_x \rangle^\alpha J_x u) + (G_{25}, \langle D_x \rangle^\alpha J_x u). \end{aligned}$$

By Theorem A.1 (Appendix)

$$|F_{10}| \le C\Bigg(\|D_y^\alpha J_x f\| \left\| \int_y^\infty (\partial_x g)\bar{h}dy' \right\|_{L^\infty} + \left\| \|D_y^{\frac{1}{2}} J_x f\|_{L_y^{p_1}} \left\| D_y^\epsilon \int_y^\infty (\partial_x g)\bar{h}dy' \right\|_{L_y^{p_2}} \right\|_{L_x^2} \Bigg) \|u\|_{Y_1} + \left| \left((J_x f) D_y^\epsilon \int_y^\infty (\partial_x g)\bar{h}dy', D_y^{\frac{1}{2}+\alpha} J_x u \right) \right|, \tag{2.68}$$

where $1/p_1 = 1/2 - \epsilon, 1/p_2 = \epsilon$. By Sobolev's inequality

$$|F_{10}| \le C\Bigg(\|f\|_{Y_1} \|(\partial_x g)\bar{h}\|_{L_y^1 L_x^\infty} + \left\| \|D_y^\alpha J_x f\|_{L_y^2} \left\| D_y^\epsilon \int_y^\infty (\partial_x g)\bar{h}dy' \right\|_{L_y^{p_2}} \right\|_{L_x^2} \Bigg) \|u\|_{Y_2} + \left\| \|\langle y\rangle^\gamma J_x f\|_{L_y^{p_3}} \left\| D_y^\epsilon \int_y^\infty (\partial_x g)\bar{h}dy' \right\|_{L_y^{p_4}} \right\|_{L_x^2} \Bigg) \times \|\langle y\rangle^{-\gamma} D_y^{\frac{1}{2}+\alpha} J_x u\|, \tag{2.69}$$

where $1/p_3 = 1/2 - \epsilon/2, 1/p_4 = \epsilon/2$. We again use the Sobolev's inequality to see that

$$\begin{cases} \|D_y^\epsilon \int_y^\infty (\partial_x g)\bar{h}dy'\|_{L_y^{p_2}} \le C\|\partial_y \int_y^\infty (\partial_x g)\bar{h}dy'\|_{L_y^1} \le C\|(\partial_x g)\bar{h}\|_{L_y^1} \\ \|D_y^\epsilon \int_y^\infty (\partial_x g)\bar{h}dy'\|_{L_y^{p_4}} \le C\|\partial_y \int_y^\infty (\partial_x g)\bar{h}dy'\|_{L_y^{\frac{2}{2-\epsilon}}} \le C\|(\partial_x g)\bar{h}\|_{L_y^{\frac{2}{2-\epsilon}}} \end{cases} \tag{2.70}$$

By (2.69) and (2.70)

$$\begin{aligned} |F_{10}| &\le C\|f\|_{Y_1} \|(\partial_x g)\bar{h}\|_{L_x^\infty L_y^1} \|u\|_{Y_1} + C\|f\|_{Y_1} \|(\partial_x g)\bar{h}\|_{L_x^\infty L_y^{\frac{2}{2-\epsilon}}} \|u\|_{Y_2} \\ &\le C\|f\|_{Y_1} (\|(\partial_x g)\bar{h}\|_{L_y^1 L_x^\infty} + \|(\partial_x g)\bar{h}\|_{L_y^{\frac{2}{2-\epsilon}} L_x^\infty})(\|u\|_{Y_1} + \|u\|_{Y_2}). \end{aligned} \tag{2.71}$$

In the same way as in the proof of (2.56)

$$\|(\partial_x g)\bar{h}\|_{L_y^1 L_x^\infty} \le Ct^{-1+\epsilon} \|g\|_{Y_2} \|h\|_{Y_1} \tag{2.72}$$

By Hölder's inequality

$$\begin{aligned}
&\| \| |(\partial_x g)\bar{h}\|_{L_x^\infty} \|_{L_y^{\frac{2}{2-\epsilon}}} \\
&\le C \| \| |(\partial_x g)\bar{h}\|_{L_x^2}^{1/2} \|\partial_x((\partial_x g)\bar{h})\|_{L_x^2}^{1/2} \|_{L_y^{\frac{2}{2-\epsilon}}} \\
&\le C \|\partial_x((\partial_x g)\bar{h})\|_{L_y^1 L_x^2}^{1/2} \|(\partial_x g)\bar{h}\|_{L_y^{\frac{1}{1-\epsilon}} L_x^2}^{1/2} \\
&\le C \|\partial_x((\partial_x g)\bar{h})\|_{L_y^1 L_x^2}^{1/2} \|(\partial_x g)\bar{h}\|_{L_y^\infty L_x^2}^{\epsilon/2} \|(\partial_x g)\bar{h}\|_{L_y^1 L_x^2}^{1/2-\epsilon/2}. \qquad (2.73)
\end{aligned}$$

By (2.54) and (2.55) we see that

$$\begin{cases} \|\partial_y((\partial_x g)\bar{h})\|_{L_y^1 L_x^2} \le C t^{-\frac{3}{2}+\epsilon} \|g\|_{Y_2} \|h\|_{Y_1} \\ \|(\partial_x g)\bar{h}\|_{L_y^1 L_x^2} \le C t^{-\frac{1}{2}+\epsilon} \|g\|_{Y_2} \|h\|_{Y_1} \end{cases} \qquad (2.74)$$

and

$$\begin{aligned}
&\|(\partial_x g)\bar{h}\|_{L_y^\infty L_x^2} \\
&\le \|(\partial_x g)\bar{h}\|_{L_x^2 L_y^\infty} \le C \|\partial_y((\partial_x g)\bar{h})\|_{L_x^2 L_y^1} \quad \text{(by Sobolev)} \\
&\le C t^{-1} \| \| |(J_y \partial_x g)\bar{h}\|_{L_y^1} + \|(\partial_x g)\overline{J_y h}\|_{L_y^1} \|_{L_x^2} \\
&\le C t^{-1} (\| \| |\langle x\rangle^{-\gamma} J_y \partial_x g\|_{L_y^2} \|\langle x\rangle^{\gamma} h\|_{L_y^2} \|_{L_x^2} \\
&\quad + \| \| |\langle x\rangle^{-\gamma} \partial_x g\|_{L_y^2} \|\langle x\rangle^{\gamma} J_y h\|_{L_y^2} \|_{L_x^2}) \quad \text{(by Hölder)} \\
&\le C t^{-1} (\| \langle x\rangle^{-\gamma} \partial_x J_y g\| \| \langle x\rangle^{\gamma} h\|_{L_x^\infty L_y^2} \\
&\quad + \|\langle x\rangle^{-\gamma} \partial_x g\|_{L_x^\infty L_y^2} \|\langle x\rangle^{\gamma} J_y h\|) \\
&\le C t^{-1} (\|g\|_{Y_2} \|\langle x\rangle^{\gamma} h\|_{L_y^2 L_x^\infty} + \|h\|_{Y_1} \|\langle x\rangle^{-\gamma} \partial_x g\|_{L_y^2 L_x^\infty}) \\
&\le C t^{-\frac{3}{2}} \|g\|_{Y_2} \|h\|_{Y_1} \quad \text{(by (2.53))}. \qquad (2.75)
\end{aligned}$$

We apply (2.74) and (2.75) to the right hand side of (2.73) to get

$$\|(\partial_x g)\bar{h}\|_{L_y^{\frac{2}{2-\epsilon}} L_x^\infty} \le C t^{-1+\frac{\epsilon^2}{2}} \|g\|_{Y_2} \|h\|_{Y_1}. \qquad (2.76)$$

We use (2.72) and (2.76) in the right hand side of (2.71). Then we have

$$|F_{10}| \le C t^{-1+\frac{\epsilon^2}{2}} \|f\|_{Y_1} \|g\|_{Y_2} \|h\|_{Y_1} (\|u\|_{Y_1} + \|u\|_{Y_2}). \qquad (2.77)$$

We now consider F_{11}. We have by Theorem A.1 and Sobolev's inequality

$$\begin{aligned}
&\left\| D_y^\alpha \left(f \int_y^\infty J_x \partial_x g) \bar{h} dy' \right) \right\| \\
&\leq C \left(\| D_y^\alpha f \|_{L_y^2} \left\| \int_y^\infty (J_x \partial_x) \bar{h} dy' \right\|_{L_y^\infty} \right. \\
&\qquad \left. + \| f \|_{L_y^\infty} \left\| D_y^\alpha \int_y^\infty (J_x \partial_x g) \bar{h} dy' \right\|_{L_y^2} \right) \\
&\leq C \left(\| D_y^\alpha f \|_{L_y^2} \| (J_x \partial_x) \bar{h} \|_{L_y^1} \right. \\
&\qquad \left. + \| f \|_{L_y^\infty} \left\| \partial_y^\alpha \int_y^\infty (J_x \partial_x g) \bar{h} dy' \right\|_{L_y^r} \right), \quad 1/r = 1 - \epsilon \\
&\leq C (\| D_y^\alpha f \|_{L_y^2} \| (J_x \partial_x) \bar{h} \|_{L_y^1} + \| f \|_{L_y^\infty} \| (J_x \partial_x g) \bar{h} dy' \|_{L_y^r}) \\
&\leq C (\| D_y^\alpha f \|_{L_y^2} \| \langle x \rangle^{-\gamma} J_x \partial_x g \|_{L_y^2} \| \langle x \rangle^\gamma h \|_{L_y^2} \\
&\quad + \| f \|_{L_y^\infty} \| \langle x \rangle^{-\gamma} J_x \partial_x g \|_{L_y^{r'}} \| \langle x \rangle^\gamma h \|_{L_y^2}). \qquad (2.78)
\end{aligned}$$

Hence by Sobolev's inequality

$$\begin{aligned}
|F_{11}| &\leq C t^{-\frac{1}{2}} \sum_{|\beta| \leq 1} \| J^\beta \langle D_y \rangle^\alpha f \| \| \langle x \rangle^{-\gamma} \langle D_y \rangle^\epsilon J_x \partial_x g \| \| \langle x \rangle^\gamma h \|_{L_x^\infty L_y^2} \\
&\leq C t^{-\frac{1}{2}} \| f \|_{Y_1} \| g \|_{Y_2} \| \langle x \rangle^\gamma h \|_{L_y^2 L_x^\infty} \qquad (2.79)
\end{aligned}$$

We find that by Sobolev's inequality

$$\begin{aligned}
\| f \|_{L_x^\infty} &\leq C t^{-\frac{1}{p}} \| J_x f \|_{L_x^p}^{1/p} \| f \|_{L_x^p}^{1-1/p} \\
&\leq C t^{-\frac{1}{p}} \| D_x^{\frac{1}{2} - \frac{1}{p}} J_x f \|_{L_x^2}^{1/p} \| D_x^{\frac{1}{2} - \frac{1}{p}} f \|_{L_x^2}^{1-1/p} \quad (p \geq 2) \qquad (2.80)
\end{aligned}$$

From (2.80) with $1/2 - 1/p = \epsilon/2$ we see that

$$\| \langle x \rangle^\gamma h \|_{L_y^2 L_x^\infty} \leq C t^{-\frac{1}{2}(1-\epsilon)} \| h \|_{Y_1}. \qquad (2.81)$$

Applying (2.81) to (2.79), we obtain

$$|F_{11}| \leq C t^{-1+\frac{\epsilon}{2}} \| f \|_{Y_1} \| g \|_{Y_2} \| h \|_{Y_1}. \qquad (2.82)$$

We finally consider F_{12}. In the same way as in the proof of (2.78)

$$\begin{aligned}&\left\|D_y^\alpha\left(f\int_y^\infty(\partial_x g)(\overline{J_x h})dy'\right\|_{L_y^2}\\&\le C(\|D_y^\alpha f\|_{L_y^2}\|\langle x\rangle^{-\gamma}\partial_x g\|_{L_y^2}\|\langle x\rangle^\gamma J_x h\|_{L_y^2}\\&\quad+\|f\|_{L_y^2}\|\langle x\rangle^{-\gamma}\partial_x g\|_{L_y^{2r}}\|\langle x\rangle^\gamma J_x h\|_{L_y^{2r}}),\end{aligned}$$

where $1/r = 1-\epsilon$. Hence we have

$$\begin{aligned}|F_{12}|&\le C\|\langle D_y\rangle^\alpha f\|_{L_x^\infty L_y^2}\|\langle x\rangle^{-\gamma}\langle D_y\rangle^{\epsilon/2}\partial_x g\|_{L_x^\infty L_y^2}\\&\quad\times\|\langle x\rangle^\gamma\langle D_y\rangle^{\epsilon/2}J_x h\|\\&\le Ct^{-1/2}\|f\|_{Y_1}\|h\|_{Y_1}\|\langle x\rangle^{-\gamma}\langle D_y\rangle^{\epsilon/2}\partial_x g\|_{L_y^2 L_x^\infty}.\end{aligned}\tag{2.83}$$

We use (2.80) with $1/2-1/p=\epsilon/2$ in (2.83) to get

$$|F_{12}|\le Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|h\|_{Y_1}\|g\|_{Y_2}.\tag{2.84}$$

From (2.77), (2.82) and (2.84) the lemma follows.

In the same way as in the proof of Lemma 2.8 we have

LEMMA 2.9. – *We let $\alpha=\delta-\frac{1}{2}$, $f,\bar{h}\in Y_1$ and $g\in Y_2$. Then we have*

$$\begin{aligned}&\left|\left(D_y^\alpha J_y\left(f\int_y^\infty(\partial_x g)\bar{h}dy'\right),D_y^\alpha J_y u\right)\right|\\&\le Ct^{-1+\frac{\epsilon^2}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1}+\|u\|_{Y_2}).\end{aligned}$$

LEMMA 2.10. – *We let $f,g,h\in H^{\delta,0}\cap H^{0,\delta}$*

$$\|fhg\|_{H^{\delta,0}}\le C\|f\|_{H^{\delta,0}}\|g\|_{H^{\delta,0}}\|h\|_{H^{\delta,0}},$$

$$\|fhg\|_{H^{0,\delta}}\le C\|f\|_{H^{\delta,0}}\|g\|_{H^{\delta,0}}\|h\|_{H^{0,\delta}}.$$

Proof. – We have the lemma by Sobolev's inequality and Theorem A.1.

LEMMA 2.11. – *We have for $f,h\in X_1$, and $g,u\in X_1\cap\tilde{X}_2$*

$$\begin{aligned}&\left|\left(\tilde{K}_x\langle D_x\rangle^\epsilon\partial_x\left(f\int_y^\infty(\partial_x g)hdy'\right),\tilde{K}_x\langle D_x\rangle^\epsilon\partial_x u\right)\right|\\&\le C\|f\|_{X_1}\|h\|_{X_1}(\|g\|_{X_1}+\|g\|_{\tilde{X}_2})(\|u\|_{X_1}+\|u\|_{\tilde{X}_2}).\end{aligned}$$

Proof. – We have

$$
\begin{aligned}
&\left|\left(\tilde K_x\langle D_x\rangle^\epsilon\partial_x\left(f\int_y^\infty(\partial_x g)h\,dy'\right),\tilde K_x\langle D_x\rangle^\epsilon\partial_x u\right)\right|\\
&\le\left|\left(\tilde K_x\langle D_x\rangle^\epsilon\left((\partial_x f)\int_y^\infty(\partial_x g)h\,dy'\right),\tilde K_x\langle D_x\rangle^\epsilon\partial_x u\right)\right|\\
&+\left|\left(\tilde K_x\langle D_x\rangle^\epsilon\left(f\int_y^\infty(\partial_x g)(\partial_x h)\,dy'\right),\tilde K_x\langle D_x\rangle^\epsilon\partial_x u\right)\right|\\
&+\left|\left(\tilde K_x\langle D_x\rangle^\epsilon\left(f\int_y^\infty(\partial_x^2 g)h\,dy'\right),\tilde K_x\langle D_x\rangle^\epsilon\partial_x u\right)\right| \qquad (2.85)
\end{aligned}
$$

By the definition of the operator $\tilde K_x$ we see that

$$
\|\tilde K_x f\|_{L^p}\le C\|f\|_{L^p},\quad [\tilde K_x,\langle D_x\rangle^\epsilon]=0,\quad [\tilde K_x,\langle y\rangle^\gamma]=0. \qquad (2.86)
$$

We apply (2.86) to (2.85) to get

$$
\begin{aligned}
&\left\|\left(\tilde K_x\langle D_x\rangle^\epsilon\partial_x\left(f\int_y^\infty(\partial_x g)h\,dy'\right),\tilde K_x\langle D_x\rangle^\epsilon\partial_x u\right)\right|\\
&\le C\left(\left\|\langle D_x\rangle^\epsilon\left((\partial_x f)\int_y^\infty(\partial_x g)h\,dy'\right\|\right\|\langle D_x\rangle^\epsilon\partial_x u\|\right.\\
&+\left\|\langle y\rangle^\gamma f\int_y^\infty(\partial_x g)(\partial_x h)\,dy'\right\|_{L_y^2L_x^{p_1}}\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_y^2L_x^{p_2}}\\
&+\left\|\langle y\rangle^\gamma f\int_y^\infty(\partial_x^2 g)h\,dy'\right\|_{L_y^2L_x^{p_1}}\|\langle y\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|_{L_y^2L_x^{p_2}}, \qquad (2.87)
\end{aligned}
$$

where $1/p_2=\epsilon$, $1/p_1=1-\epsilon$. The rest of the proof is obtained in the same way as in the proof of Lemma 2.1 and so we omit it.

LEMMA 2.12. – *We have for $f,h\in X_1$ and $g\in X_1\cap\tilde X_2$*

$$
\left\|\tilde K_y\langle D_y\rangle^\epsilon\partial_y\left(f\int_y^\infty(\partial_x g)h\,dy'\right)\right\|\le C\|f\|_{X_1}(\|g\|_{X_1}+\|g\|_{\tilde X_2})\|h\|_{X_1}.
$$

Proof. – The lemma is obtained by Sobolev's inequality and (2.86).

LEMMA 2.13. – *We have for $f,h\in X_1$ and $g,u\in X_1\cap X_2$*

$$
\begin{aligned}
&\left|\left(K_x\langle D_x\rangle^\epsilon\partial_x\left(f\int_y^\infty(\partial_x g)h\,dy'\right),K_x\langle D_x\rangle^\epsilon\partial_x u\right)\right|\\
&\le C\|f\|_{X_1}\|h\|_{X_1}(\|g\|_{X_1}+\|g\|_{X_2})(\|u\|_{X_1}+\|u\|_{X_2}).
\end{aligned}
$$

Proof. – We have

$$\begin{aligned}(K_x\langle D_x\rangle^\epsilon A, K_x\langle D_x\rangle^\epsilon B)\\ = ([K_x, \langle D_x\rangle^\epsilon]A, K_x\langle D_x\rangle^\epsilon B)\\ + (K_x A, [\langle D_x\rangle^\epsilon, K_x]\langle D_x\rangle^\epsilon B) + (K_x A, K_x\langle D_x\rangle^{2\epsilon} B)\end{aligned} \tag{2.88}$$

and

$$\begin{aligned}(K_x A, K_x\langle D_x\rangle^{2\epsilon} B)\\ = (K_x\langle x\rangle^{-\gamma}\langle x\rangle^\gamma A, K_x\langle x\rangle^\gamma\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon} B)\\ = (\langle x\rangle^\gamma K_x^* K_x\langle x\rangle^{-\gamma}\langle x\rangle^\gamma A, \langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon} B),\end{aligned} \tag{2.89}$$

where K_x^* is the dual operator of K_x. By Lemma A.2

$$\begin{cases} \|\langle x\rangle^\gamma K_x^* K_x\langle x\rangle^{-\gamma} f\|_{L_x^p} \le C\|f\|_{L_x^p}, \\ \|[\langle D_x\rangle^\epsilon, K_x]f\|_{L_x^2} \le C\|f\|_{L_x^2}. \end{cases} \tag{2.90}$$

From (2.88)-(2.90) it follows that

$$\begin{aligned}|(K_x\langle D_x\rangle^\epsilon A, K_x\langle D_x\rangle^\epsilon B)|\\ \le C(\|A\|\|\langle D_x\rangle^\epsilon B\| + \|\langle x\rangle^\gamma A\|_{L_y^2 L_x^{p_1}}\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon} B\|_{L_y^2 L_x^{p_2}})\\ \le C(\|A\|\|\langle D_x\rangle^\epsilon B\|\\ + \|\langle x\rangle^\gamma A\|_{L_y^2 L_x^{p_1}}\|\langle D_x\rangle^{\frac{1}{2}-\epsilon}\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon} B\|),\end{aligned} \tag{2.91}$$

where $1/p_1 = 1 - \epsilon$, $1/p_2 = \epsilon$. We use (2.91) with

$$A = f\int_y^\infty (\partial_x g)(\partial_x h)dy' \quad \text{or} \quad f\int_y^\infty (\partial_x^2 g)h dy'$$

and $B = \partial_x u$. Then we have

$$\begin{aligned}&\left|\left(K_x\langle D_x\rangle^\epsilon\partial_x\left(f\int_y^\infty(\partial_x g)hdy'\right), K_x\langle D_x\rangle^\epsilon\partial_x u\right)\right|\\ &\le C\Bigg(\Bigg(\left\|\langle D_x\rangle^\epsilon\left((\partial_x f)\int_y^\infty(\partial_x g)hdy'\right)\right\|\\ &\quad+\left\|f\int_y^\infty(\partial_x g)(\partial_x h)dy'\right\| + \left\|f\int_y^\infty(\partial_x^2 g)hdy'\right\|\Bigg)\|\langle D_x\rangle^\epsilon\partial_x u\|\\ &\quad+\Bigg(\left\|\langle x\rangle^\gamma f\int_y^\infty(\partial_x g)(\partial_x h)dy'\right\|_{L_y^2 L_x^{p_1}}\\ &\quad+\left\|\langle x\rangle^\gamma f\int_y^\infty(\partial_x^2 g)hdy'\right\|_{L_y^2 L_x^{p_1}}\Bigg)\|\langle D_x\rangle^{\frac{1}{2}-\epsilon}\langle x\rangle^{-\gamma}\langle D_x\rangle^{2\epsilon}\partial_x u\|\Bigg).\end{aligned} \tag{2.92}$$

In the same way as in the proof of Lemma 2.1 we have the lemma from (2.92).

LEMMA 2.14. – *We have for $f, h \in X_1$, and $g \in X_1 \cap X_2$*

$$\left\|K_y \langle D_y\rangle^\epsilon \partial_y \left(f \int_y^\infty (\partial_x g) h dy'\right)\right\| \leq C\|f\|_{X_1}(\|g\|_{X_1} + \|g\|_{X_2})\|h\|_{X_1}.$$

Proof. – The lemma follows from Lemma 2.5 and the fact that $\|K_y f\| \leq C\|f\|$.

LEMMA 2.15. – *We let $f, \bar{h} \in Y_1$, $g \in Y_2$ and $\alpha = \delta - \frac{1}{2}$. Then we have*

$$\left|\left(K_x\langle D_x\rangle^\alpha J^\beta \left(f \int_y^\infty (\partial_x g)\bar{h} dy'\right), K_x \langle D_x\rangle^\alpha J^\beta u\right)\right|$$
$$\leq Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1} + \|u\|_{Y_2}).$$

Proof. – We have

$$\begin{aligned}
&\left(K_x\langle D_x\rangle^\alpha J_x \left(f \int_y^\infty (\partial_x g)\bar{h} dy'\right), K_x\langle D_x\rangle^\alpha J_x u\right) \\
&= \left(K_x\langle D_x\rangle^\alpha \left((J_x f) \int_y^\infty (\partial_x g)\bar{h} dy'\right), K_x\langle D_x\rangle^\alpha J_x u\right) \\
&+ \left(K_x\langle D_x\rangle^\alpha \left(f \int_y^\infty (J_x \partial_x g)\bar{h} dy'\right), K_x\langle D_x\rangle^\alpha J_x u\right) \\
&- \left(K_x\langle D_x\rangle^\alpha \left(f \int_y^\infty (\partial_x g)(\overline{J_x h}) dy'\right), K_x\langle D_x\rangle^\alpha J_x u\right) \\
&\equiv \sum_{j=7}^{9} F'_j. \qquad (2.93)
\end{aligned}$$

In the same way as in the proofs of (2.58) and (2.59)

$$|F'_7| + |F'_9| \leq Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1} + \|u\|_{Y_2}). \qquad (2.94)$$

We now consider F'_8. We have

$$|F'_8| \leq \left|\left(K_x\langle D_x\rangle^\alpha \left(f \int_y^\infty (J_x \partial_x g)\bar{h} dy'\right), K_x\langle D_x\rangle^\alpha J_x u\right)\right| \qquad (2.95)$$

Since

$$K_x\langle D_x\rangle^\alpha = [K_x, \langle D_x\rangle^{\frac{1}{2}}]\langle D_x\rangle^\epsilon + \langle D_x\rangle^{\frac{1}{2}} K_x \langle D_x\rangle^\epsilon$$

and

$$\|[K_x, \langle D_x\rangle^{\frac{1}{2}}]f\| \le C\|f\| \quad \text{(by Lemma A.2)} \tag{2.96}$$

we see that the right hand side of (2.95) is bounded from above by

$$\begin{aligned}
&C\left\|\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\|u\|_{Y_2}\\
&\quad+\left|\left(K_x\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right), \langle D_x\rangle^{\frac{1}{2}}K_x\langle D_x\rangle^{\alpha}J_x u\right)\right|.\\
&= C\left\|\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\|u\|_{Y_2}\\
&\quad+\left|\left(K_x\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right),\right.\right.\\
&\qquad\left.\left.[\langle D_x\rangle^{\frac{1}{2}}, K_x]\langle D_x\rangle^{\frac{1}{2}}J_x u + K_x\langle D_x\rangle^{1+\epsilon}J_x u\right)\right|\\
&\le C\left(\left\|\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\|u\|_{Y_2}\right.\\
&\quad+\left\|\langle x\rangle^{\gamma}K_x\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\\
&\quad\left.\times\left(\|\langle x\rangle^{-\gamma}K_x\langle D_x\rangle^{1+\epsilon}J_x u)\| + \|\langle D_x\rangle^{\frac{1}{2}}J_x u\|\right)\right),
\end{aligned} \tag{2.97}$$

where we have used (2.96). By Lemma A.2 we see that the second term of the right hand side of the last inequality of (2.97) is estimated by

$$\begin{aligned}
&\left\|\langle x\rangle^{\gamma}\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\\
&\qquad\times\left(\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{1+\epsilon}J_x u)\| + \|u\|_{Y_2}\right)\\
&\quad\le\left\|\langle x\rangle^{\gamma}\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\|u\|_{Y_2}.
\end{aligned} \tag{2.98}$$

Hence (2.95), (2.97) and (2.98) give

$$|F_8'| \le \left\|\langle x\rangle^{\gamma}\langle D_x\rangle^{\epsilon}\left(f\int_y^{\infty}(J_x\partial_x g)\bar{h}dy'\right)\right\|\|u\|_{Y_2}.$$

In the same way as in the proof of (2.66) we have by this inequality

$$|F_8'| \le Ct^{-1+\epsilon}\|f\|_{Y_1}\|h\|_{Y_1}\|g\|_{Y_2}\|u\|_{Y_2}. \tag{2.99}$$

From (2.94) and (2.99) it follows that

$$\left|\left(K_x\langle D_x\rangle^\alpha J_x\left(f\int_y^\infty(\partial_x g)\bar{h}dy'\right), K_x\langle D_x\rangle^\alpha J_x u\right)\right| \le Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|h\|_{Y_1}\|g\|_{Y_2}\|u\|_{Y_2}. \tag{2.100}$$

In the same way as in the proof of (2.100) we obtain

$$\left|\left(K_x\langle D_x\rangle^\alpha J_y\left(f\int_y^\infty(\partial_x g)\bar{h}dy'\right), K_x\langle D_x\rangle^\alpha J_y u\right)\right| \le Ct^{-1+\frac{\epsilon}{2}}\|f\|_{Y_1}\|h\|_{Y_1}\|g\|_{Y_2}\|u\|_{Y_2}. \tag{2.101}$$

The lemma follows from (2.100) and (2.101).

LEMMA 2.16. – *We let $\alpha = \delta - \frac{1}{2}$, $f, \bar{h} \in Y_1$, and $g \in Y_2$. Then we have*

$$\left|\left(K_x\langle D_y\rangle^\alpha J^\beta\left(f\int_y^\infty(\partial_x g)\bar{h}dy'\right), K_x\langle D_y\rangle^\alpha J^\beta u\right)\right| \le Ct^{-1+\frac{\epsilon^2}{2}}\|f\|_{Y_1}\|g\|_{Y_2}\|h\|_{Y_1}(\|u\|_{Y_1}+\|u\|_{Y_2}).$$

Proof. – The lemma follows from Lemma 2.8, Lemma 2.9 and the facts that K_x commutes with $\langle D_y\rangle^\alpha$ and bouded operator from L_x^2 to itself.

3. LINEAR SCHRÖDINGER EQUATIONS

In this section we consider the inhomogeneous Schrödinger equations

$$\begin{cases} i\partial_t u + Hu = f, & (x,y,t) \in \mathbf{R}^3, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{3.1}$$

where $H = \partial_x^2 + \partial_y^2$ or $\partial_x\partial_y$. Our purpose is to prove Lemma 3.6.

We first prove

LEMMA 3.1. – *Let u be the solution of (3.1) with $\partial_x\partial_y$. Then we have*

$$\begin{aligned} &\|u(t)\|^2 + \int_0^t \|\langle x\rangle^{-\gamma}\langle D_y\rangle^{1/2}u(\tau)\|^2 + \|\langle y\rangle^{-\gamma}\langle D_x\rangle^{1/2}u(\tau)\|^2 d\tau \\ &\le C\Bigg(\|u_0\|^2 + \int_0^t \|\langle x\rangle^{-\gamma}\langle D_y\rangle^{-1/2}u(\tau)\|^2 + \|\langle y\rangle^{-\gamma}\langle D_x\rangle^{-1/2}u(\tau)\|^2 d\tau \\ &\quad + \sum_{j=x,y}\int_0^t |(\tilde{K}_j f(\tau), \tilde{K}_j u(\tau))| d\tau\Bigg). \end{aligned}$$

Proof. – Applying the operator $\tilde{K}_x$ to both sides of (3.1), we obtain

$$i\partial_t \tilde{K}_x u + \partial_x \partial_y \tilde{K}_x + i\frac{1}{\langle y\rangle^{2\gamma}}\frac{D_x^2}{\langle D_x\rangle}\tilde{K}_x u = \tilde{K}_x f. \tag{3.2}$$

We multiply both sides of (3.2) by $\overline{\tilde{K}_x u}$, integrate over $\mathbf{R}^2$ and take the imaginary part to get

$$\frac{d}{dt}\|\tilde{K}_x u\|^2 + 2\left\|\langle y\rangle^{-\gamma}\tilde{K}_x\frac{D_x}{\langle D_x\rangle^{1/2}}u\right\|^2 = 2Im(\tilde{K}_x f, \tilde{K}_x u). \tag{3.3}$$

Since

$$\left\|\langle y\rangle^{-\gamma}\tilde{K}_x\frac{D_x}{\langle D_x\rangle^{1/2}}u\right\|^2 = \|\langle y\rangle^{-\gamma}\tilde{K}_x\langle D_x\rangle^{1/2}u\|^2 - \|\langle y\rangle^{-\gamma}\tilde{K}_x\langle D_x\rangle^{-1/2}u\|^2 \cdot$$

we have by (3.3)

$$\begin{aligned}\|\tilde{K}_x u(t)\|^2 &+ 2\int_0^t \|\tilde{K}_x\langle y\rangle^{-\gamma}\langle D_x\rangle^{1/2}u(\tau)\|^2 d\tau \\ &= \|\tilde{K}_x u_0\|^2 + 2Im\int_0^t (\tilde{K}_x f(\tau), \tilde{K}_x u(\tau))d\tau \\ &\quad + 2\int_0^t \|\tilde{K}_x\langle y\rangle^{-\gamma}\langle D_x\rangle^{-1/2}u(\tau)\|^2 d\tau.\end{aligned} \tag{3.4}$$

From (3.4) and the fact that there exist positive constants C_1, C_2 such that

$$C_1\|f\| \le \|\tilde{K}_j f\| \le C_2\|f\|, \quad (j = x, y),$$

it follows that

$$\begin{aligned}\|u(t)\|^2 &+ 2C_1\int_0^t \|\langle y\rangle^{-\gamma}\langle D_x\rangle^{1/2}u(\tau)\|^2 d\tau \\ &\le C_2(\|u_0\|^2 + 2\int_0^t \|\langle y\rangle^{-\gamma}\langle D_x\rangle^{-1/2}u(\tau)\|^2 d\tau \\ &\quad + 2\int_0^t |(\tilde{K}_x f(\tau), \tilde{K}_x u(\tau))| d\tau).\end{aligned} \tag{3.5}$$

In the same way as in the proof of (3.5) we have

$$\begin{aligned}\|u(t)\|^2 &+ 2C_1\int_0^t \|\langle x\rangle^{-\gamma}\langle D_y\rangle^{1/2}u(\tau)\|^2 d\tau \\ &\le C_2(\|u_0\|^2 + 2\int_0^t \|\langle x\rangle^{-\gamma}\langle D_y\rangle^{-1/2}u(\tau)\|^2 d\tau \\ &\quad + 2\int_0^t |(\tilde{K}_y f(\tau), \tilde{K}_y u(\tau))| d\tau).\end{aligned} \tag{3.6}$$

The lemma follows from (3.5) and (3.6) immediately.

LEMMA 3.2. – *Let u be the solution of (3.1) with $H = \partial_x^2 + \partial_y^2$. Then we have*

$$\|u(t)\|^2 + \sum_{j=x,y} \int_0^t \|\langle j\rangle^{-\gamma} \langle D_j\rangle^{1/2} u(\tau)\|^2 d\tau$$
$$\leq C\Big(\|u_0\|^2 + \sum_{j=x,y} \int_0^t \|\langle j\rangle^{-\gamma} u(\tau)\| d\tau + \sum_{j=x,y} \int_0^t |(K_j f(\tau), K_j u(\tau))| d\tau \Big).$$

Proof. – Applying the operator K_x to both sides of (3.1), we obtain

$$i\partial_t K_x u + \Delta K_x u + i[i\partial_x^2, K_x]u = K_x f, \tag{3.7}$$

where $\Delta = \partial_x^2 + \partial_y^2$. We multiply both sides of (3.7) by $\overline{K_x u}$, integrate over $\mathbf{R}^2$ and take the imaginary part to get

$$\frac{d}{dt}\|K_x u\|^2 = -2Re([i\partial_x^2, K_x]u, K_x u) + 2Im(K_x f, K_x u). \tag{3.8}$$

A direct caluculation gives

$$[i\partial_x^2, K_x] = QK_x + K_x R + \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-2} [P^{m-m_1-1}, Q] P^{m_1} + \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=1}^{m-1} P^{m-m_1-1} [R, P^{m_1}] \tag{3.9}$$

with

$$P = \Big(\int_0^x \langle \tau\rangle^{-2\gamma} d\tau \Big) \frac{D_x}{\langle D_x\rangle}, \quad Q = \langle x\rangle^{-2\gamma} \frac{D_x^2}{\langle D_x\rangle},$$
$$R = -2i\gamma \langle x\rangle^{-2\gamma-2} x \frac{D_x}{\langle D_x\rangle}.$$

By Lemma A.3 and Lemma A.4 we find that there exist positive constants C_1 and C_2 such that

$$\frac{d}{dt}\|K_x u\|^2 + C_1 \|\langle x\rangle^{-\gamma} \langle D_x\rangle^{1/2} u\|^2 \leq C_2 \|\langle x\rangle^{-\gamma} u\|^2 + 2Im(K_x f, K_x u). \tag{3.10}$$

Integrating in time variable t and using (1.5), we get

$$M_2^{-1}\|u\|^2 + C_1 \int_0^t \|\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2} u(\tau)\|^2 d\tau$$
$$\leq M_2\|u_0\|^2 + C_2 \int_0^t \|\langle x\rangle^{-\gamma} u(\tau)\| d\tau$$
$$+ 2\int_0^t |(K_x f(\tau), K_x u(\tau))| d\tau. \tag{3.11}$$

In the same way as in the proof of (3.11)

$$\|u\|^2 + \int_0^t \|\langle y\rangle^{-\gamma}\langle D_y\rangle^{1/2} u(\tau)\|^2 d\tau$$
$$\leq C\biggl(\|u_0\|^2$$
$$+ \int_0^t \|\langle y\rangle^{-\gamma} u(\tau)\| d\tau + \int_0^t |(K_y f(\tau), K_y u(\tau))| d\tau \biggr). \tag{3.12}$$

The lemma follows from (3.11) and (3.12).

LEMMA 3.3. – *Let u be the solution of (3.1) with $\partial_x\partial_y$. Then we have*

$$\|u(t)\|^2_{H^{0,\delta}}$$
$$\leq C\biggl(\|u_0\|^2_{H^{0,\delta}} + \int_0^t (\|\langle x\rangle^{\delta-1}\partial_y u(\tau)\| + \|\langle y\rangle^{\delta-1}\partial_x u(\tau)\|$$
$$+ \|f(\tau)\|_{H^{0,\delta}})\|u(\tau)\|_{H^{0,\delta}} d\tau \biggr) \quad \textit{for} \quad \delta \in \mathbf{R}.$$

Proof. – By (3.1) we have

$$i\partial_t\langle x\rangle^\delta u + \partial_x\partial_y\langle x\rangle^\delta u - \delta\langle x\rangle^{\delta-2} x\partial_y u = \langle x\rangle^\delta f. \tag{3.13}$$

Multiplying both sides of (3.13) by $\langle x\rangle^\delta \bar{u}$, integrating over $\mathbf{R}^2$ and taking the imaginary part, we get

$$\frac{d}{dt}\|\langle x\rangle^\delta u\|^2 - 2Im\delta(\langle x\rangle^{\delta-2} x\partial_y u, \langle x\rangle^\delta u) = 2Im(\langle x\rangle^\delta f, \langle x\rangle^\delta u)$$

from which it follows that

$$\|\langle x\rangle^\delta u(t)\|^2$$
$$\leq \|\langle x\rangle^\delta u_0\|^2$$
$$+ C\int_0^t (\|\langle x\rangle^{\delta-1}\partial_y u(\tau)\| + \|\langle x\rangle^\delta f(\tau)\|)\|\langle x\rangle^\delta u(\tau)\| d\tau. \tag{3.14}$$

In the same way as in the proof of (3.14) we have

$$\begin{aligned}
&\|\langle y\rangle^{\delta}u(t)\|^2\\
&\quad\le \|\langle y\rangle^{\delta}u_0\|^2\\
&\qquad + C\int_0^t (\|\langle y\rangle^{\delta-1}\partial_x u(\tau)\| + \|\langle y\rangle^{\delta}f(\tau)\|)\|\langle y\rangle^{\delta}u(\tau)\|d\tau. \quad (3.15)
\end{aligned}$$

From (3.14) and (3.15) the lemma follows.

In the same way as in the proof of Lemma 3.3 we have

LEMMA 3.4. – *Let u be the solution of (3.1) with $H = \partial_x^2 + \partial_y^2$. Then we have*

$$\begin{aligned}
&\|u(t)\|^2_{H^{0,\delta}}\\
&\le C\Big(\|u_0\|^2_{H^{0,\delta}} + \int_0^t (\|\langle x\rangle^{\delta-1}\partial_x u(\tau)\|\\
&\qquad + \|\langle y\rangle^{\delta-1}\partial_y u(\tau)\| + \|f(\tau)\|_{H^{0,\delta}})\|u(\tau)\|_{H^{0,\delta}}d\tau\Big) \quad \textit{for} \quad \delta\in\mathbf{R}.
\end{aligned}$$

To prove Theorem 2, we need

LEMMA 3.5. – *Let u be the solution of (3.1) with $H = \partial_x^2 + \partial_y^2$. Then we have for $1/2 < \beta < 1$*

$$\begin{aligned}
&\|u(t)\|^2_{H^{0,\beta}}\\
&\le C\Big(\|u_0\|^2_{H^{0,\beta}} + \int_0^t (\|u(\tau)\|^2_{H^{\beta,0}} + \|u(\tau)\|^2_{H^{0,\beta}}\\
&\qquad + \|f(\tau)\|_{H^{0,\beta}}\|u(\tau)\|_{H^{0,\beta}}\Big)d\tau
\end{aligned}$$

Proof. – By (3.1) we have

$$i\partial_t\langle x\rangle^{\beta}u + \Delta\langle x\rangle^{\beta}u - \beta\langle x\rangle^{\beta-2}x\partial_x u = \langle x\rangle^{\beta}f \quad (3.16)$$

Multiplying both sides of (3.16) by $\langle x\rangle^{\beta}\bar{u}$, integrating over $\mathbf{R}^2$ and taking the imaginary part, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\langle x\rangle^{\beta}u\| - 2Im\beta(\langle x\rangle^{\beta-2}x\partial_x u, \langle x\rangle^{\beta}u)\\
&\quad = 2Im(\langle x\rangle^{\beta}f, \langle x\rangle^{\beta}u). \quad (3.17)
\end{aligned}$$

We have

$$\begin{aligned}(\langle x\rangle^{\beta-2}x\partial_x u, \langle x\rangle^{\beta}u) &= (\langle x\rangle^{\beta-\frac{3}{2}}x\partial_x u, \langle x\rangle^{\beta-\frac{1}{2}}u)\\ &= (\partial_x(\langle x\rangle^{\beta-\frac{3}{2}}xu), \langle x\rangle^{\beta-\frac{1}{2}}u)\\ &\quad - ((\partial_x\langle x\rangle^{\beta-\frac{3}{2}}x)u, \langle x\rangle^{\beta-\frac{1}{2}}u)\\ &= \left(\frac{\partial_x}{\langle D_x\rangle^{\frac{1}{2}}}(\langle x\rangle^{\beta-\frac{3}{2}}xu), \langle D_x\rangle^{\frac{1}{2}}\langle x\rangle^{\beta-\frac{1}{2}}u\right)\\ &\quad - ((\partial_x\langle x\rangle^{\beta-\frac{3}{2}}x)u, \langle x\rangle^{\beta-\frac{1}{2}}u)\end{aligned}$$

from which we see that

$$\begin{aligned}&|(\langle x\rangle^{\beta-2}x\partial_x u, \langle x\rangle^{\beta}u)|\\ &\quad \le C(\|\langle D_x\rangle^{\frac{1}{2}}(\langle x\rangle^{\beta-\frac{3}{2}}xu)\|\|\langle D_x\rangle^{\frac{1}{2}}\langle x\rangle^{\beta-\frac{1}{2}}u\|\\ &\qquad + \|\langle x\rangle^{\beta-\frac{3}{2}}u\|\|\langle x\rangle^{\beta-\frac{1}{2}}u\|)\\ &\quad \le C(\|\langle x\rangle^{\beta}u\|^2 + \|\langle D_x\rangle^{\beta}u\|^2) \quad \text{(by Lemma A.1).}\end{aligned} \tag{3.18}$$

By (3.17) and (3.18)

$$\begin{aligned}&\|\langle x\rangle^{\beta}u(t)\|^2\\ &\le C\bigg(\|u_0\|^2_{H^{0,\beta}} + \int_0^t (\|u(\tau)\|^2_{H^{\beta,0}} + \|u(\tau)\|^2_{H^{0,\beta}}\\ &\qquad + \|f(\tau)\|_{H^{0,\beta}}\|u(\tau)\|_{H^{0,\beta}}\bigg)d\tau.\end{aligned}$$

Similarly, we find that $\|\langle y\rangle^{\beta}u\|^2$ is bounded from above by the right hand side of (3.19). This completes the proof of the lemma.

From Lemma 3.1-Lemma 3.5 we have

LEMMA 3.6. – *Let u be the solution of (3.1). Then we have for $\delta > 1$*

$$\begin{aligned}&\|u\|^2_{\tilde{X}(T)}\\ &\le C\bigg(\|u_0\|^2_{X_1} + \int_0^t \|u(\tau)\|^2_{X_1}d\tau\\ &\qquad + \sum_{j=x,y}\int_0^t |(\tilde{K}_j\langle D_j\rangle^{\delta}f(\tau), \tilde{K}_j\langle D_j\rangle^{\delta}u(\tau))|\\ &\qquad + |(\langle D_j\rangle^{\delta}f(\tau), \langle D_j\rangle^{\delta}u(\tau))|\\ &\qquad + \|f(\tau)\|_{H^{0,\delta}}\|u(\tau)\|_{X_1}d\tau\bigg)\\ &\quad \textit{for}\quad t\in[0,T],\quad H = \partial_x\partial_y,\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&\|u\|^2_{X(T)} \\
&\quad \le C\Big(\|u_0\|^2_{X_1} + \int_0^t \|u(\tau)\|^2_{X_1} d\tau \\
&\qquad + \sum_{j=x,y} \int_0^t |(K_j \langle D_j\rangle^\delta f(\tau), K_j \langle D_j \rangle^\delta u(\tau))| \\
&\qquad + |(\langle D_j\rangle^\delta f(\tau), \langle D_j\rangle^\delta u(\tau))| \\
&\qquad + \|f(\tau)\|_{H^{0,\delta}} \|u(\tau)\|_{X_1} d\tau \Big) \\
&\textit{for} \quad t \in [0,T], \quad H = \partial_x^2 + \partial_y^2,
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
&\|u\|_{Y(T)} \\
&\le C\Big(\|\langle D\rangle^{\delta - \frac{1}{2}} \langle \bar{x} \rangle u_0\|^2 + \int_0^t \|u(\tau)\|_{Y_1} d\tau \\
&\quad + \sum_{\substack{j=x,y \\ |\alpha| \le 1}} \int_0^t |(K_j \langle D_j\rangle^{\delta-\frac{1}{2}} J^\alpha f(\tau), K_j \langle D_j\rangle^{\delta - \frac{1}{2}} J^\alpha u(\tau))| \\
&\quad + |(\langle D_j\rangle^{\delta - \frac{1}{2}} J^\alpha f(\tau), \langle D_j\rangle^{\delta-\frac{1}{2}} J^\alpha u(\tau))| \\
&\quad + \|J^\alpha f(\tau)\|_{H^{0,\delta-\frac{1}{2}}} \|u(\tau)\|_{Y_1} d\tau.
\end{aligned} \tag{3.21}
$$

4. PROOF OF THEOREMS

In this section we prove Theorems. To prove Theorems we consider the linearized equation of (1.4) :

$$
\begin{cases} i\partial_t u + Hu = K(v), & (x,y,t) \in \mathbf{R}^3, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{4.1}
$$

where

$$
\begin{aligned}
K(v) =& d_1 |v|^2 v + d_2 v \int_y^\infty \partial_x |v|^2 dy' d_3 v \int_x^\infty \partial_y |v|^2 dx' \\
&+ d_4 v \partial_x \varphi_1 + d_5 v \partial_x \varphi_2.
\end{aligned}
$$

We define the closed balls $\tilde{X}_\rho(T)$, $X_\rho(T)$ and $Y_\rho(T)$ as follows.

$$
\tilde{X}_\rho(T) = \{ f \in \tilde{X}(T); \|f\|_{\tilde{X}(T)} \le \rho \},
$$

$$X_\rho(T) = \{f \in X(T); \|f\|_{X(T)} \le \rho\}$$

and

$$Y_\rho(T) = \{f \in Y(T); \|f\|_{Y(T)} \le \rho\},$$

where ρ is a sufficiently small positive constant. We first prove Theorem 1 with $H = \partial_x\partial_y$. We assume that $v \in \tilde{X}_\rho(T)$ and difine the mappting M by $u = Mv$. It is sufficient to prove that M is a contraction mapping from $\tilde{X}_\rho(T)$ into itself for some time $T > 0$. We apply Lemmas 2.1-2.3, Lemma 2.10, Lemma 2.11 and Lemma 2.12 to Lemma 3.6 (3.19) with $f = K(v)$. Then we have

$$\begin{aligned}\|u\|^2_{\tilde{X}(T)} \le & C(\|u_0\|_{X_1} + \|u\|^2_{\tilde{X}(T)}T \\ & + (\sup_{t\in[0,T]} \|v(t)\|_{X_1})^2 \|v\|_{\tilde{X}(T)}\|u\|_{\tilde{X}(T)}),\end{aligned} \tag{4.2}$$

from which it follows that

$$(1 - CT)\|u\|^2_{\tilde{X}(T)} \le C(\|u_0\|^2_{X_1} + \rho^3\|u\|_{\tilde{X}(T)}).$$

By Schwarz' inequality

$$\left(\frac{1}{2} - CT\right)\|u\|^2_{\tilde{X}(T)} \le C(\|u_0\|^2_{X_1} + \rho^6)$$

which implies

$$\|u\|^2_{\tilde{X}(T)} \le \frac{2C}{1 - 2CT}(\|u_0\|^2_{X_1} + \rho^6). \tag{4.3}$$

Then by (4.3) we see that there exist positive constants T and ρ such that

$$\|u\|_{\tilde{X}(T)} \le \rho. \tag{4.4}$$

We let u_j ($j = 1, 2$) be the solutions of

$$\begin{cases} i\partial_t u_j + \partial_x\partial_y u_j = H(v_j), \\ u_j(x, y, 0) = u_0(x, y), \end{cases} \tag{4.5}$$

where $v_j \in \tilde{X}_\rho$. In the same way as in the proof of (4.4) we see that there exist positive constants ρ and T such that

$$\|u_1 - u_2\|_{\tilde{X}(T)} \le \frac{1}{2}\|v_1 - v_2\|_{\tilde{X}(T)}. \tag{4.6}$$

It is clear that (4.4) and (4.6) yield Theorem 1 with $H = \partial_x\partial_y$. In the same way as in the proof of Theorem 1 with $H = \partial_x\partial_y$, Theorem 1 with $\partial_x^2 + \partial_y^2$ is obtained by applying Lemmas 2.3-2.5, Lemma 2.10, Lemmas 2.13-2.14 to Lemma 3.6 (3.20) and Theorem 2 is obtained by applying Lemmas 2.6-2.10, Lemmas 2.15-2.16 to Lemma 3.6 (3.21).

5. APPENDIX

First we state inequalities concerning fractional derivatives which are needed to handle the nonlinear terms with fractional derivatives.

THEOREM A.1 [18]. – *Let* $0 < \alpha < 1$ *and* $1 < p < \infty$. *Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p} \leq C\|g\|_{L_x^\infty}\|D_x^\alpha f\|_{L_x^p}.$$

Let $p, p_1, p_2 \in (1,\infty)$ *be such that* $1/p = 1/p_1 + 1/p_2$. *Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p} \leq C\|g\|_{L_x^{p_1}}\|D_x^\alpha f\|_{L_x^{p_2}}.$$

For the proof of Theorem A.1, see Appendix of [18].

LEMMA A.1. – *We have for* $0 \leq \alpha \leq 1$, $0 \leq a < 1 - 2\epsilon$

$$\|[\langle D_x\rangle^\alpha, f]g\|_{L_x^2}, \quad \left\|\left[\frac{D_x}{\langle D_x\rangle}, f\right]g\right\|_{L_x^2} \leq C\begin{cases} \|\langle D_x\rangle^{2\epsilon}D_x^{3/2-\epsilon}f\|_{L_x^2}\|g\|_{L_x^2}, \\ (\|\langle D_x\rangle^{2\epsilon}D_x^{3/2-\epsilon}(\langle x\rangle^a f)\|_{L_x^2} \\ \quad +\|D_x^{3/2-\epsilon}\langle x\rangle^a\|_{L_x^2}\|f\|_{L_x^\infty})\|\langle x\rangle^{-a}g\|_{L_x^2}. \end{cases} \tag{a.1}$$

We have for $0 < 2^{m+1}\alpha < 1, m \in \mathbf{N}\cup 0$

$$\|\langle D_x\rangle^\alpha\langle x\rangle^{(2^{m+1}-1)\alpha}f\|_{L_x^2} \leq C(\|\langle x\rangle^{2^{m+1}\alpha}f\|_{L_x^2} + \|\langle D_x\rangle^{2^{m+1}\alpha}f\|_{L_x^2}. \tag{a.2}$$

Proof. – We have

$$[\langle D_x\rangle^\alpha, f]g = \begin{cases} \langle D_x\rangle^\alpha(fg) - f\langle D_x\rangle^\alpha g, \\ \langle D_x\rangle^\alpha(\langle x\rangle^a f\langle x\rangle^{-a}g) \\ \quad -\langle x\rangle^a f\langle D_x\rangle^\alpha\langle x\rangle^{-a}g - f[\langle x\rangle^a, \langle D_x\rangle^\alpha]\langle x\rangle^{-a}g. \end{cases} \tag{a.3}$$

By the Planchelel theorem and the Schwarz inequality

$$\begin{aligned}
&\|[\langle D_x\rangle^\alpha, f_1]g_1\|_{L_x^2} \\
&= \left(\int_{\mathbf{R}_{\xi_x}}\left|\int_{\mathbf{R}_{\eta_x}}(\langle\xi_x\rangle^\alpha - \langle\eta_x\rangle^\alpha)\hat{f}_1(\xi_x - \eta_x)\hat{g}_1(\eta_x)d\eta_x\right|^2 d\xi_x\right)^{1/2} \\
&\leq C\left(\int_{\mathbf{R}_{\xi_x}}\left|\int_{\mathbf{R}_{\eta_x}}|\xi_x - \eta_x||\hat{f}_1(\xi_x - \eta_x)||\hat{g}_1(\eta_x)|d\eta_x\right|^2 d\xi_x\right)^{1/2} \\
&\leq C\|\xi_x\hat{f}_1\|_{L_{\xi_x}^1}\|\hat{g}_1\|_{L_{\xi_x}^2} \\
&\leq C\|\langle\xi_x\rangle^{2\epsilon}\xi_x^{3/2-\epsilon}\hat{f}_1\|_{L_{\xi_x}^2}\|\hat{g}_1\|_{L_{\xi_x}^2} \\
&\leq C\|\langle D_x\rangle^{2\epsilon}D_x^{3/2-\epsilon}f_1\|_{L_x^2}\|g_1\|_{L_x^2}.
\end{aligned} \tag{a.4}$$

The first inequality of (a.1) is obtained by putting $f_1 = f$ and $g_1 = g$ in (a.4). By (a.3)

$$\begin{aligned} &\|[\langle D_x\rangle^\alpha, f]g\|_{L^2_x} \\ &\le \|[\langle D_x\rangle^\alpha, \langle x\rangle^a f]\langle x\rangle^{-a} g\|_{L^2_x} + \|f[\langle D_x\rangle^\alpha, \langle x\rangle^a]\langle x\rangle^{-a} g\|_{L^2_x}. \quad (a.5)\end{aligned}$$

We put $f_1 = \langle x\rangle^a f$ and $g_1 = \langle x\rangle^{-a} g$ in (a.4) and apply it to the first term of the right hand side of (a.5), and we put $f_1 = \langle x\rangle^a$ and $g_1 = \langle x\rangle^{-a} g$ in (a.4) and apply it to the second term of the right hand side of (a.5) to find that $\|[\langle D_x\rangle^\alpha, f]g\|_{L^2_x}$ is bounded from above by the right hand side of (a.1). Similarly, it can be shown that $\|[\frac{D_x}{\langle D_x\rangle}, f]g\|_{L^2_x}$ is estimated from above by the right hand side of (a.1). Thus we have (a.1). We next prove (a.2). We have by (a.1) and the Schwarz inequality

$$\begin{aligned} &\|\langle D_x\rangle^\alpha \langle x\rangle^{(2^{m+1}-1)\alpha} f\|^2_{L^2_x} \\ &\le C\|\langle x\rangle^{2^{m+1}\alpha} f\|^2_{L^2_x} + \frac{1}{2}\|\langle D_x\rangle^{2\alpha}\langle x\rangle^{(2^{m+1}-2)\alpha} f\|^2_{L^2_x} \\ &\le C(m)\|\langle x\rangle^{2^{m+1}\alpha} f\|^2_{L^2_x} \\ &\quad + \frac{1}{2^m}\|\langle D_x\rangle^{2^{m+1}\alpha}\langle x\rangle^{2^{m+1}-2-2(1+2+\cdots+2^{m-1})} f\|_{L^2_x} \\ &= C(m)\|\langle x\rangle^{2^{m+1}\alpha} f\|^2_{L^2_x} + \frac{1}{2^m}\|\langle D_x\rangle^{2^{m+1}\alpha} f\|_{L^2_x}.\end{aligned}$$

This completes the proof of Lemma A.1.

LEMMA A.2. – *We have for* $2 \le p < \infty$, $1/2 < \gamma < 1$ *and* $0 < \alpha < 1$

$$\|\langle x\rangle^\gamma K_x^* K_x \langle x\rangle^{-\gamma} f\|_{L^p_x} \le C\|f\|_{L^p_x}, \quad (a.6)$$

$$\|[\langle D_x\rangle^\alpha, K_x] f\|_{L^2_x} \le C\|f\|_{L^2_x}. \quad (a.7)$$

Proof. – By a direct calculation we have

$$P^m\langle x\rangle^\gamma g = \sum_{m_1=0}^{m-1} P^{m-m_1-1} A P^{m_1} g + \langle x\rangle^\gamma P^m g, \quad (a.8)$$

where

$$P = \left(\int_0^x \langle\tau\rangle^{-2\gamma} d\tau\right)\frac{D_x}{\langle D_x\rangle}, \quad \text{and} \quad A = [P, \langle x\rangle^\gamma].$$

In the same way as in the proof of (a.8)

$$P^{*l}P^m\langle x\rangle^\gamma g = P^{*l}\Big(\sum_{m_1=0}^{m-1} P^{m-m_1-1}AP^{m_1}g\Big) + \sum_{l_1=0}^{l-1} P^{*l-l_1-1}A^*P^{l_1}P^m g + \langle x\rangle^\gamma P^{*l}P^m g, \quad (a.9)$$

where

$$P^* = \frac{D_x}{\langle D_x\rangle}\Big(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\Big), \quad \text{and} \quad A^* = [P^*, \langle x\rangle^\gamma].$$

Hence by (a.8) and (a.9)

$$\begin{aligned} K_x^*K_x g &= \sum_{l=0}^{\infty}\frac{1}{l!}P^{*l}\Big(\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{m_1=0}^{m-1}P^{m-m_1-1}AP^{m_1}g\Big) \\ &\quad + \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{l_1=0}^{l-1}P^{*l-l_1-1}A^*P^{l_1}\sum_{m=0}^{\infty}\frac{1}{m!}P^m g + \langle x\rangle^\gamma K_x^*K_x g. \\ &= K_x^*\Big(\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{m_1=0}^{m-1}P^{m-m_1-1}AP^{m_1}g\Big) \\ &\quad + \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{l_1=0}^{l-1}P^{*l-l_1-1}A^*P^{l_1}K_x g + \langle x\rangle^\gamma K_x^*K_x g. \qquad (a.10) \end{aligned}$$

Therefore we obtain by (1.5) and (1.6)

$$\begin{aligned} &\|[K_x^*K_x, \langle x\rangle^\gamma]g\|_{L_x^p} \\ &\le M_p\Big\|\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{m_1=0}^{m-1}P^{m-m_1-1}AP^{m_1}g\Big\|_{L_x^p} \\ &\quad + \Big\|\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{l_1=0}^{l-1}P^{*l-l_1-1}A^*P^{l_1}K_x g\Big\|_{L_x^p} \\ &\le M_p\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{m_1=0}^{m-1}\Big(C_p\int_0^\infty\langle\tau\rangle^{-2\gamma}d\tau\Big)^{m-m_1-1}\|AP^{m_1}g\|_{L_x^p} \\ &+ \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{l_1=0}^{l-1}\Big(C_p\int_0^\infty\langle\tau\rangle^{-2\gamma}d\tau\Big)^{l-l_1-1}\|A^*P^{l_1}K_x g\|_{L_x^p}. \qquad (a.11) \end{aligned}$$

By the way we easily see that

$$\|Ah\|_{L^p_x} = \left\| \int_0^x \langle\tau\rangle^{-2\gamma} d\tau \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{L^p_x}$$
$$\leq \left(\int_0^\infty \langle\tau\rangle^{-2\gamma} d\tau \right) \left\| \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{L^p_x}. \qquad (a.12)$$

Applying the Sobolev's inequality, we obtain

$$\left\| \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{L^p_x}$$
$$\leq C \left\| \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{H^1_x}$$
$$\leq C \left(\left\| \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{L^2_x} + \left\| D_x \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{L^2_x} \right)$$
$$\leq C \left(\left\| \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{L^2_x} \right.$$
$$\left. + \left\| [\langle D_x\rangle, \langle x\rangle^\gamma] h - \left[\frac{1}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h - \gamma \langle x\rangle^{\gamma-2} x \frac{D_x}{\langle D_x\rangle} h \right\|_{L^2_x} \right). \qquad (a.13)$$

We use Lemma A.1 in the right hand side of (a.13) to get

$$\left\| \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] h \right\|_{L^p_x} \leq C \|h\|_{L^2_x}, \qquad (a.14)$$

since

$$\|\langle D_x\rangle^{2\epsilon} D_x^{3/2-\epsilon} \langle x\rangle^\gamma\|_{L^2_x} < \infty.$$

Hence by (a.12)

$$\|Ah\|_{L^p_x} \leq C\|h\|_{L^2_x}. \qquad (a.15)$$

From the definition of A^* such that

$$A^* g = \left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^\gamma \right] \left(\int_0^x \langle\tau\rangle^{-2\gamma} d\tau \right) g$$

and (a.14) it follows that

$$\|A^* h\|_{L^p_x} \leq C\|h\|_{L^2_x}. \qquad (a.16)$$

We apply (a.15), (a.16) and (1.5)-(1.6) to (a.11) to get

$$\begin{aligned}
&\|[K_x^* K_x, \langle x\rangle^\gamma]g\|_{L_x^p} \\
&\le C\Big(\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{m_1=0}^{m-1}\Big(C\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau\Big)^{m-1}\|g\|_{L_x^2} \\
&\qquad + \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{l_1=0}^{l-1}\Big(C\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau\Big)^{l-1}\|K_x g\|_{L_x^2}\Big) \\
&\le C\|g\|_{L_x^2}. \qquad (a.17)
\end{aligned}$$

We put $g = \langle x\rangle^{-\gamma} f$ in (a.17), and use (1.5)-(1.6) then it follows that

$$\begin{aligned}
\|\langle x\rangle^\gamma K_x^* K_x \langle x\rangle^{-\gamma} f\|_{L_x^p} &\le \|K_x^* K_x f\|_{L_x^p} + C\|\langle x\rangle^{-\gamma} f\|_{L_x^2} \\
&\le C(\|f\|_{L_x^p} + \|\langle x\rangle^{-\gamma} f\|_{L_x^2}) \le C\|f\|_{L_x^p}
\end{aligned}$$

which implies (a.6). We next prove (a.7).

We have

$$[K_x, \langle D_x\rangle^\alpha]f = \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{m_1=0}^{m-1} P^{m-m_1-1} B P^{m_1} f$$

with

$$B = [P, \langle D_x\rangle^\alpha] = \Big[\int_0^x \langle\tau\rangle^{-2\gamma}d\tau, \langle D_x\rangle^\alpha\Big]\frac{D_x}{\langle D_x\rangle}.$$

By Lemma A.1 we find that

$$\|[K_x, \langle D_x\rangle^\alpha]f\|_{L_x^2} \le C\sum_{m=1}^{\infty}\frac{1}{m!}m\Big(\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau\Big)^{m-1}\|f\|_{L_x^2} \le C\|f\|_{L_x^2}.$$

This shows (a.7).

LEMMA A.3. – *We have for* $1/2 < \gamma < 1$

$$\Big\|\langle x\rangle^\gamma \sum_{m=2}^{\infty}\frac{1}{m!}\sum_{m_1=0}^{m-2}[P^{m-m_1-1}, Q]P^{m_1} f\Big\|_{L_x^2} \le C\|\langle x\rangle^{-\gamma} f\|_{L_x^2}, \qquad (a.18)$$

$$\Big\|\langle x\rangle^\gamma \sum_{m=2}^{\infty}\frac{1}{m!}\sum_{m_1=1}^{m-1}P^{m-m_1-1}[R, P^{m_1}]f\Big\|_{L_x^2} \le C\|\langle x\rangle^{-\gamma} f\|_{L_x^2}, \qquad (a.19)$$

where

$$Q = \langle x \rangle^{-2\gamma} \frac{D_x^2}{\langle D_x \rangle}, \quad R = -2i\gamma \langle x \rangle^{-2\gamma-2} x \frac{D_x}{\langle D_x \rangle}.$$

Proof. – Since

$$[P^l, Q] = \sum_{l_1=0}^{l-1} P^{l-l_1-1}[P, Q] P^{l_1}, \quad [P^l, R] = \sum_{l_1=0}^{l-1} P^{l-l_1-1}[P, R] P^{l_1},$$

we have

$$\begin{aligned} &\langle x \rangle^{\gamma} \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-2} [P^{m-m_1-1}, Q] P^{m_1} \\ &= \langle x \rangle^{\gamma} \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-2} \left(\sum_{l_1=0}^{m-m_1-2} P^{m-m_1-2-l_1} [P, Q] P^{l_1} \right) \\ &= \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-2} \left(\sum_{l_1=0}^{m-m_1-2} P^{m-m_1-2-l_1} \langle x \rangle^{\gamma} [P, Q] P^{l_1} \right) \\ &\quad + \sum_{m=3}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-3} \\ &\quad \times \left(\sum_{l_1=0}^{m-m_1-3} \sum_{n_1=0}^{m-m_1-3-l_1} P^{m-m_1-3-l_1-n_1} A P^{n_1} [P, Q] P^{l_1} \right) \end{aligned} \qquad (a.20)$$

and

$$\begin{aligned} &\langle x \rangle^{\gamma} \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=1}^{m-1} P^{m-m_1-1} [R, P^{m_1}] \\ &= -\langle x \rangle^{\gamma} \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=1}^{m-1} \left(\sum_{l_1=0}^{m_1-1} P^{m_1-1-l_1} [P, R] P^{l_1} \right) \\ &= -\sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=1}^{m-1} \left(\sum_{l_1=0}^{m_1-1} P^{m_1-1-l_1} \langle x \rangle^{\gamma} [P, R] P^{l_1} \right) \\ &- \sum_{m=3}^{\infty} \frac{1}{m!} \sum_{m_1=2}^{m-1} \left(\sum_{l_1=0}^{m_1-2} \sum_{n_1=0}^{m_1-2-l_1} P^{m_1-2-l_1-n_1} A P^{n_1} [P, R] P^{l_1} \right) \end{aligned} \qquad (a.21)$$

with $A = [P, \langle x\rangle^\gamma]$. Taking L_x^2 norm in both sides of (a.20) and (a.21), we obtain

$$\begin{aligned}
&\left\| \langle x\rangle^\gamma \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-2} [P^{m-m_1-1}, Q] P^{m_1} f \right\|_{L_x^2} \\
&\leq \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-2} \\
&\quad \times \left(\sum_{l_1=0}^{m-m_1-2} \int_0^\infty \langle \tau\rangle^{-2\gamma} d\tau \right)^{m-m_1-2-l_1} \|\langle x\rangle^\gamma [P, Q] P^{l_1} f\|_{L_x^2} \\
&\quad + C \sum_{m=3}^{\infty} \frac{1}{m!} \sum_{m_1=0}^{m-3} \sum_{l_1=0}^{m-m_1-3} \sum_{n_1=0}^{m-m_1-3-l_1} \\
&\quad \times \left(\int_0^\infty \langle \tau\rangle^{-2\gamma} d\tau \right)^{m-m_1-3-l_1} \|[P, Q] P^{l_1} f\|_{L_x^2} \qquad (a.22)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \langle x\rangle^\gamma \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=1}^{m-1} P^{m-m_1-1} [R, P^{m_1}] f \right\|_{L_x^2} \\
&\leq \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{m_1=1}^{m-1} \sum_{l_1=0}^{m_1-1} \\
&\quad \times \left(\int_0^\infty \langle \tau\rangle^{-2\gamma} d\tau \right)^{m_1-1-l_1} \|\langle x\rangle^\gamma [P, R] P^{l_1} f\|_{L_x^2} \\
&\quad + C \sum_{m=3}^{\infty} \frac{1}{m!} \sum_{m_1=2}^{m-1} \sum_{l_1=0}^{m_1-2} \sum_{n_1=0}^{m_1-2-l_1} \\
&\quad \times \left(\int_0^\infty \langle \tau\rangle^{-2\gamma} d\tau \right)^{m_1-2-l_1} \|[P, R] P^{l_1} f\|_{L_x^2}. \qquad (a.23)
\end{aligned}$$

We have by a simple calculation

$$\begin{aligned}
\langle x\rangle^{\gamma}[P,Q] &= \langle x\rangle^{\gamma}\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle}\langle x\rangle^{-2\gamma}\frac{D_x^2}{\langle D_x\rangle} \\
&\quad - \langle x\rangle^{-\gamma}\frac{D_x^2}{\langle D_x\rangle}\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle} \\
&= \langle x\rangle^{\gamma}\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x^2}{\langle D_x\rangle}\langle x\rangle^{-2\gamma}\frac{D_x}{\langle D_x\rangle} \\
&\quad + 2\gamma\langle x\rangle^{\gamma}\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle}\langle x\rangle^{-2\gamma-2}x\frac{D_x}{\langle D_x\rangle} \\
&\quad - \langle x\rangle^{-\gamma}\frac{D_x^2}{\langle D_x\rangle}\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle} \\
&= \left[\langle x\rangle^{\gamma}\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right), \frac{D_x^2}{\langle D_x\rangle}\right]\langle x\rangle^{-2\gamma}\frac{D_x}{\langle D_x\rangle} \\
&\quad + 2\gamma\left[\langle x\rangle^{\gamma}\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right), \frac{D_x}{\langle D_x\rangle}\right]\langle x\rangle^{-2\gamma-2}x\frac{D_x}{\langle D_x\rangle} \\
&\quad + 2\gamma\frac{D_x}{\langle D_x\rangle}\langle x\rangle^{-2\gamma-2}x\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle} \\
&\quad + \left[\frac{D_x^2}{\langle D_x\rangle}, \langle x\rangle^{-\gamma}\right]\left(\int_0^x \langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle}.
\end{aligned}$$

Hence by Lemma A.1

$$\begin{aligned}
&\|\langle x\rangle^{\gamma}[P,Q]P^{l_1}f\|_{L_x^2} \\
&\le C\|\langle x\rangle^{-\gamma}P^{l_1}f\|_{L_x^2} \\
&\le C\left(\left\|\langle x\rangle^{-2\gamma}\frac{D_x}{\langle D_x\rangle}P^{l_1}f\right\|_{L_x^2} + \left\|\langle x\rangle^{-\gamma}\frac{D_x}{\langle D_x\rangle}P^{l_1}f\right\|_{L_x^2}\right) \\
&\le C\|\langle x\rangle^{-\gamma}P^{l_1}f\|_{L_x^2}. \qquad (a.24)
\end{aligned}$$

From the equality

$$\begin{aligned}
&\langle x\rangle^{-\gamma}P^{l_1}\langle x\rangle^{\gamma} \\
&= \sum_{k=0}^{l_1-1} P^{l_1-k-1}g\left(\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau\right)\left[\frac{D_x}{\langle D_x\rangle}, \langle x\rangle^{\gamma}\right]P^k g + P^{l_1}g \qquad (a.25)
\end{aligned}$$

and (a.25) with $g = \langle x\rangle^{-\gamma}f$ it follows that

$$\|\langle x\rangle^{\gamma}[P,Q]P^{l_1}f\|_{L_x^2}$$

$$
\begin{aligned}
&\leq C\Big(\sum_{k=0}^{l_1-1}\Big(\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau\Big)^{l_1}\|\langle x\rangle^{-\gamma}f\|_{L_x^2}\\
&\qquad+\Big(\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau\Big)^{l_1}\|\langle x\rangle^{-\gamma}f\|_{L_x^2}\Big)\\
&= C(l_1+1)\Big(\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau\Big)^{l_1}\|\langle x\rangle^{-\gamma}f\|_{L_x^2}. \qquad (a.26)
\end{aligned}
$$

In the same way as in the proof of (a.26) we find that $\|[P,Q]P^{l_1}f\|_{L_x^2}$ is bounded from above by the right hand side of (a.26). Hence by applying (a.26) to (a.22), we obtain

$$
\begin{aligned}
&\Big\|\langle x\rangle^\gamma\sum_{m=2}^\infty\frac{1}{m!}\sum_{m_1=0}^{m-2}[P^{m-m_1-1},Q]P^{m_1}f\Big\|_{L_x^2}\\
&\leq C\sum_{m=2}^\infty\frac{1}{m!}\sum_{m_1=0}^{m-2}\Big(\sum_{l_1=0}^{m-m_1-2}\\
&\quad\times\Big(\int_0^\infty\langle\tau\rangle^{-2\gamma}d\tau\Big)^{m-m_1-2}(l_1+1)\|\langle x\rangle^{-\gamma}f\|_{L_x^2}\Big)\\
&\quad+C\sum_{m=3}^\infty\frac{1}{m!}\sum_{m_1=0}^{m-3}\Big(\sum_{l_1=0}^{m-m_1-3}\sum_{n_1=0}^{m-m_1-3-l_1}\\
&\quad\times\Big(\int_0^\infty\langle\tau\rangle^{-2\gamma}d\tau\Big)^{m-m_1-3}(l_1+1)\|\langle x\rangle^{-\gamma}f\|_{L_x^2}\Big)\\
&\leq C\|\langle x\rangle^{-\gamma}f\|_{L_x^2}.
\end{aligned}
$$

This shows (a.18). We next prove (a.19). Since

$$
\begin{aligned}
\langle x\rangle^\gamma[P,R] &= \langle x\rangle^\gamma\Big(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\Big)\frac{D_x}{\langle D_x\rangle}2\gamma\langle x\rangle^{-2\gamma-2}x\frac{D_x}{\langle D_x\rangle}\\
&\quad-2\gamma\langle x\rangle^{-\gamma-2}x\frac{D_x}{\langle D_x\rangle}\Big(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\Big)\frac{D_x}{\langle D_x\rangle}\\
&= -\Big[\frac{D_x}{\langle D_x\rangle},\langle x\rangle^\gamma\Big(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\Big)\Big]2\gamma\langle x\rangle^{-2\gamma-2}x\frac{D_x}{\langle D_x\rangle}\\
&\quad+\frac{D_x}{\langle D_x\rangle}2\gamma\langle x\rangle^{-\gamma-2}x\Big(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\Big)\frac{D_x}{\langle D_x\rangle}\\
&\quad-2\gamma\langle x\rangle^{-\gamma-2}x\frac{D_x}{\langle D_x\rangle}\Big(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\Big)\frac{D_x}{\langle D_x\rangle}.
\end{aligned}
$$

We have by Lemma A.1

$$\|\langle x\rangle^{\gamma}[P,R]g\|_{L^2_x} \le C\|\langle x\rangle^{-\gamma}g\|_{L^2_x} \tag{a.27}$$

In the same way as in the proof of (a.26) we obtain by (a.27)

$$\begin{aligned}&\|\langle x\rangle^{\gamma}[P,R]P^{l_1}f\|_{L^2_x} + \|[P,R]P^{l_1}f\|_{L^2_x}\\ &\le C(\int_0^\infty \langle\tau\rangle^{-2\gamma}d\tau)^{l_1}(l_1+1)\|\langle x\rangle^{-\gamma}f\|_{L^2_x}.\end{aligned} \tag{a.28}$$

We apply (a.28) to (a.23) to get

$$\begin{aligned}&\left\|\langle x\rangle^{\gamma}\sum_{m=2}^{\infty}\frac{1}{m!}\sum_{m_1=1}^{m-1}P^{m-m_1-1}[R,P^{m_1}]f\right\|_{L^2_x}\\ &\le \sum_{m=2}^{\infty}\frac{1}{m!}\sum_{m_1=1}^{m-1}\left(\sum_{l_1=0}^{m_1-1}\left(\int_0^\infty\langle\tau\rangle^{-2\gamma}d\tau\right)^{m_1-1}(l_1+1)\|\langle x\rangle^{-\gamma}f\|_{L^2_x}\right)\\ &\quad + C\sum_{m=3}^{\infty}\frac{1}{m!}\sum_{m_1=2}^{m-1}\\ &\quad\times\left(\sum_{l_1=0}^{m_1-2}\sum_{n_1=0}^{m_1-2-l_1}\left(\int_0^\infty\langle\tau\rangle^{-2\gamma}d\tau\right)^{m_1-2}(l_1+1)\|\langle x\rangle^{-\gamma}f\|_{L^2_x}\right).\\ &\le C\|\langle x\rangle^{-\gamma}f\|_{L^2_x}.\end{aligned} \tag{a.29}$$

This completes the proof of Lemma A.3.

LEMMA A.4. – *We assume that $\gamma > -1$. Then there exist positive constants C_1 and C_2 such that*

$$\begin{aligned}&\left(\langle x\rangle^{-\gamma}\frac{-D_x^2}{\langle D_x\rangle}K_xf,\langle x\rangle^{-\gamma}K_xf\right)_{L^2_x}\\ &\ge C_1\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}f\|^2_{L^2_x} - C_2\|\langle x\rangle^{-\gamma}f\|^2_{L^2_x}.\end{aligned}$$

Proof. – It is easy to see that

$$\begin{aligned}&\left(\langle x\rangle^{-\gamma}\frac{-D_x^2}{\langle D_x\rangle}K_xf,\langle x\rangle^{-\gamma}K_xf\right)_{L^2_x}\\ &= (\langle x\rangle^{-\gamma}\langle D_x\rangle K_xf,\langle x\rangle^{-\gamma}K_xf)_{L^2_x} - \|\langle x\rangle^{-\gamma}K_xf\|_{L^2_x}\\ &= ([\langle x\rangle^{-\gamma},\langle D_x\rangle^{1/2}]\langle D_x\rangle^{1/2}K_xf,\langle x\rangle^{-\gamma}K_xf)_{L^2_x}\\ &\quad + (\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}K_xf,[\langle D_x\rangle^{1/2},\langle x\rangle^{-\gamma}]K_xf)_{L^2_x}\\ &\quad + \|\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}f\|^2_{L^2_x} - \|\langle x\rangle^{-\gamma}f\|^2_{L^2_x}\end{aligned}$$

We use the Schwarz inequality in the right hand side of the above equality to get

$$\left(\langle x\rangle^{-\gamma}\frac{-D_x^2}{\langle D_x\rangle}K_xf,\langle x\rangle^{-\gamma}K_xf\right)_{L_x^2}$$
$$\geq\frac{1}{2}\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}K_xf\|_{L_x^2}^2-C\|\langle x\rangle^{-\gamma}K_xf\|_{L_x^2}^2 \qquad (a.30)$$

We next prove that

$$\|[K_x,\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}]f\|_{L_x^2}+\|[K_x,\langle x\rangle^{-\gamma}]f\|_{L_x^2}\leq C\|\langle x\rangle^{-\gamma}f\|_{L_x^2}. \quad (a.31)$$

We have

$$[K_x,\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}]f=\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{m_1}^{m-1}P^{m-m_1-1}A_1P^{m_1}f, \qquad (a.32)$$

where

$$\begin{aligned}A_1&=[P,\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}]\\&=\left(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle}\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}\\&\quad-\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}\left(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle}\\&=\left(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\right)\frac{D_x}{\langle D_x\rangle}[\langle x\rangle^{-\gamma},\langle D_x\rangle^{1/2}]\\&\quad+\left(\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\right)\left[\frac{D_x}{\langle D_x\rangle^{1/2}},\langle x\rangle^{-\gamma}\right]\\&\quad-\langle x\rangle^{-\gamma}\left[\langle D_x\rangle^{1/2},\int_0^x\langle\tau\rangle^{-2\gamma}d\tau\right]\frac{D_x}{\langle D_x\rangle}.\end{aligned}$$

By Lemma A.1

$$\|A_1g\|_{L_x^2}\leq C\|\langle x\rangle^{-\gamma}f\|_{L_x^2}. \qquad (a.33)$$

We apply (a.33) to (a.32). Then we obain

$$\|[K_x,\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}]f\|_{L_x^2}$$
$$\leq\sum_{m=1}^{\infty}\sum_{m_1=0}^{m-1}\left(\int_0^{\infty}\langle\tau\rangle^{-2\gamma}d\tau\right)^{m-m_1-1}\|\langle x\rangle^{-\gamma}P^{m_1}f\|_{L_x^2}. \quad (a.34)$$

We again use Lemma A.1 and (a.25) in the right hand side of (a.34) to see that

$$\begin{aligned} &\|[K_x, \langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}]f\|_{L_x^2} \\ &\leq \sum_{m=1}^{\infty}\sum_{m_1=0}^{m-1}\left(\int_0^{\infty}\langle\tau\rangle^{-2\gamma}d\tau\right)^{m-1}(m_1+1)\|\langle x\rangle^{-\gamma}f\|_{L_x^2} \\ &\leq CM_2\|\langle x\rangle^{-\gamma}f\|_{L_x^2}. \end{aligned} \tag{a.35}$$

In the same way as in the proof of (a.35)

$$\|[K_x, \langle x\rangle^{-\gamma}]f\|_{L_x^2} \leq CM_2\|\langle x\rangle^{-\gamma}f\|_{L_x^2}. \tag{a.36}$$

From (a.35) and (a.36) the inequality (a.31) follows. By (a.30), (a.31) and (1.5)-(1.6)

$$\begin{aligned} &\left(\langle x\rangle^{-\gamma}\frac{-D_x^2}{\langle D_x\rangle}K_x f, \langle x\rangle^{-\gamma}K_x f\right)_{L_x^2} \\ &\geq \frac{1}{2}\|K_x\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}f\|_{L_x^2}^2 \\ &\quad - C\|K_x\langle x\rangle^{-\gamma}f\|_{L_x^2}^2 - CM_2\|\langle x\rangle^{-\gamma}f\|_{L_x^2}^2 \\ &\geq \frac{1}{2M_2}\|\langle x\rangle^{-\gamma}\langle D_x\rangle^{1/2}f\|_{L_x^2}^2 - CM_2\|\langle x\rangle^{-\gamma}f\|_{L_x^2}^2. \end{aligned} \tag{a.37}$$

This completes the proof of the lemma.

ACKNOWLEDGMENT

The author would like to thank the referee for useful comments.

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(Manuscript received on September 20th, 1995;
Revised version received on February 26th, 1996.)