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Scattering for a dispersive charge transfer model

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ABSTRACT. – We prove the asymptotic completeness of wave operators associated with the scattering of a quantum particle in a field of classical particles in the dispersive case when the free hamiltonian has the form $H_0 = p(D)$ with p elliptic of degree $m \geq 1$ and satisfying some convexity hypotheses.

RÉSUMÉ. – L'article présente une preuve de la complétude asymptotique des opérateurs d'onde associés à la diffusion d'une particule quantique soumise au champ de particules classiques dans le cas dispersif où l'hamiltonien libre est de la forme $H_0 = p(D)$ avec p elliptique de degré $m \geq 1$ et satisfaisant certaines hypothèses de convexité.

1. PRELIMINARIES

1.1. Introduction

This paper is motivated by the spectral and scattering theory of many-body quantum hamiltonians $H = H_0 + V(x)$ and their recent developments. In the classical case $H_0 = p(D)$ with p being a positive quadratic form we refer to I. M. Sigal-A. Soffer [15], [16] and J. Derezinski [5] (*cf.* also [11], [20], [1]). In the case of a more general kinetic energy satisfactory results are known only in the 2-body case ([13], [7], [14], [2], [3], [1], [21]) and in the N-body case with $N \geq 3$ many questions remain open (*cf.* [9], [4], [1]). Following C. Gérard [9] we use the name of "dispersive systems" and note the particular importance of the relativistic model with $p(\xi) = (m^2 + |\xi|^2)^{1/2}$.

The aim of this paper is to consider the 3-body problem in a simplified version called the charge transfer model. The mathematical simplification of the model allows to develop a geometrical approach based on Deift-Simon wave operators similarly as in the paper of G. M. Graf [11] (*cf.* also [19], [20], [21]) and to give a rigorous mathematical proof of the asymptotic completeness in the framework of dispersive systems. We have tried to give a detailed and self-contained presentation comprehensible without any knowledge of other results of the scattering theory.

Concerning the charge transfer model itself we note that it originates from a description of motions of a light particle (e.g. an electron) in collisions with some heavy particles (e.g. some ions). It is assumed that only the light particle follows the quantum-mechanical laws, while the heavy particles follow some classical trajectories $\mathbb{R} \ni t \mapsto \chi_k(t) \in \mathbb{R}^d$ and are called classical particles. If V_k denotes the quantum interaction potential between the quantum particle and the classical particle number k, then the total quantum time-dependent interaction $V(t)$ is the operator of multiplication by

(1.1) $$V(t, x) = \sum_{1 \leq k \leq N} V_k(x - \chi_k(t)),$$

and the total time-dependent hamiltonian $H(t)$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$,

(1.1′) $$H(t) = H_0 + V(t, x).$$

In the case $H_0 = -\Delta$ the existence and asymptotic completeness of associated wave operators for 2 classical particles with linear trajectories χ_k have been studied in [18] and [12] in the situation of short range interactions and in [17] in the situation of long range interactions. The case of an arbitrary number of classical particles for a large class of classical trajectories has been studied in [10] in the situation of short range interactions and in [19] in the situation of long range interactions. The approach we present here allows to treat short-range interactions of Enss type, *i.e.* $|V_k(x)| \leq \Phi(|x|)$ with Φ decreasing and integrable on $[1; \infty[$ (*cf.* [7]). We note that [18], [12], [10], consider only the case $\Phi(t) = \mathrm{C}t^{-1-\varepsilon}$ with $\varepsilon > 0$.

Further on $\dot{\chi}_k$ and $\ddot{\chi}_k$ denote the first and second time derivative of χ_k and we assume that the motion of classical particles is asymptotically free, *i.e.* there exist

$$v_k = \lim_{t \to \infty} \dot{\chi}_k(t) \tag{1.2}$$

with $v_j \neq v_k$ for $j \neq k$ and we are interested in the large time behaviour of the associated family $\{U(t, s)\}_{t \geq s}$ of unitary operators in $L^2(\mathbb{R}^d)$ satisfying

$$i \frac{d}{dt} U(t, s) \varphi = H(t) U(t, s) \varphi, \qquad U(s, s) \varphi = \varphi \tag{1.3}$$

for $\varphi \in D(H_0)$, where $D(H_0)$ denotes the domain of H_0.

The associated wave operators are given by

$$\Omega_0(s) = s\text{-}\lim_{t \to \infty} U(t, s)^* e^{-it H_0}, \tag{1.4}$$

$$\Omega_k(s) = s \lim_{t \to \infty} U(t, s)^* e^{-i \chi_k(t) \cdot D} e^{-it H_k} E_p(H_k),$$
$$\text{where } H_k = p_k(D) + V_k(x) \quad \text{with } p_k(\xi) = p(\xi) - v_k \cdot \xi,$$

and $E_p(H_k)$ denotes the orthogonal projection on the linear subspace $\mathcal{H}_p(H_k)$ generated by all eigenvectors of H_k. If $\{\Omega_k(s)\}_{0 \leq k \leq N}$ exist then we say that the asymptotic completeness holds if the ranges of wave operators form a direct decomposition of $L^2(\mathbb{R}^d)$, *i.e.*

$$L^2(\mathbb{R}^d) = \bigoplus_{0 \leq k \leq N} \operatorname{Ran} \Omega_k(s). \tag{1.5}$$

If the asymptotic completeness holds then the wave operators allow to give a simple asymptotic description of the large time behaviour of $U(t, s) \varphi$ for every $\varphi \in L^2(\mathbb{R}^d)$. Indeed, (1.5) allows to decompose $\varphi = \varphi_0 + \varphi_1 + ... + \varphi_N$ with $\varphi_k = \Omega_k(s) \psi_k$ and $\psi_k \in L^2(\mathbb{R}^d)$ for $k = 0, 1, ..., N$. However $\varphi_0 = \Omega_0(s) \psi_0$ means $U(t, s) \varphi_0 - e^{-it H_0} \psi_0 \to 0$ and for $k = 1, ..., N$, $\varphi_k = \Omega_k(s) \psi_k$ means $U(t, s) \varphi_k - e^{-i \chi_k(t) \cdot D} e^{-it H_k} E_p(H_k) \psi_k \to 0$ when $t \to \infty$. Let $\mathcal{B}_k = \{\psi_{k,n}\}_{n=1,2,...}$ be an orthonormal basis of $\mathcal{H}_p(H_k)$ composed of eigenvectors associated with eigenvalues $\{\lambda_{k,n}\}_{n=1,2,...}$, *i.e.* $H \psi_{k,n} = \lambda_{k,n} \psi_{k,n}$ and let $\{c_{k,n}\}_{n=1,2,...}$ be the coefficients of $E_p(H_k) \psi_k$ in the basis $\mathcal{B}_k$, *i.e.* for every $\varepsilon > 0$ there is $n(\varepsilon) \in \mathbb{N}$ such that $\|c_{k,1} \psi_{k,1} + c_{k,2} \psi_{k,2} + ... + c_{k,n(\varepsilon)} \psi_{k,n(\varepsilon)} - E_p(H_k) \psi_k\| < \varepsilon/(2N)$. Since

$$(e^{-i \chi_k(t) \cdot D} e^{-it H_k} E_p(H_k) \psi_{k,n})(x) = e^{-it \lambda_{k,n}} \psi_{k,n}(x - \chi_k(t)),$$

we may conclude that for every $\varepsilon > 0$ we have $\lim_{t\to\infty} \sup \|U(t, s)\varphi - \psi_t\| \leq \varepsilon$ with

$$\psi_t(x) = e^{-itH_0}\psi_0(x) + \sum_{1\leq k\leq N,\, 1\leq n\leq n(\varepsilon)} c_{k,n}\, e^{-it\lambda_{k,n}}\psi_{k,n}(x - \chi_k(t)).$$

This conclusion may be interpreted as the fact that the only possible asymptotic behaviour of the quantum particle is either a free motion or a motion localized near one of the classical particle.

1.2. Precise formulation of the result

We assume $N = 2$ in (1.1).

I. Hypotheses concerning the free hamiltonian H_0

We consider $H_0 = p(D)$ where p is a smooth real-valued elliptic symbol of degree m, being convex function and satisfying an additional hypothesis of microhyperbolic type, which holds e.g. if the principal part of p is homogeneous.

More precisely $H_0 = p(D)$ [where $D = (-i\,\partial_{x_1}, \ldots, -i\,\partial_{x_d})$] is self-adjoint in $L^2(\mathbb{R}^d)$ with the domain $D(H_0) = H^m(\mathbb{R}^d)$ [the Sobolev space of degree m on $L^2(\mathbb{R}^d)$] and

a) p is a symbol of degree $m \geq 1$ [which will be denoted $p \in S^m(\mathbb{R}^d)$], *i.e.*

$$\text{(1.6a)}\quad \forall\,\alpha \in \mathbb{N}^d\ \exists\, C_\alpha > 0, \quad |p^{(\alpha)}(\xi)| \leq C_\alpha \langle\xi\rangle^{m-|\alpha|} \text{ for every } \xi \in \mathbb{R}^d,$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.

b) p is elliptic of degree m, *i.e.*

$$\text{(1.6b)}\qquad \exists\, C_0,\, c_0 > 0,\ p(\xi) \geq c_0\langle\xi\rangle^m \qquad \text{for}\quad |\xi| \geq C_0.$$

In the case $m = 1$ we assume moreover that $p_k(\xi) = p(\xi) - v_k \cdot \xi$ are elliptic (of degree 1).

c) p satisfies a microhyperbolic type hypothesis of the form

$$\text{(1.6c)}\qquad \exists\, C_0,\, c_0 > 0,\ \xi\cdot\nabla p(\xi) \geq c_0\langle\xi\rangle^m \qquad \text{for}\quad |\xi| \geq C_0,$$

where $\nabla p(\xi) = (\nabla_1 p(\xi), \ldots, \nabla_d p(\xi)) = (\partial_{\xi_1} p(\xi), \ldots, \partial_{\xi_d} p(\xi))$.

d) p is strictly convex.

II. Hypotheses concerning quantum interactions

We assume that V_k is H_0-compact and there is a decreasing, integrable function on $[1; \infty[$, $\Phi \in L^1([1; \infty[, dt)$, such that

$$|V_k(x)| + |\nabla V_k(x)| \leq \Phi(|x|) \quad \text{for} \quad |x| \geq C_0. \tag{1.7}$$

III. Hypotheses concerning the classical trajectories

We assume that

a) there exist constants C, $\varepsilon_0 > 0$ such that

$$|\ddot{\chi}_k(t)| \leq C t^{-1-\varepsilon_0} \quad \text{for every } t \geq 1. \tag{1.8a}$$

We note that this hypothesis implies the existence of the limits (1.2) defining v_k.

b) We assume moreover that $v_1 \neq v_2$ and there exists a decreasing and integrable function on $[1; \infty[$, $\Phi \in L^1([1; \infty[, dt)$ such that

$$|\dot{\chi}_k(t) - v_k| \leq \Phi(t) \quad \text{for every } t \geq 1. \tag{1.8b}$$

IV. Hypotheses concerning the total evolution

We say that the time dependent hamiltonian $H(t)$ is H_0-admissible if there exists a family $\{U(t, s)\}_{t \geq s}$ of unitary operators in $L^2(\mathbb{R}^d)$ such that

$$t \mapsto U(t, s)\varphi \text{ is continuous } [s; \infty) \to L^2(\mathbb{R}^d) \text{ for every } \varphi \in L^2(\mathbb{R}^d), \tag{1.9a}$$

$$t \mapsto U(t, s)\varphi \text{ is continuous } [s; \infty) \to D(H_0) \text{ for every } \varphi \in D(H_0), \tag{1.9b}$$

$$t \mapsto U(t, s)\varphi \text{ is continuously differentiable } (s; \infty) \to L^2(\mathbb{R}^d) \text{ for every } \varphi \in D(H_0), \tag{1.9c}$$

where $D(H_0)$ is equipped with the graph norm $\|\varphi\|_{D(H_0)} = \|\varphi\| + \|H_0\varphi\|$ and satisfying (1.3).

Further on we assume that $H(t)$ given by (1.1) is H_0-admissible. We have

THEOREM 1. – *Assume that $H(t)$ is given by* (1.1) *with $N = 2$ and the hypotheses* (1.6-9) *hold. Then the wave operators* (1.4) *exist and the asymptotic completeness* (1.5) *holds.*

We note that the hypotheses on H_0 may be slightly weakened without any change in the proof. The weaker version of these hypotheses is described below.

1.3. Remarks concerning the hypotheses of Theorem 1

Let us fix $v \in \mathbb{R}^d$ and set

$$\chi_k^v(t) = \chi_k(t) - v\,t \qquad \text{for} \quad k = 1,\, 2, \tag{1.10}$$

$$H_k^v = H_k - v \cdot D \qquad \text{for} \quad k = 0,\, 1,\, 2. \tag{1.11}$$

Then the evolution

$$U^v(t,\, s) = e^{it\,v\cdot D}\, U(t,\, s)\, e^{-is\,v\cdot D} \tag{1.12}$$

is associated with the hamiltonian

$$\begin{gathered} H^v(t) = H_0^v + V^v(t,\, x), \\ \text{where } V^v(t,\, x) = \sum_{1\leq k\leq 2} V_k(x - \chi_k^v(t)), \end{gathered} \tag{1.13}$$

and it is clear that the existence and completeness of wave operators for $H(t)$ is equivalent to the analogous question for $H^v(t)$.

It is clear that the way of formulations of the hypotheses of Theorem 1 guarantees that they hold for $H(t)$ if and only if they hold for $H^v(t)$. Further on we take

$$v = (v_1 + v_2)/2 \tag{1.14}$$

which allows to reduce the proof of Theorem 1 to the situation when $v_1 + v_2 = 0$.

Now we note that the hypotheses are invariant with respect to rotations, hence choosing the suitable system of coordonates we may assume that $v_1 - v_2$ and $e_1 = (1, 0, \ldots, 0)$ are colinear. Moreover changing the scale we may assume that

$$v_1 = e_1 \qquad \text{and} \qquad v_2 = -e_1. \tag{1.15}$$

The entire proof that follows will treat only the situation (1.15) and considering only this case we may replace the hypotheses on H_0 by weaker ones, adopted to the situation (1.15).

Weaker version of hypotheses on H_0 in the situation $v_1 = e_1$, $v_2 = -e_1$

We keep the hypotheses (1.6a, b) and replace (1.6c, d) by (1.6′c, d, e) of the form

$$\exists\, C_0,\ c_0 > 0,\quad |\nabla p(\xi)| \geq c_0 \langle \xi \rangle^{m-1} \qquad \text{for} \qquad |\xi| \geq C_0, \tag{1.6′c}$$

$$\mathcal{C}_p = \{p(\xi) : \nabla p(\xi) = 0\} \text{ is a finite or countable subset of } \mathbb{R}, \tag{1.6′d}$$

$$\begin{cases} \exists\, r_0 > 0,\ c_1 > 0,\ \xi^{cr} = (\xi_1^{cr}, .., \xi_d^{cr}) \in \mathbb{R}^d \\ \text{such that } (\xi_1 - \xi_1^{cr}) \cdot \nabla_1 p(\xi) \geq c_1 \langle \xi \rangle^m \\ \text{holds for } \xi \in \{\xi \in \mathbb{R}^d : |\nabla_1 p(\xi)| \geq \dfrac{1}{2} \\ \text{and } |\nabla_j p(\xi))| \leq r_0 |\nabla p(\xi)| \text{ for } j = 2, ..., d\}. \end{cases} \tag{1.6′e}$$

Let us check that the hypotheses (1.6c, d) imply (1.6′c, d, e). Indeed, (1.6′c) follows from (1.6c) due to $|\nabla p(\xi)| \geq |(\xi/|\xi|) \cdot \nabla p(\xi)|$. Moreover the strict convexity of p guarantees the existence of a single critical point $\xi^{cr} = (\xi_1^{cr}, ..., \xi_d^{cr})$ and the strict positivity of the radial derivative around ξ^{cr}, *i.e.*

$$\nabla p(\xi) = 0 \Leftrightarrow \xi = \xi^{cr}, \tag{1.16}$$

$$(\xi - \xi^{cr}) \cdot \nabla p(\xi) > 0 \qquad \text{for} \quad \xi \in \mathbb{R}^d \backslash \{\xi^{cr}\}. \tag{1.17}$$

Hence (1.6′d) holds and it remains to show (1.6′e). However (1.6c) and (1.16-17) imply the existence of $c_1 > 0$ such that

$$(\xi - \xi^{cr}) \cdot \nabla p(\xi) \geq 2\, c_1 \langle \xi \rangle^m \tag{1.18}$$

holds for $\xi \in \mathbb{R}^d$ such that $|\nabla_1 p(\xi)| \geq \frac{1}{2}$. Moreover $|\nabla_j p(\xi))| \leq r_0 |\nabla p(\xi)|$ for $j = 2, ..., d$ implies

(1.19)
$$|(\xi_j - \xi_j^{cr}) (\nabla_j p(\xi))| \leq C_1 \langle \xi \rangle r_0 |\nabla p(\xi)| \leq C_2 r_0 \langle \xi \rangle^m \quad \text{for } j = 2, ..., d.$$

Now it is clear that (1.18-19) with $r_0 > 0$ small enough imply $(\xi_1 - \xi_1^{cr}) \cdot \nabla_1 p(\xi) \geq c_1 \langle \xi \rangle^m$.

2. SCHEME OF THE PROOF

To simplify the notations further on we consider only $s = 1$ and denote $U(t, 1) = U(t)$. Further on t denotes always a parameter belonging to $[1; \infty[$ and we denote $H = \{H(t)\}_{t \geq 1}$. For $\mathcal{U} \subset \mathbb{R}^d$ we denote by $C_0^\infty(\mathcal{U})$ the set of smooth functions having the support compact in $\mathcal{U}$. For $x \in \mathbb{R}^d$, $r > 0$, we denote $B(x, r) = \{v \in \mathbb{R}^d : |x - v| < r\}$. We define the Banach algebra

$$(2.1) \qquad C_\ell(\mathbb{R}^d) = \{J \in C(\mathbb{R}^d) : \lim_{|x| \to \infty} J(x) \text{ exists}\}$$

with the norm of the uniform convergence in $\mathbb{R}^d$, *i.e.* $\|J\|_{C_\ell(\mathbb{R}^d)} = \sup\{|J(x)| : x \in \mathbb{R}^d\}$. We have

THEOREM 2.1. – *For every $J \in C_\ell(\mathbb{R}^d)$ the following limits exist*

$$(2.2) \qquad \Omega(H, J) = s\text{-}\lim_{t \to \infty} U(t)^* J\left(\frac{x}{t}\right) U(t),$$

$$(2.2') \qquad \Omega(H_k, J) = s\text{-}\lim_{t \to \infty} e^{it H_k} J\left(\frac{x}{t}\right) e^{-it H_k} \qquad \textit{for} \quad k = 0, 1, 2.$$

Using the chain rule of Remark A.1a) from Appendix we get

$$(2.3) \qquad \begin{cases} \Omega(H, J_1 + J_2) = \Omega(H, J_1) + \Omega(H, J_2) \\ \text{and} \quad \Omega(H, J_1 J_2) = \Omega(H, J_1)\, \Omega(H, J_2), \end{cases}$$

for every $J_1, J_2 \in C_\ell(\mathbb{R}^d)$. Thus $\{\Omega(H, J)\}_{J \in C_\ell(\mathbb{R}^d)}$ is a subalgebra of $B(L^2(\mathbb{R}^d))$ [we denote by $B(\mathcal{X})$ the algebra of linear bounded operators on the Banach space $\mathcal{X}$]. If Z is a compact set of $\mathbb{R}^d$, then we define $E_Z(\vec{\mathcal{V}}(H))$ as the orthogonal projection with

$$(2.4) \qquad \operatorname{Ran} E_Z(\vec{\mathcal{V}}(H)) = \cap\{\operatorname{Ran}\Omega(H, J) : J \in C_0^\infty(\mathbb{R}^d),\ J \geq 0,$$
$$J = 1 \text{ in a neighbourhood of Z}\}.$$

In our proof of Theorem 1 we shall use the definition of $E_Z(\vec{\mathcal{V}}(H))$ only in the case when Z is a finite subset of $\mathbb{R}^d$. However we would like to add here a remark concerning the above definition and the interpretation of $\vec{\mathcal{V}}(H)$. In fact, following [5] it is easy to see that for every $\varphi, \psi \in L^2(\mathbb{R}^d)$ there is an extension $B \to (E_B(\vec{\mathcal{V}}(H))\varphi, \psi)$ being a complex Borel measure of $\mathbb{R}^d$ and $J(\vec{\mathcal{V}}(H)) = \Omega(H, J)$ holds for every $J \in C_0^\infty(\mathbb{R}^d)$, where

$$(2.5) \qquad (J(\vec{\mathcal{V}}(H))\varphi, \psi) = \int_{\mathbb{R}^d} J(\lambda_1, \ldots, \lambda_1)\, d(E_{\lambda_1, \ldots, \lambda_d}(\vec{\mathcal{V}}(H))\varphi, \psi),$$

which allows to interpret $\vec{\mathcal{V}}(H)$ as a commutative family of self-adjoint operators $(\mathcal{V}_1(H), \ldots, \mathcal{V}_d(H))$.

Since $\mathcal{V}_j(H)\varphi$ may be obtained as $\lim_{t\to\infty} U(t)^*(x_j/t)U(t)\varphi$, following [5], $\vec{\mathcal{V}}(H)$ will be called the asymptotic velocity for H.

Instead of H we may use H_k for $k = 0, 1, 2$, and similarly as above we define $E_Z(\vec{\mathcal{V}}(H_k))$. Denoting

$$\hat{U}_k(t) = e^{i\chi_k(t)\cdot D}U(t), \tag{2.6}$$

it is easy to check that for $\varphi \in D(H_0)$

$$\begin{aligned} &i\frac{d}{dt}\hat{U}_k(t)\varphi = \hat{H}_k(t)\hat{U}_k(t)\varphi \text{ with} \\ &\hat{H}_k(t) = H_0 - \dot{\chi}_k(t)\cdot D + V(t, x+\chi_k(t)). \end{aligned} \tag{2.6'}$$

Due to (1.8a),

$$\begin{aligned} |[\chi_k(\tau) - \tau\dot{\chi}_k(\tau)]_{\tau=1}^{\tau=t}| &\le \int_1^t d\tau\left|\frac{d}{d\tau}(\chi_k(\tau) - \tau\dot{\chi}_k(\tau))\right| \\ &= \int_1^t d\tau\,|\tau\ddot{\chi}_k(\tau))| \le C_0 t^{1-\varepsilon_0}, \end{aligned} \tag{2.7}$$

hence due to (1.8b) we have

$$\left|\frac{\chi_k(t)}{t} - v_k\right| \le \frac{1}{t}|\chi_k(t) - t\dot{\chi}_k(t)| + |\dot{\chi}_k(t) - v_k| \le Ct^{-\varepsilon_0} \tag{2.8}$$

and $J((x-\chi_k(t))/t) = J(x/t - v_k) + O(t^{-\varepsilon_0})$ for every $J \in C_0^\infty(\mathbb{R}^d)$.

If $J, J_k \in C_0^\infty(\mathbb{R}^d)$ are related by $J_k(v) = J(v - v_k)$, then

$$\begin{aligned} \Omega(\hat{H}_k, J) &= s\text{-}\lim_{t\to\infty} \hat{U}_k(t)^* J\left(\frac{x}{t}\right)\hat{U}_k(t) \\ &= s\text{-}\lim_{t\to\infty} U(t)^* J\left(\frac{x-\chi_k(t)}{t}\right)U(t) = \Omega(H_k, J_k). \end{aligned} \tag{2.9}$$

hence $\vec{\mathcal{V}}(\hat{H}_k) = \vec{\mathcal{V}}(H) - v_k$, *i.e.* denoting $Z + \{v_k\} = \{v + v_k : v \in Z\}$ we have

$$E_Z(\vec{\mathcal{V}}(\hat{H}_k)) = E_{Z+\{v_k\}}(\vec{\mathcal{V}}(H)). \tag{2.10}$$

We shall denote $\hat{H}_k = \{\hat{H}_k(t)\}_{t\ge1}$. We have

THEOREM 2.2. – *a) If* $J \in C_0^\infty(B(0, r_0))$ *with* $r_0 < |v_1 - v_2|$, *then the following limits exist*

$$\Omega(\hat{H}_k, H_k, J) = s\text{-}\lim_{t\to\infty} \hat{U}_k(t)^* J\left(\frac{x}{t}\right) e^{-it H_k}, \tag{2.11}$$

$$\Omega(H_k, \hat{H}_k, J) = s\text{-}\lim_{t\to\infty} e^{it H_k} J\left(\frac{x}{t}\right) \hat{U}_k(t). \tag{2.11'}$$

b) If $\overline{J} \in C_\ell(\mathbb{R}^d)$ *is such that* $\overline{J}(v_k) = 0$ *for* $k = 1, 2$, *then the following limits exist*

$$\Omega(H, H_0, \overline{J}) = s\text{-}\lim_{t\to\infty} U(t)^* \overline{J}\left(\frac{x}{t}\right) e^{-it H_0}, \tag{2.12}$$

$$\Omega(H_0, H, \overline{J}) = s\text{-}\lim_{t\to\infty} e^{it H_0} \overline{J}\left(\frac{x}{t}\right) U(t). \tag{2.12'}$$

Theorem 2.1 and 2.2 have the following consequence

COROLLARY 2.3. – *a) An isometric bijection*

$$\Omega(\hat{H}_k, H_k) : \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(H_k)) \to \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(\hat{H}_k)), \tag{2.13}$$

is well defined by the formula

$$\Omega(\hat{H}_k, H_k)\varphi = \lim_{t\to\infty} \hat{U}_k(t)^* e^{-it H_k} \varphi, \tag{2.13'}$$

where the limit (2.13′) exists for every $\varphi \in \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(H_k))$.

b) An isometric bijection

$$\Omega(H, H_0) : (\mathit{Ran}\, E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H_0)))^\perp \to (\operatorname{Ran} E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H)))^\perp \tag{2.14}$$

is well defined by the formula

$$\Omega(H, H_0)\varphi = \lim_{t\to\infty} U(t)^* e^{-it H_0} \varphi, \tag{2.14'}$$

where the limit (2.14′) *exists for every* $\varphi \in (\operatorname{Ran} E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H_0)))^\perp$.

Proof. – a) For $0 < r \le r_0 < |v_1 - v_2|$ let $J_r \in C_0^\infty(B(0, r))$ be such that $J_r = 1$ on $B(0, r/2)$. Then

$$\Omega(H_k, J_r) E_{\{0\}}(\vec{\mathcal{V}}(H_k)) = E_{\{0\}}(\vec{\mathcal{V}}(H_k)), \tag{2.15}$$

and we may write

(2.16)
$$\varphi \in \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(H_k)) \Rightarrow \Omega(\hat{H}_k, H_k)\varphi = \Omega(\hat{H}_k, H_k)\Omega(H_k, J_r)\varphi = \Omega(\hat{H}_k, H_k, J_r)\varphi.$$

Indeed, (2.15) implies $\varphi = \Omega(H_k, J_r)\varphi$ for $\varphi \in \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(H_k))$, *i.e.* $A_1(t)\varphi \to \varphi$ with $A_1(t) = \hat{U}_k(t)^* J_r(x/t)\hat{U}_k(t)$ and denoting $A_2(t) = U(t)^* \hat{U}_k(t)$, we can see that Theorem 2.2 a implies the existence of $\lim_{t\to\infty} A_2(t) A_1(t)\varphi = \lim_{t\to\infty} U(t)^* J_r(x/t)\hat{U}_k(t)\varphi = \psi$. Then clearly Remark A.1a implies the convergence $A_2(t)\varphi \to \psi = \Omega(\hat{H}_k, H_k, J_r)\varphi$ in $L^2(\mathbb{R}^d)$ when $t \to \infty$. In order to show that $\psi \in \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(\hat{H}_k))$ we should check that for an arbitrary function $J \in C_0^\infty(\mathbb{R}^d)$ such that $J = 1$ in a neighbourhood of 0, we have $\psi \in \operatorname{Ran}\Omega(\hat{H}_k, J)$. However, if $J = 1$ in a neighbourhood of 0, then taking J_r defined as above with $r > 0$ small enough we have $J_r = JJ_r$ and

$$\begin{aligned}\psi &= \Omega(\hat{H}_k, H_k, J_r)\varphi = \Omega(\hat{H}_k, H_k, JJ_r)\varphi \\ &= \Omega(\hat{H}_k, J)\Omega(\hat{H}_k, H_k, J_r)\varphi \in \operatorname{Ran}\Omega(\hat{H}_k, J).\end{aligned} \tag{2.17}$$

Interchanging $\hat{H}_k$ and H_k in the reasoning above we obtain that

$$\Omega(H_k, \hat{H}_k)\psi = \lim_{t\to\infty} e^{itH_k}\hat{U}_k(t)\psi \tag{2.18}$$

exists for every $\psi \in \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(\hat{H}_k))$ and belongs to Ran $E_{\{0\}}(\vec{\mathcal{V}}(H_k))$. Therefore (2.18) defines an isometric injection,

$$\Omega(H_k, \hat{H}_k) : \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(\hat{H}_k)) \to \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(H_k)). \tag{2.18'}$$

Since for every $\varphi \in \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(H_k))$ and $\psi \in \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}(\hat{H}_k))$ we have

$$\Omega(H_k, \hat{H}_k)\Omega(\hat{H}_k, H_k)\varphi = \varphi, \qquad \Omega(\hat{H}_k, H_k)\Omega(H_k, \hat{H}_k)\psi = \psi,$$

it is clear that $\Omega(H_k, \hat{H}_k)$ is the inverse of $\Omega(\hat{H}_k, H_k)$.

b) As before we show first the existence of (2.14′) with the fact that (2.14) is an isometric injection and then we show the existence of

$$\Omega(H_0, H)\psi = \lim_{t\to\infty} e^{itH_0}U(t)\psi \tag{2.19}$$

for every $\psi \in (\operatorname{Ran} E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H)))^{\perp}$, defining an isometric injection

(2.19′)
$$\Omega(H_0, H) : (\operatorname{Ran} E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H)))^{\perp} \to (\operatorname{Ran} E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H_0)))^{\perp},$$

which is the inverse of $\Omega(H, H_0)$.

If $J_r \in C_0^\infty(\mathbb{R}^d)$ are as in the proof of $a)$ above, then

$$(\operatorname{Ran} E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H)))^{\perp} = \text{the closure of} \bigcup_{r>0} \operatorname{Ran} \Omega(H, \overline{J}_r), \quad \text{where } \overline{J}_r(v) = 1 - J_r(v - v_1) - J_r(v - v_2), \tag{2.20}$$

and the similar relation holds with H_0 instead of H. Thus it suffices to prove the existence of the limit (2.14′) for $\varphi \in \operatorname{Ran} \Omega(H_0, \overline{J}_r)$ and the existence of the limit (2.19) for $\psi \in \operatorname{Ran} \Omega(H, \overline{J}_r)$ with certain $0 < r \leq 1$. However similarly as before we have

$$\varphi = \Omega(H_0, \overline{J}_r)\tilde{\varphi} \Rightarrow \Omega(H, H_0)\varphi = \Omega(H, H_0)\Omega(H_0, \overline{J}_r)\tilde{\varphi} = \Omega(H, H_0, \overline{J}_r)\tilde{\varphi}, \tag{2.21}$$

hence Theorem 2.2b implies the existence of the limit $\Omega(H, H_0)\varphi \in (\operatorname{Ran} E_{\{v_1, v_2\}}(\vec{\mathcal{V}}(H)))^{\perp}$ and clearly the analogical reasoning holds with H and H_0 interchanged. ■

Now we describe how Theorem 1 follows from the existence of Deift-Simon wave operators described in Theorem 2.1, 2.2 and from the following property of the asymptotic velocity.

THEOREM 2.4. – *The space $\mathcal{H}_p(H_k)$ generated by eigenvectors of H_k is the space of vectors with zero asymptotic velocity,* i.e.

$$E_p(H_k) = E_{\{0\}}(\vec{\mathcal{V}}(H_k)). \tag{2.22}$$

It is clear that Theorem 2.4 and Corollary 2.3a imply the existence of wave operators Ω_k for $k = 1, 2$. Introducing

$$\mathcal{H}_k = \operatorname{Ran} E_{\{v_k\}}(\vec{\mathcal{V}}(H)) \quad \text{for} \quad k = 1, 2, \quad \text{and } \mathcal{H}_0 = (\mathcal{H}_1 \oplus \mathcal{H}_2)^{\perp}, \tag{2.23}$$

and noting that

$$E_{\{0\}}(\vec{\mathcal{V}}(\hat{H}_k)) = E_{\{v_k\}}(\vec{\mathcal{V}}(H)), \tag{2.24}$$

due to (2.10), we have

$$\text{Ran}\, \Omega_k = \mathcal{H}_k \tag{2.25}$$

for $k = 1, 2$. Let $v \in \mathbb{R}^d$. Then replacing $t \mapsto \chi_k(t)$, $\hat{H}_k$ by $t \mapsto vt$, H^v [defined as in (1.13)] in the reasoning (2.9) we obtain (2.10) with $Z + \{v_k\}$, $\hat{H}_k$ replaced by $Z + \{v\}$, H^v. In the particular case when the potential $V(t, x) = 0$ identically we obtain

$$E_{\{v\}}(\vec{\mathcal{V}}(H_0)) = E_{\{0\}}(\vec{\mathcal{V}}(H_0 - v \cdot D)) = E_p(H_0 - v \cdot D) = 0. \tag{2.26}$$

Now it is clear that Corollary 2.3b implies the existence of the wave operator Ω_0 and (2.25) holds for $k = 0$ as well. Thus the asymptotic completeness follows from (2.25) and $L^2(\mathbb{R}^d) = \bigoplus_{0 \le k \le 2} \mathcal{H}_k$, hence Theorem 1 follows from Theorem 2.1, 2.2 and 2.4. ■

3. SCHEME OF THE PROOF OF THE EXISTENCE OF DEIFT-SIMON WAVE OPERATORS

Before describing the method of proving of Theorem 2.1 and 2.2 we explain some notational conventions used further on. We denote by $(\cdot, \cdot)$ the scalar product of $L^2(\mathbb{R}^d)$. We shall often treat linear operators or sesquilinear forms in $L^2(\mathbb{R}^d)$ without precisions about their domains. If A is a sesquilinear form and $\tilde{A}$ an operator with the domain $D(\tilde{A})$ dense in $L^2(\mathbb{R}^d)$ such that $A[\varphi, \psi] = (\tilde{A}\varphi, \psi)$ for $\varphi, \psi \in D(\tilde{A})$, then we shall often write A instead of $\tilde{A}$, *i.e.* the same letter may denote the operator and the sesquilinear form. We write $A \le B$ if and only if $D(A) \cap D(B)$ is dense in $L^2(\mathbb{R}^d)$ and $(B - A)[\varphi, \varphi] \ge 0$ for every $\varphi \in D(A) \cap D(B)$ in the case of sesquilinear forms [respectively $((B - A)\varphi, \varphi) \ge 0$ in the case of operators].

We denote $\|A\| = \sup\{\|A\varphi\| : \varphi \in D(A), \|\varphi\| \le 1\}$ if A is an operator with the domain $D(A)$ dense in $L^2(\mathbb{R}^d)$ and $\|A\| = \sup\{|A[\varphi, \psi]| : \varphi, \psi \in D(A), \|\varphi\| \le 1, \|\psi\| \le 1\}$ if A is a sesquilinear form with the domain $D(A)$ dense in $L^2(\mathbb{R}^d)$. Clearly $\|A\| < \infty$ means that A has the extension by continuity to a bounded operator or a bounded form on $L^2(\mathbb{R}^d)$.

Given a function $f : [1; \infty[\to]0; \infty[$ we use the notation $A_1(t) = A_2(t) + O(f(t))$, which means that $D(A_1(t)) \cap D(A_2(t))$ is dense in $L^2(\mathbb{R}^d)$ for all $t \ge 1$ and there is a constant $C > 0$ such that

$\|A_1(t) - A_2(t)\| \leq Cf(t)$ for all $t \geq 1$. Assume now that for every $t \geq 1$, $A_1(t)$, $A_2(t)$, $B(t)$ are operators on $\mathcal{S}(\mathbb{R}^d)$ [the Schwartz space of rapidly decreasing functions in $\mathbb{R}^d$] and $B(t)$ has the inverse $B(t)^{-1}$ on $\mathcal{S}(\mathbb{R}^d)$. Then the notation $A_1(t) = A_2(t) + O(B(t))$ means that $t \to \|B(t)^{-1}(A_1(t) - A_2(t))\| + \|(A_1(t) - A_2(t))B(t)^{-1}\|$ is a bounded function $[1; \infty[\to [0; \infty[$. In the particular case when $A_1(t) - A_2(t)$ is symmetric for all $t \in [1; \infty[$ (recall that A symmetric means $D(A) \subseteq D(A^*)$), we have

(3.1)
$$A_1(t) = A_2(t) + O(B(t)) \Leftrightarrow \exists\, C > 0,\ -CB(t) \leq A_1(t) - A_2(t) \leq CB(t) \quad \text{for all } t \geq 1.$$

We denote by $A + hc$ the symmetrization of the sequilinear form A, *i.e.*

$$(A + hc)[\varphi, \psi] = \frac{1}{2}(A[\varphi, \psi] + \overline{A[\psi, \varphi]}) \tag{3.2}$$

and similarly for an operator A such that $D(A) \cap D(A^*)$ is dense in $L^2(\mathbb{R}^d)$ [where A^* denotes the adjoint of A] we denote $A + hc = \frac{1}{2}(A + A^*)$. If A and B are operators such that $D(A^*) \cap D(B)$ is dense in $L^2(\mathbb{R}^d)$, then we may treat AB as a sesquilinear form

$$AB[\varphi, \psi] = (B\varphi, A^*\psi) \qquad \text{for} \quad \varphi, \psi \in D(A^*) \cap D(B), \tag{3.3}$$

and similarly we may treat the commutator $[A, B] = AB - BA$ as a sesquilinear linear form on $D(A) \cap D(A^*) \cap D(B) \cap D(B^*)$ if this subspace is dense in $L^2(\mathbb{R}^d)$.

We denote by $\mathcal{G}(H)$ the set of functions $t \to M(t)$ satisfying the following two conditions

$$t \to M(t)[\varphi, \psi] \text{ is continuous on } [1; \infty) \text{ for every } \varphi, \psi \in D(H), \tag{3.4}$$

$$\begin{cases} \exists\, C > 0, \quad \displaystyle\int_1^T dt\,(M(t) + hc)[U(t)\varphi, U(t)\varphi] \leq C\|\varphi\|^2 \\ \qquad\qquad \text{for every } \varphi \in D(H_0) \text{ and } T \geq 1. \end{cases} \tag{3.5}$$

As indicated in [21] it is easy to see that (3.4) implies the continuity of the function integrated in (3.5). Sometimes we shall write $M(t) \in \mathcal{G}(H(t))$ instead of $M \in \mathcal{G}(H)$.

If M satisfies (3.4), then it is clear that

$$M(t) = O(\Phi(t)) \text{ with } \Phi \in L^1([1; \infty); dt) \Rightarrow M \in \mathcal{G}(H), \tag{3.6}$$

$$(\tilde{M} \in \mathcal{G}(H) \text{ and } M(t) \leq \tilde{M}(t) \text{ for all } t \geq 1) \Rightarrow M \in \mathcal{G}(H). \tag{3.7}$$

If $t \to (M(t)\varphi, \psi)$ is C^1 on $[1; \infty)$ for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d) \subseteq D(H(t))$ for every $t \geq 1$, then we may define the following sesquilinear form on $\mathcal{S}(\mathbb{R}^d)$,

$$\mathbb{D}_{H(t)} M(t) = [i H(t), M(t)] + M'(t) \tag{3.8}$$

Assume further on that $t \to (M(t)\varphi, \psi)$ is C^1 on $[1; \infty)$ for every $\varphi, \psi \in D(H_0)$ and $H(\cdot)$ is H_0-admissible [*i.e.* (1.9a, b, c) hold]. Then

$$\frac{d}{dt}(M(t) U(t)\varphi, U(t)\varphi) = (\mathbb{D}_{H(t)} M(t))[U(t)\varphi, U(t)\varphi] \tag{3.9}$$

holds for every $\varphi \in D(H_0)$ and we have

$$M(t) = O(1) \Rightarrow \mathbb{D}_{H(t)} M(t) \in \mathcal{G}(H(t)), \tag{3.10}$$

because (3.9) implies

$$\begin{aligned} &\int_1^T dt\,(\mathbb{D}_{H(t)} M(t))[U(t)\varphi, U(t)\varphi] \\ &\quad = (M(T) U(T)\varphi, U(T)\varphi) - (M(1)\varphi, \varphi). \end{aligned}$$

We have (*cf.* Appendix for the proof).

PROPOSITION 3.1. – *For $k = 1, 2$, let $U_k(\cdot)$ be the evolution associated with H_0-admissible hamiltonian $H_k(\cdot)$ in $L^2(\mathbb{R}^d)$ and let $M \in C^1(]1; \infty[; B(L^2(\mathbb{R}^d)) \cap B(D(H_0)))$. Assume that there exists $\Phi \in L^1([1; \infty); dt)$, $\tilde{M}$ and $\tilde{M}_0$ such that*

$$i(H_2(t) M(t) - M(t) H_1(t)) + M'(t) = \tilde{M}(t) + O(\Phi(t)) \tag{3.11}$$

$$-\tilde{M}_0(t) \leq \tilde{M}(t) \leq \tilde{M}_0(t) \qquad \textit{and} \qquad \tilde{M}_0(t) \geq 0 \textit{ for all } t \geq 1, \tag{3.12}$$

$$\tilde{M}_0 \in \mathcal{G}(H_1) \cap \mathcal{G}(H_2). \tag{3.13}$$

Then the following limit exists

$$\Omega = s\text{-}\lim_{t \to \infty} U_2(t)^* M(t) U_1(t). \tag{3.14}$$

In our proof of Theorem 2.1 and 2.2, based on Proposition 3.1, we use the commutators of $J(x/t)$ and $H_0 = p(D)$. We recall that the standard symbolic calculus allows to state

Remark 3.2. – Let $\nu \in \mathbb{R}$, $\eta \in S^\nu(\mathbb{R}^d)$, $J \in S^1(\mathbb{R}^d)$ and $N \in \mathbb{N}$. Then

$$(3.15) \quad \begin{cases} \eta(D) J\left(\frac{x}{t}\right) = \sum_{|\alpha| \le N} (it)^{-|\alpha|} J^{(\alpha)}\left(\frac{x}{t}\right) \eta^{(\alpha)}(D)/\alpha! + R_N(t) \\ \quad \text{with } \langle D \rangle^s R_N(t) \langle D \rangle^{s'} = O(t^{-N-1} \langle D \rangle^{s+s'+\nu-N-1}) \\ \qquad \text{for every } s,\ s' \in \mathbb{R}, \end{cases}$$

where the operators $\eta(D)$ and $J(x/t)$ are considered on $\mathcal{S}(\mathbb{R}^d)$. In particular

(3.15′)

$$i\left[\eta(D),\ J\left(\frac{x}{t}\right)\right] = \sum_{1 \le j \le d} \frac{1}{t} \nabla_j J\left(\frac{x}{t}\right) \nabla_j \eta(D) + O(t^{-2} \langle D \rangle^{\nu-2}),$$

Proof of the existence of $\Omega(H_k, J)$ *given by* (2.2′). – It is based on

THEOREM 3.3 (Propagation estimates for H_k). – *Let* $J_0 \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$, $s \le 0$ *and*

$$M_{j, J_0}^{s, k}(t) = \frac{1}{t} J_0\left(\frac{x}{t}\right) \langle D \rangle^{s/2} \left| \frac{x_j}{t} - \nabla_j p_k(D) \right| \langle D \rangle^{s/2} J_0\left(\frac{x}{t}\right) \tag{3.16}$$

for $k = 1, 2$, $j = 1, \ldots, d$. *If s is sufficiently negative, then* $M_{j, J_0}^{s, k}(t) \in \mathcal{G}(H_k)$.

Let $\mathcal{T}_0$ denote the set of smooth functions which are constant in a neighbourhood of 0 and outside a bounded region, *i.e.*

$$\mathcal{T}_0 = \{J \in C^\infty(\mathbb{R}^d) : \text{supp}\, \nabla_j J \text{ is compact and } 0 \notin \text{supp}\, \nabla_j J \text{ for } j = 1, \ldots, d\}. \tag{3.17}$$

Then it is clear that $\mathcal{T}_0$ is dense is $C_\ell(\mathbb{R}^d)$ [with respect to the norm of $C_\ell(\mathbb{R}^d)$ being the norm of the uniform convergence on $\mathbb{R}^d$] and due to Remark A.1b of Appendix it suffices to show the existence of $\Omega(H_k, J)$ for $J \in \mathcal{T}_0$.

Let $\nu \in \mathbb{N}$ be large enough, $C_0 > 0$ such that $H_k + C_0 \geq I$ and set

$$h_r(\lambda) = (1 + r(C_0 + \lambda))^{-\nu}. \tag{3.18}$$

Then it suffices to prove the existence of $\Omega(H_k, J)$ on vectors $\varphi \in L^2(\mathbb{R}^d)$ such that $\varphi = h_r^2(H_k)\varphi$ for a certain $r > 0$ (such vectors are dense in $L^2(\mathbb{R}^d)$ because $h_r^2(H_k) \to I$ strongly when $r \to 0$), *i.e.* it suffices to prove the existence of the limit $\Omega(H_k, J)\, h_r^2(H_k)$ expressed by the right hand side of (3.14) with $H_1(t) = H_2(t) = H_k$ and

$$M(t) = J\left(\frac{x}{t}\right) h_r^2(H_k). \tag{3.19}$$

Using Remark 3.2 we get

(3.20)
$$\mathbb{D}_{H_k} M(t) = \sum_{1 \leq j \leq d} \frac{1}{t}\left(\nabla_j p_k(D) - \frac{x_j}{t}\right) \nabla_j J\left(\frac{x}{t}\right) h_r^2(H_k) + O(t^{-2})$$

If $\overline{J} \in S^0(\mathbb{R}^d)$ and supp $\overline{J} \subset \mathbb{R}^d\backslash\{0\}$, then $[p_k(D), \overline{J}(x/t)] = O(t^{-1}\langle D\rangle^{m-1})$ due to (3.15′) and $\overline{J}(x/t)\, V_k(x) = O(\Phi(t)) = O(t^{-1})$, hence

$$\begin{aligned} (3.21)\quad & (1 + r(C_0 + H_k))^{-1}\overline{J}\left(\frac{x}{t}\right) - \overline{J}\left(\frac{x}{t}\right)(1 + r(C_0 + p_k(D)))^{-1} \\ &= -r(1 + r(C_0 + H_k))^{-1}\left(H_k \overline{J}\left(\frac{x}{t}\right) - \overline{J}\left(\frac{x}{t}\right) p_k(D)\right) \\ &\quad \times (1 + r(C_0 + p_k(D)))^{-1} = O(t^{-1}) \end{aligned}$$

and consequently

(3.22)
$$\begin{cases} h_r(H_k)\overline{J}\left(\frac{x}{t}\right) = \overline{J}\left(\frac{x}{t}\right)\eta(D) + O(t^{-1}) = \eta(D)\overline{J}\left(\frac{x}{t}\right) + O(t^{-1}), \\ \qquad \text{where} \quad \eta \in S^{-m\nu}(\mathbb{R}^d) \text{ has the form } \eta(\xi) = h_r(p_k(\xi)). \end{cases}$$

Due to the definition of $\mathcal{T}_0$ we have supp $\nabla_j J \subset \mathbb{R}^d\backslash\{0\}$ for $j = 1, \dots, d$, and using (3.22) with $\overline{J} = \nabla_j J$ we may replace $h_r(H_k)$ by $\eta(D)$ in (3.20). Moreover introducing

$$A(t) = \nabla_j p_k(D) - \frac{x_j}{t}, \tag{3.23}$$

$$(3.24) \qquad B(t) = \langle D\rangle^{s/2}\,\eta(D)\,\nabla_j J\left(\frac{x}{t}\right) + hc,$$

and choosing $J_0 \in C_0^\infty(\mathbb{R}^d\backslash\{0\})$ such that $J_0 = 1$ on supp $\nabla_j J$ for $j = 1, \ldots, d$, we may also write

$$(3.25) \qquad \mathbb{D}_{H_k} M(t) = \tilde{M}(t) + O(t^{-2}) \text{ with}$$

$$\tilde{M}(t) = \sum_{1\le j\le d} \frac{1}{t} J_0\left(\frac{x}{t}\right) \langle D\rangle^{s/2} (B(t)\,A(t) + hc)\,\langle D\rangle^{s/2} J_0\left(\frac{x}{t}\right).$$

It remains to show that (3.12) holds with $\tilde{M}$ given by (3.25) and

$$(3.26) \qquad \tilde{M}_0(t) = C \sum_{1\le j\le d} M_{j,\,J_0}^{s,\,k}(t) + Ct^{-2},$$

where $C > 0$ is a constant large enough. Indeed, the condition (3.13) holds due to Theorem 3.3, hence the existence of $\Omega(H_k, J)\,h_r^2(H_k)$ follows from Proposition 3.1. Thus it remains to show that there is a constant $C > 0$ such that

$$(3.27) \qquad B(t)\,A(t) + hc \le C\,|A(t)| + Ct^{-1}$$

and (3.27) follows from Lemma A.2 of Appendix with $A = A(t)$, $B = B(t)$, because

$$(3.28) \qquad \begin{cases} B(t) = O(1), \quad [A(t), B(t)] = O(t^{-1}), \\ \qquad [A(t), [A(t), B(t)]] = O(t^{-2}) \end{cases}$$

due to Remark 3.2. ■

The idea of the proof of the existence of the limit (2.2) will be similar and instead of Theorem 3.3 we shall use

THEOREM 3.4 (Propagation estimates for H). – *Let* $J_0 \in C_0^\infty(\mathbb{R}^d\backslash\{v_1, v_2\})$, $s \le 0$ *and*

$$(3.29) \quad M_{j,\,J_0}^{s}(t) = \frac{1}{t} J_0\left(\frac{x}{t}\right) \langle D\rangle^{s/2} \left|\frac{x_j}{t} - \nabla_j\,p(D)\right| \langle D\rangle^{s/2} J_0\left(\frac{x}{t}\right).$$

for $j = 1, \ldots, d$. *If* s *is sufficiently negative, then* $M_{j,\,J_0}^{s} \in \mathcal{G}(H)$.

In the proof of Theorem 3.4 we shall consider

$$M^{\eta}_{j,\,J_0}(t) = \frac{1}{t}\, J_0\left(\frac{x}{t}\right) \eta(D) \left| \frac{x_j}{t} - \nabla_j\, p(D) \right| \eta(D)\, J_0\left(\frac{x}{t}\right), \tag{3.30}$$

$$M^{s}_{g,\,J_0}(t) = \frac{1}{t}\, J_0\left(\frac{x}{t}\right) g\,(\nabla p(D))^2 \langle D\rangle^s J_0\left(\frac{x}{t}\right), \tag{3.31}$$

where $g,\, \eta \in S^0(\mathbb{R}^d)$. Let $1 = g + g_0$ with $g_0 \in C_0^\infty(\mathbb{R}^d)$ such that $g_0 = 1$ on a neighbourhood of supp J_0, hence supp $g \cap \operatorname{supp} J_0 = \varnothing$. Then

$$\begin{gathered} \langle D\rangle^{s/2} = \eta_0(D) + \eta(D) \ \text{ with} \\ \eta_0(\xi) = g_0(\nabla p(\xi))\,\langle\xi\rangle^{s/2}, \quad \eta(\xi) = g(\nabla p(\xi))\,\langle\xi\rangle^{s/2}. \end{gathered} \tag{3.32}$$

and using the inequality

$$A \geq 0 \Rightarrow (B_1 + B_2)^*\, A\,(B_1 + B_2) \leq 2\, B_1^*\, AB_1 + 2\, B_2^*\, AB_2, \tag{3.33}$$

we get

$$M^{s}_{j,\,J_0}(t) \leq 2\, M^{\eta_0}_{j,\,J_0}(t) + 2\, M^{\eta}_{j,\,J_0}(t). \tag{3.34}$$

Since there is a constant $C > 0$ such that

$$M^{\eta}_{j,\,J_0}(t) \leq C M^{s'}_{g,\,J_0}(t) \quad \text{where } s' = s + (m-1), \tag{3.35}$$

in order to prove the propagation estimates $M^{s}_{j,\,J_0} \in \mathcal{G}(H)$ of Theorem 3.4, it suffices to prove $M^{\eta_0}_{j,\,J_0} \in \mathcal{G}(H)$ with $\eta_0 \in C_0^\infty(\mathbb{R}^d)$ as in (3.32) and

THEOREM 3.5 (Velocity estimates for H). – *Let* $J_0 \in C_0^\infty(\mathbb{R}^d \backslash \{v_1,\, v_2\})$, $g \in S^0(\mathbb{R}^d)$ *and* $s < 0$. *If* supp $J_0 \cap \operatorname{supp} g = \varnothing$, *then* $M^{s}_{g,\,J_0} \in \mathcal{G}(H)$.

4. VELOCITY ESTIMATES

As indicated at the end of the section 1, it is sufficient to consider the situation when $v_1 = e_1$, $v_2 = -e_1$. We make this assumption in the entire proof presented further on and we shall write χ_{-1}, V_{-1}, v_{-1}, p_{-1}, H_{-1} instead of χ_2, V_2, v_2, p_2, H_2.

For $\overline{v}_0,\, v_0 \in \mathbb{R}^d$ we denote by $\ell(\overline{v}_0,\, v_0)$ the straight line passing through $\overline{v}_0$ and v_0, *i.e.*

$$\ell(\overline{v}_0,\, v_0) = \{\overline{v}_0 + \lambda\,(v_0 - \overline{v}_0) \in \mathbb{R}^d : \lambda \in \mathbb{R}\} \tag{4.1}$$

and we denote by $[\overline{v}_0;\, v_0]$ (respectively $]\overline{v}_0;\, v_0[$) the closed (respectively open) segment of $\ell(\overline{v}_0,\, v_0)$ between $\overline{v}_0$ and v_0, *i.e.*

$$\left\{\begin{array}{l}[\overline{v}_0;\, v_0] = \{\overline{v}_0 + \lambda\,(v_0 - \overline{v}_0) \in \mathbb{R}^d : 0 \leq \lambda \leq 1\},\\ \qquad\qquad]\overline{v}_0;\, v_0[= [\overline{v}_0;\, v_0]\backslash\{\overline{v}_0,\, v_0\}.\end{array}\right. \tag{4.2}$$

Further on x, v, $\overline{v}$ denote always the variables belonging to $\mathbb{R}^d$ and identified with $(x_1, \ldots, x_d)$, $(v^1, \ldots, v^d)$, $(\overline{v}^1, \ldots, \overline{v}^d) \in \mathbb{R}^d$. For $-\infty \leq \kappa_1 < \kappa_2 \leq \infty$ and $j = 1, \ldots, d$, we denote

$$\mathcal{U}_j\,(\kappa_1;\, \kappa_2) = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \kappa_1 < x_j < \kappa_2\}. \tag{4.3}$$

For $\mathcal{U} \subset \mathbb{R}^d$ we denote $S^m\,(\mathcal{U}) = \{g \in S^m\,(\mathbb{R}^d) : \operatorname{supp} g \subset \mathcal{U}\}$.

PROPOSITION 4.1. – *Let $s \in \mathbb{R}$ and $J_0 \in S^0\,(\mathcal{U}_j\,(\overline{\kappa}_-;\, \overline{\kappa}_+))$, $g \in C_0^\infty\,(\mathbb{R}^d)$, $\operatorname{supp} g$ and the closure of $\mathcal{U}_j\,(\overline{\kappa}_-;\, \overline{\kappa}_+)$ are disjoint for certain $\overline{\kappa}_-$, $\overline{\kappa}_+ \in \mathbb{R}$ and $1 \leq j \leq d$. Then $M^s_{g,\, J_0} \in \mathcal{G}\,(H)$ if one of the following conditions holds*

$$j = 1, \quad [\overline{\kappa}_-;\, \overline{\kappa}_+] \subset]-1;\, 1[\quad \textit{and} \quad \operatorname{supp} g \subset \mathcal{U}_1\,(-1;\, 1), \tag{4.4i}$$

$$j = 1 \quad \textit{and} \quad [\overline{\kappa}_-;\, \overline{\kappa}_+] \cap [-1;\, 1] = \varnothing \tag{4.4ii}$$

$$j \neq 1 \qquad \textit{and} \qquad 0 \notin [\overline{\kappa}_-;\, \overline{\kappa}_+]. \tag{4.4iii}$$

Instead of $J_0 \in S^0\,(\mathcal{U}_j\,(\overline{\kappa}_-;\, \overline{\kappa}_+))$ we may assume that $J_0\,(x) = \tilde{J}_0\,(x_j)$ with $\tilde{J}_0 \in C_0^\infty\,(]\overline{\kappa}_-;\, \overline{\kappa}_+[)$.

Proof. – Let κ_-, $\kappa_+ \in \mathbb{R}$ be such that $\operatorname{supp}\, g \subset \mathcal{U}_j\,(\kappa_-;\, \kappa_+)$ and $[\kappa_-;\, \kappa_+] \cap [\overline{\kappa}_-;\, \overline{\kappa}_+] = \varnothing$.

We shall say that $J_\pm$ are associated with $\overline{\kappa}_-$, $\overline{\kappa}_+$, κ_-, κ_+ if and only if $J_\pm \in S^0\,(\mathbb{R})$ are such that $J_\pm \geq 0$ with the derivatives $J'_+ \geq 0$, $J'_- \leq 0$, satisfying

$$\operatorname{supp} J'_- \subset]\kappa_+;\, \infty[\text{ and } \operatorname{supp} J'_+ \subset]-\infty;\, \kappa_-[\text{ with } J'_+ > 0 \text{ on } [\overline{\kappa}_-;\, \overline{\kappa}_+] \tag{4.5}$$

in the case $\overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+$, or satisfying

$$\operatorname{supp} J'_+ \subset]-\infty;\, \kappa_-[\text{ and } \operatorname{supp} J'_- \subset]\kappa_+;\, \infty[\text{ with } J'_- < 0 \text{ on } [\overline{\kappa}_-;\, \overline{\kappa}_+] \tag{4.5'}$$

in the case $\kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+$.

1a) Consider first the case $j = 1$ and $-1 < \overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+ < 1$. Set

$$\eta_s(\xi) = g(\nabla p(\xi))^2 \langle \xi \rangle^s \tag{4.6}$$

and note that $g \in C_0^\infty(\mathbb{R}^d)$ with (1.6b) imply $\eta_s \in C_0^\infty(\mathbb{R}^d)$. Let $J_\pm$ be associated with $\overline{\kappa}_-$, $\overline{\kappa}_+$, κ_-, κ_+ and such that supp $J_+ \subset]-1; \infty[$, supp $J_- \subset]-\infty; 1[$. Setting

$$M(t) = J_- J_+ \left(\frac{x_1}{t}\right) \eta_s(D) J_- J_+ \left(\frac{x_1}{t}\right) \tag{4.7}$$

we have

$$\begin{cases} \mathbb{D}_{H(t)} M(t) = \mathbb{D}_{H_0} M(t) + [iV(t), M(t)], \\ \text{with } [iV(t), M(t)] = O(\Phi(t)) \quad \text{and } \mathbb{D}_{H_0} M = M^+ + M^-, \end{cases} \tag{4.8}$$

where

(4.9)
$$M^+(t) = -\frac{2}{t} J_- J_+' \left(\frac{x_1}{t}\right) \left(\frac{x_1}{t} - \nabla_1 p(D)\right) \eta_s(D) J_+ J_- \left(\frac{x_1}{t}\right) + hc + O(t^{-2}),$$

(4.9′)
$$M^-(t) = -\frac{2}{t} J_-' J_+ \left(\frac{x_1}{t}\right) \left(\frac{x_1}{t} - \nabla_1 p(D)\right) \eta_s(D) J_+ J_- \left(\frac{x_1}{t}\right) + hc + O(t^{-2}).$$

Since $\eta_s \geq 0$ and $-J_+^2 J_- J_-'(x_1/t)(x_1/t - \kappa_+) \geq 0$, Corollary A.4 of Appendix gives

(4.10)
$$M_1^-(t) = -\frac{2}{t} J_+^2 J_- J_-' \left(\frac{x_1}{t}\right) \left(\frac{x_1}{t} - \kappa_+\right) \eta_s(D) + hc \geq -C_1 t^{-2}.$$

Since $(\kappa_+ - v^1)\, g(v)^2 \geq 0 \Rightarrow (\kappa_+ - \nabla_1 p(D))\, \eta_s(D) \geq 0$ and $-J_+^2\, J_-\, J_-' \geq 0$, we have

(4.11)
$$M_2^-(t) = -\frac{2}{t}\, J_+^2\, J_-\, J_-' \left(\frac{x_1}{t}\right) (\kappa_+ - \nabla_1 p(D))\, \eta_s(D) + hc \geq -C_2\, t^{-2},$$

hence

$$M^-(t) = M_1^-(t) + M_2^-(t) + O(t^{-2}) \geq -C_3\, t^{-2}. \tag{4.12}$$

Next $\eta_s \geq 0$ and $-J_-^2\, J_+\, J_+'(x_1/t)\,(x_1/t - \kappa_-) \geq 0$ imply

(4.13)
$$M_1^+(t) = -\frac{2}{t}\, J_-^2\, J_+\, J_+' \left(\frac{x_1}{t}\right) \left(\frac{x_1}{t} - \kappa_-\right) \eta_s(D) + hc \geq -C_4\, t^{-2}.$$

For $c_0 > 0$ small enough we have $(\kappa_- + c_0 - v^1)\, g(v)^2 \leq 0 \Rightarrow -(\kappa_- + c_0 - \nabla_1 p(D))\, \eta_s(D) \geq 0$ and using $J_-^2\, J_+\, J_+' \geq 0$ we get

(4.14)
$$M_2^+(t) = -\frac{2}{t}\, J_-^2\, J_+\, J_+' \left(\frac{x_1}{t}\right) (\kappa_- + c_0 - \nabla_1 p(D))\, \eta_s(D) + hc \geq -C_5\, t^{-2}$$

hence

$$\begin{cases} M^+(t) = M_1^+(t) + M_2^+(t) + c_0\, M_0^+(t) + O(t^{-2}) \\ \qquad\qquad \geq c_0\, M_0^+(t) - C_6\, t^{-2}, \\ \text{where } M_0^+(t) = \dfrac{2}{t}\, J_-^2\, J_+\, J_+' \left(\dfrac{x_1}{t}\right) \eta_s(D) + hc. \end{cases} \tag{4.15}$$

Since there is a constant $C > 0$ such that $J_0^2(x/t) \leq 2\, C J_-^2\, J_+\, J_+'(x_1/t)$, we get

$$\begin{aligned} M_{g,\, J_0}^s(t) &= \frac{1}{t}\, J_0^2 \left(\frac{x}{t}\right) \eta_s(D) + hc + O(t^{-2}) \\ &\leq C M_0^+(t) + C_7\, t^{-2} \\ &\leq C_8\, (\mathbb{D}_{H(t)}\, M(t) + \Phi(t) + t^{-2}) \end{aligned} \tag{4.16}$$

which gives $M_{g,\, J_0}^s \in \mathcal{G}(H)$, because the right hand side of (4.16) belongs to $\mathcal{G}(H)$.

1b) In the case $j = 1$ and $-1 < \kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+ < 1$ we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$ and supp $J_+ \subset] - 1; \infty[$, supp $J_- \subset] - \infty; 1[$. It remains to follow the analogical reasoning as above with one change: in (4.16) instead of M_0^+ we use $M_0^- (t) = -2\, J_+^2\, J_-\, J_-' \,(x_1/t)\, \eta_s\,(D)/t + hc.$

2a) In the case $j = 1$ and $1 < \overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_+ \subset]1; \infty[$, $J_- = 1$ identically and follow the same reasoning as in 1a above.

2b) In the case $j = 1$ and $\kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+ < -1$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_- \subset] - \infty; -1[$, $J_+ = 1$ identically and follow the same reasoning as in 1b above.

2c) In the case $j = 1$, $\kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+$ and $\overline{\kappa}_- > 1$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_+ \subset]1; \infty[$, $J_- = 1$ identically and follow the analogical reasoning as in 1a with the modification of the definition of M consisting in the change of sign of the right hand side of (4.7), *i.e.* we get (4.16) using $M\,(t) = -J_-\, J_+\,(x_1/t)\, \eta_s\,(D)\, J_-\, J_+\,(x_1/t)$.

2d) In the case $j = 1$, $\overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+$ and $\overline{\kappa}_+ < -1$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_- \subset] - \infty; -1[$, $J_+ = 1$ identically and follow the analogical reasoning using M as in 2c and getting (4.16) with M_0^- described in 1b instead of M_0^+.

3a) In the case $j \neq 1$ and $0 < \overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_+ \subset]0; \infty[$, $J_- = 1$ identically and follow the reasoning of 1a with x_j/t instead of x_1/t.

3b) In the case $j \neq 1$ and $\kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+ < 0$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_- \subset] - \infty; 0[$, $J_+ = 1$ identically and follow the reasoning of 1b with x_j/t instead of x_1/t.

3c) In the case $j \neq 1$, $\kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+$ and $\overline{\kappa}_- > 0$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_+ \subset]0; \infty[$, $J_- = 1$ identically and follow the reasoning which uses the modified definition of M from 2c and x_1/t is replaced by x_j/t.

3d) In the case $j \neq 1$, $\overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+$ and $\overline{\kappa}_+ < 0$, we take $J_\pm$ associated with $\overline{\kappa}_-, \overline{\kappa}_+, \kappa_-, \kappa_+$, supp $J_- \subset] - \infty; 0[$, $J_+ = 1$ identically and follow the analogical reasoning with x_j/t instead of x_1/t, M as in 3c and M_0^- instead of M_0^+ in (4.16). ■

PROPOSITION 4.2. – *Let* $\overline{v}_0 \in] - e_1; e_1[$, $r > 0$ *and* $g \in S^0\,(\mathcal{U}\,(r)\backslash\{\overline{v}_0\})$, *where*

(4.17)

$$\mathcal{U}\,(r) = \{(x_1, ..., x_d) \in \mathbb{R}^d : |x_2| + ... + |x_d| > r\,|x_1|\} \cup \mathcal{U}_1\,(-1 + r;\, 1 - r).$$

If $r_0 > 0$ *is small enough, then* $M^s_{g, J_0} \in \mathcal{G}(H)$ *holds for every* $J_0 \in C_0^\infty(B(\overline{v}_0; r_0))$ *and* $s \leq \min\{1, m-1\}$.

Proof. – Denote by $\ell^+(\overline{v}_0, v_0)$ the half-line

$$\ell^+(\overline{v}_0, v_0) = \{v_0 + \lambda(v_0 - \overline{v}_0) \in \mathbb{R}^d : \lambda > 0\}, \tag{4.18}$$

and for $r_0 > 0$ denote by $\ell^+_{r_0}(\overline{v}_0, v_0)$ the conical neighbourhood of $\ell^+(\overline{v}_0, v_0)$,

$$\ell^+_{r_0}(\overline{v}_0, v_0) = \left\{v \in \mathbb{R}^d : \left| \frac{v - v_0}{|v - v_0|} - \frac{v_0 - \overline{v}_0}{|v_0 - \overline{v}_0|} \right| < r_0 \right\}. \tag{4.19}$$

Due to Proposition 4.1 it suffices to consider $g \in S^0(\mathcal{U}(r) \setminus [-1; 1])$. Therefore it suffices to fix $v_0 \in \mathcal{U}(r) \setminus [-1; 1]$ and prove $M^s_{g, J_0} \in \mathcal{G}(H)$ for $g \in S^0(\ell^+_{r_0}(\overline{v}_0, v_0))$ with $r_0 > 0$ small enough. Let $\sigma_0 \in \operatorname{span}\{e_1, v_0 - \overline{v}_0\}$ be orthogonal to $v_0 - \overline{v}_0$ and oriented so that $\sigma_0 \cdot e_1 > 0$. Then

$$\sigma_0 \cdot (v_0 - \overline{v}_0) = 0 \quad \text{and} \quad \pm\sigma_0 \cdot (\overline{v}_0 \pm e_1) > 0. \tag{4.20}$$

We assume moreover $|\sigma_0| = 1$ and note that (4.20) may be written as

$$-\sigma_0 \cdot e_1 < \sigma_0 \cdot \overline{v}_0 = \sigma_0 \cdot v_0 < \sigma_0 \cdot e_1. \tag{4.21}$$

It is easy to see that taking $\sigma_+ \in \operatorname{span}\{e_1, v_0 - \overline{v}_0\}$ with $|\sigma_+| = 1$, $|\sigma_+ - \sigma_0| > 0$ small enough and with the correct orientation of the pair $\{\sigma_+, \sigma_0\}$, we get

$$-\sigma_+ \cdot e_1 < \sigma_+ \cdot \overline{v}_0 < \sigma_+ \cdot v_0 < \sigma_+ \cdot e_1. \tag{4.22}$$

Moreover for $r_0 > 0$ small enough we have

$$\begin{aligned} \overline{v} \in B(\overline{v}_0, r_0) \Rightarrow -\sigma_+ \cdot e_1 &< \sigma_+ \cdot \overline{v}_0 - 2r_0 \\ &< \sigma_+ \cdot \overline{v} < \sigma_+ \cdot \overline{v}_0 + 2r_0 < \sigma_+ \cdot v_0 \end{aligned} \tag{4.23}$$

$$\begin{aligned} v \in \ell^+_{r_0}(\overline{v}_0, v_0) &\Rightarrow \operatorname{sign} \sigma_+ \cdot \frac{v - v_0}{|v - v_0|} \\ &= \operatorname{sign} \sigma_+ \cdot \frac{v_0 - \overline{v}_0}{|v_0 - \overline{v}_0|} = 1 \Rightarrow \sigma_+ \cdot v > \sigma_+ \cdot v_0. \end{aligned} \tag{4.24}$$

Now let $J_+ \in C^\infty(\mathbb{R})$ be such that $J_+ = 0$ on $]-\infty; \sigma_+ \cdot \overline{v}_0 - 2\,r_0]$, $J_+ = \text{const}$ on $[\sigma_+ \cdot \overline{v}_0 + 2\,r_0; \infty[$, $J'_+ \geq 0$ and $J'_+ \geq 1$ on $[\sigma_+ \cdot \overline{v}_0 - r_0; \sigma_+ \cdot \overline{v}_0 + r_0]$. Then there is $c_0 > 0$ such that

$$(4.25)\quad J'_+(\sigma_+ \cdot \overline{v}) \neq 0,\, g(v) \neq 0 \Rightarrow \sigma_+ \cdot (v - \overline{v}) \geq \sigma_+ \cdot (v - v_0) \geq c_0 \langle v \rangle.$$

Similarly taking $\sigma_- \in \operatorname{span}\{e_1, v_0 - \overline{v}_0\}$ with $|\sigma_-| = 1$, $|\sigma_- - \sigma_0| > 0$ small enough and with the correct orientation, we get

$$(4.26)\qquad -\sigma_- \cdot e_1 < \sigma_- \cdot v_0 < \sigma_- \cdot \overline{v}_0 < \sigma_- \cdot e_1.$$

Then for $r_0 > 0$ small enough we have

$$(4.27)\qquad \begin{aligned} &\overline{v} \in B(\overline{v}_0, r_0)\\ &\Rightarrow \sigma_- \cdot v_0 < \sigma_- \cdot \overline{v}_0 - 2\,r_0\\ &< \sigma_- \cdot \overline{v} < \sigma_- \cdot \overline{v}_0 + 2\,r_0 < \sigma_- \cdot e_1 \end{aligned}$$

$$(4.28)\qquad v \in \ell^+_{r_0}(\overline{v}_0, v_0) \Rightarrow \sigma_- \cdot v < \sigma_- \cdot v_0.$$

Now let $J_- \in C^\infty(\mathbb{R})$ be such that $J_- = \text{const}$ on $]-\infty; \sigma_- \cdot \overline{v}_0 - 2\,r_0]$, $J_- = 0$ on $[\sigma_- \cdot \overline{v}_0 + 2\,r_0; \infty[$, $J'_- \leq 0$ and $J'_- \leq -1$ on $[\sigma_- \cdot \overline{v}_0 - r_0; \sigma_- \cdot \overline{v}_0 + r_0]$. Then there is $c_0 > 0$ such that

$$(4.29)\quad J'_-(\sigma_- \cdot \overline{v}) \neq 0,\, g(v) \neq 0 \Rightarrow \sigma_- \cdot (\overline{v} - v) \geq \sigma_- \cdot (v_0 - v) \geq c_0 \langle v \rangle.$$

Taking

$$(4.30)\qquad \begin{cases} M(t) = \dfrac{1}{t}\, J_-\left(\dfrac{\sigma_- \cdot x}{t}\right) J_+\left(\dfrac{\sigma_+ \cdot x}{t}\right) \eta_{s'}(D)\\ \qquad \times J_+\left(\dfrac{\sigma_+ \cdot x}{t}\right) J_-\left(\dfrac{\sigma_- \cdot x}{t}\right),\\ \text{where } \eta_{s'}(\xi) = g(\nabla p(\xi))^2 \langle \xi \rangle^{s'} \quad \text{with } s' = s - m + 1, \end{cases}$$

we have as before $[iV(t), M(t)] = O(\Phi(t))$ and $\mathbb{D}_{H_0} M(t) = M^+(t) + M^-(t) + O(t^{-2})$ with

$$(4.31)\qquad \begin{aligned} M^+(t) = -\frac{2}{t}\, J_-^2\left(\frac{\sigma_- \cdot x}{t}\right) J_+ J'_+\left(\frac{\sigma_+ \cdot x}{t}\right)\\ \times \left(\frac{\sigma_+ \cdot x}{t} - \sigma_+ \cdot \nabla p(D)\right) \eta_{s'}(D) + hc, \end{aligned}$$

$$(4.31') \qquad M^{-}(t) = -\frac{2}{t} J_{-} J'_{-} \left(\frac{\sigma_{-} \cdot x}{t}\right) J_{+}^{2} \left(\frac{\sigma_{+} \cdot x}{t}\right) \times \left(\frac{\sigma_{-} \cdot x}{t} - \sigma_{-} \cdot \nabla p(D)\right) \eta_{s'}(D) + hc.$$

Using first $\eta_{s'} \in S^{s'}(\mathbb{R}^d)$, $\eta_{s'} \geq 0$, $J_{+}^{2} \geq 0$, $-J_{-} J'_{-} \geq 0$ and afterward $J_{-}^{2} \geq 0$, $J_{+} J'_{+} \geq 0$ with (4.29), similarly as before we obtain the existence of constants $C_1, C_2, c_0 > 0$ with

$$(4.32) \quad M^{-}(t) \geq -C_1 t^{-2} \quad \text{and} \quad M^{+}(t) \geq c_0 M_0^{+}(t) - C_2 t^{-2},$$

where

$$(4.33) \qquad M_0^{+}(t) = \frac{2}{t} J_{-}^{2} \left(\frac{\sigma_{-} \cdot x}{t}\right) J_{+} J'_{+} \left(\frac{\sigma_{+} \cdot x}{t}\right) \eta_s(D) + hc$$

and η_s given as in (4.6). Now $J_0^2(x/t) \leq C J_{-}^{2}(\sigma_{-} \cdot x/t) J_{+} J'_{+}(\sigma_{+} \cdot x/t)$ implies

$$(4.34) \quad M_{g, J_0}^{s}(t) = \frac{1}{t} J_0^2\left(\frac{x}{t}\right) \eta_s(D) + hc + O(t^{-2}) \leq C M_0^{+}(t) + C_3 t^{-2} \leq C_4 (\mathbb{D}_{H(t)} M(t) + \Phi(t) + t^{-2})$$

i.e. $M_{g, J_0}^{s} \in \mathcal{G}(H)$, because the right hand side of (4.34) belongs to $\mathcal{G}(H)$. ■

PROPOSITION 4.3. – *Let* $\overline{v}_0 \in \mathbb{R}^d \backslash [-e_1; e_1]$ *and* $g \in S^0(\mathbb{R}^d \backslash \{\overline{v}_0\})$. *If* $r_0 > 0$ *is small enough then* $M_{g, J_0}^{s} \in \mathcal{G}(H)$ *holds for every* $J_0 \in C_0^{\infty}(B(\overline{v}_0; r_0))$ *and* $s \leq \min\{1, m-1\}$.

Proof. – We show that $M_{g, J_0}^{s} \in \mathcal{G}(H)$ for $g \in S^0(\ell_{r_0}^{+}(\overline{v}_0, v_0))$ with $r_0 > 0$ small enough if $\overline{v}_0$ and v_0 satisfy some suitable conditions.

1) Assume $\overline{v}_0 \in \ell(-e_1, e_1) \cap \mathcal{U}_1(1; \infty)$ and $v_0 \notin \ell(-e_1, e_1) \cap \mathcal{U}_1(-\infty; |\overline{v}_0|)$. In this situation, taking $\sigma_0 \in \operatorname{span}\{e_1, v_0 - \overline{v}_0\}$ such that $|\sigma_0| = 1$ and $\sigma_0 \cdot (v_0 - \overline{v}_0) = 0$ with the correct orientation we get $\pm\sigma_0 \cdot e_1 < \sigma_0 \cdot \overline{v}_0 = \sigma_0 \cdot v_0$. Modifying slightly σ_0 we may find $\sigma_{+} \in \operatorname{span}\{e_1, v_0 - \overline{v}_0\}$ such that $\pm\sigma_{+} \cdot e_1 < \sigma_{+} \cdot \overline{v}_0 < \sigma_{+} \cdot v_0$. It remains to follow the reasoning of the proof of Proposition 4.2 using J_{+} as above (with $r_0 > 0$ small enough) and $J_{-} = 1$ identically.

2) Assume $\overline{v}_0 \in \ell(-e_1, e_1) \cap \mathcal{U}_1(-\infty; -1)$ and $v_0 \notin \ell(e_1, -e_1) \cap \mathcal{U}_1(-|\overline{v}_0|; \infty)$. If σ_0 is as above then modifying slightly σ_0 we may find $\sigma_{-} \in \operatorname{span}\{e_1, v_0 - \overline{v}_0\}$ such that $\sigma_{-} \cdot v_0 < \sigma_{-} \cdot \overline{v}_0 < \pm\sigma_{-} \cdot e_1$. It

remains to follow the reasoning of the proof of Proposition 4.2 using J_- as above (with $r_0 > 0$ small enough) and $J_+ = 1$ identically.

3) Assume $\overline{v}_0 \in \ell(-e_1, e_1) \cap \mathcal{U}_1(1; \infty)$ and $v_0 \in \ell(-e_1, e_1) \cap \mathcal{U}_1(-\infty; |\overline{v}_0|)$. In this situation it suffices to follow the reasoning of the proof of Proposition 4.2 with $\sigma_+ = e_1$, $J_\pm$ as in the point 2c) of the proof of Proposition 4.1 and changing the sign in the definition of $M(t)$ (*i.e.* the proof is analogical as in the point 2c) of the proof of Proposition 4.1).

4) Assume $\overline{v}_0 \in \ell(-e_1, e_1) \cap \mathcal{U}_1(-\infty; -1)$ and $v_0 \in \ell(e_1, -e_1) \cap \mathcal{U}_1(-|\overline{v}_0|; \infty)$. In this situation it suffices to follow the reasoning of the proof of Proposition 4.2 with $\sigma_- = e_1$, $J_\pm$ as in the point 2d) of the proof of Proposition 4.1 and changing the sign in the definition of $M(t)$ (*i.e.* the proof is analogical as in the point 2d) of the proof of Proposition 4.1).

5) Let $\overline{v}_0 \notin \ell(-e_1, e_1)$ and assume moreover that the lines $\ell(\overline{v}_0, v_0)$ and $\ell(-e_1, e_1)$ are not parallel. In this situation we have $\overline{v}'_0 \neq 0$ and $\overline{v}'_0 \neq v'_0$, where $\overline{v}'_0$ and v'_0 denote the orthogonal projections of $\overline{v}_0$ and v_0 on $\{v \in \mathbb{R}^d : v \cdot e_1 = 0\}$.

In the case when $(v'_0 - \overline{v}'_0) \cdot \overline{v}'_0 > 0$ we set $\sigma_+ = \overline{v}'_0/|\overline{v}'_0|$, hence $\sigma_+ \cdot v'_0 > \sigma_+ \cdot \overline{v}'_0 > 0$. Then it is clear that (4.23-25) hold with $\pm e_1 \cdot \sigma_+ = 0$ and to obtain $M^s_{g, J_0} \in \mathcal{G}(H)$ it remains to follow the reasoning of the proof of Proposition 4.2 using the same J_+ and taking $J_- = 1$ identically.

In the case when $(v'_0 - \overline{v}'_0) \cdot \overline{v}'_0 = 0$ we have $\sigma_0 \cdot v'_0 = \sigma_0 \cdot \overline{v}'_0 > 0$ if $\sigma_0 = \overline{v}'_0/|\overline{v}'_0|$ and clearly a slight modification of σ_0 allows to find $\sigma_+ \in \mathbb{R}^d$ such that $\sigma_+ \cdot e_1 = 0$, $|\sigma_+| = 1$ and $\sigma_+ \cdot v'_0 > \sigma_+ \cdot \overline{v}'_0 > 0$. Then (4.23-25) hold with $\pm e_1 \cdot \sigma_+ = 0$ as above and $M^s_{g, J_0} \in \mathcal{G}(H)$ follows similarly.

In the case when $(v'_0 - \overline{v}'_0) \cdot \overline{v}'_0 < 0$ we set $\sigma_+ = \overline{v}'_0/|\overline{v}'_0|$, hence $\sigma_+ \cdot v'_0 < \sigma_+ \cdot \overline{v}'_0 < 0 = \pm e_1 \cdot \sigma_+$ and it remains to follow the reasoning of the proof of Proposition 4.2 replacing J_+ by $-J_+$ and taking $J_- = 1$ identically.

6) Let $\overline{v}_0 \notin \ell(-e_1, e_1)$ and assume that the lines $\ell(\overline{v}_0, v_0)$ and $\ell(-e_1, e_1)$ are parallel, *i.e.* $\overline{v}'_0 = v'_0 \neq 0$ and $\sigma_0 \cdot v'_0 = \sigma_0 \cdot \overline{v}'_0 > 0$ if $\sigma_0 = \overline{v}'_0/|\overline{v}'_0|$. Then a slight modification of σ_0 allows to find $\sigma_+ \in \mathbb{R}^d$ such that $|\sigma_+| = 1$ and $\sigma_+ \cdot v_0 > \sigma_+ \cdot \overline{v}_0 > \pm\sigma_+ \cdot e_1$, hence (4.23-25) hold and it remains to follow the reasoning of the proof of Proposition 4.2 using the same J_+ and taking $J_- = 1$ identically. ■

COROLLARY 4.4. – *Let* $s \leq \min\{1, m - 1\}$. *Then* $M^s_{g, J_0} \in \mathcal{G}(H)$ *if one of the following two conditions holds*

(4.35i)
$J_0 \in C_0^\infty(\mathbb{R}^d \backslash \{-e_1, e_1\})$ *and* $g \in S^0(\mathcal{U}(r) \backslash \operatorname{supp} J_0)$ *for a certain* $r > 0$;

(4.35ii) $\qquad J_0 \in C_0^\infty(\mathbb{R}^d \backslash [-e_1;\, e_1]) \quad and \;\; g \in S^0(\mathbb{R}^d \backslash \operatorname{supp} J_0).$

Proof. – Due to (3.33) we have $M^s_{g,\,J_1+J_2} \le 2\,M^s_{g,\,J_1} + 2\,M^s_{g,\,J_2}$ and it is clear that using a suitable partition of unity on supp J_0 we obtain $M^s_{g,\,J_0} \in \mathcal{G}(H)$ from Proposition 4.2 if (4.35i) holds or from Proposition 4.3 if (4.35ii) holds. ■

In order to prove Theorem 3.5 it remains to consider $g \in S^0(\tilde{\mathcal{U}}(r) \backslash \operatorname{supp} J_0)$, where

(4.36)
$$\tilde{\mathcal{U}}(r) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : |x_1| > 1 - 2r \text{ and } |x_2| + \ldots + |x_d| < 2r\,|x_1|\}.$$

The proof of $M^s_{g,\,J_0} \in \mathcal{G}(H)$ in this situation will be given in section 6.

5. PROPAGATION ESTIMATES

PROPOSITION 5.1. – *Assume that* $J_0 \in S^0(\mathcal{U}_j(\overline{\kappa}_-;\, \overline{\kappa}_+))$ *for certain* $\overline{\kappa}_-$, $\overline{\kappa}_+ \in \mathbb{R}$, $1 \le j \le d$. *Then* $M^\eta_{j,\,J_0} \in \mathcal{G}(H)$ *if one of the following conditions holds*

(5.1i)
$$\begin{cases} j = 1, \quad [\overline{\kappa}_-;\, \overline{\kappa}_+] \subset \mathbb{R} \backslash \{-1,\, 1\} \textit{ and } \eta(\xi) = g(\nabla p(\xi)) \\ \qquad\qquad \textit{with } g \in C_0^\infty(\mathcal{U}_1(\overline{\kappa}_-;\, \overline{\kappa}_+)), \end{cases}$$

(5.1ii)
$$j \ne 1, \quad [\overline{\kappa}_-;\, \overline{\kappa}_+] \subset \mathbb{R} \backslash \{0\} \quad \textit{and} \quad \eta \in S^{-\nu}(\mathbb{R}^d) \textit{ with } \nu \ge 0 \textit{ great enough.}$$

Instead of $J_0 \in S^0(\mathcal{U}_j(\overline{\kappa}_-;\, \overline{\kappa}_+))$ *we may assume that* $J_0(x) = \tilde{J}_0(x_j)$ *with* $\tilde{J}_0 \in C_0^\infty(]\overline{\kappa}_-;\, \overline{\kappa}_+[)$.

Before starting the proof we note that for $f \in S^0(\mathbb{R})$,

(5.2)
$$\mathbb{D}_{H_0} f\left(\frac{x_j}{t} - \nabla_j p(D)\right) = -\frac{1}{t}\left(\frac{x_j}{t} - \nabla_j p(D)\right) f'\left(\frac{x_j}{t} - \nabla_j p(D)\right)$$

Indeed, the left hand side of (5.2) may be written as

$$\frac{d}{ds}\, e^{i(s-t)\,H_0}\, f\left(\frac{x_j}{s} - \nabla_j p(D)\right) e^{i\,(t-s)\,H_0}\,\Bigg|_{s=t}$$
$$= e^{-it\,H_0}\, \frac{d}{ds}\, f\left(\frac{x_j}{s}\right)\Bigg|_{s=t} e^{it\,H_0},$$

hence we obtain the right hand side of (5.2) due to $\frac{d}{ds} f\left(\frac{x_j}{s}\right) = -\frac{x_j}{s^2} f'\left(\frac{x_j}{s}\right)$.

Further on for $\lambda \in \mathbb{R}$ by denote by $F_+(\lambda)$ the positive part of λ and by $F_-(\lambda)$ the negative part of λ as in (A.4) of Appendix. Let $f_0 \in C(\mathbb{R})$ be smooth on $\mathbb{R}\backslash\{0\}$, $0 \leq f_0 \leq 1$, $f_0' \geq 0$, $f_0(\lambda) = F_+(\lambda)$ for $\lambda \leq \frac{1}{2}$, $f_0(\lambda) = 1$ for $\lambda \geq 1$. Note that this definition of f_0 implies that

$$\lambda\,(f_0 + f_0')(\lambda) \geq c_0\, F_+(\lambda) \tag{5.3}$$

holds for a certain $c_0 > 0$. Let $\gamma_1 \in C_0^\infty(]-1;1[)$ be such that $\gamma_1 \geq 0$, $\int \gamma_1 = 1$ and $\gamma_\varepsilon(\lambda) = \gamma_1(\lambda/\varepsilon)/\varepsilon$ for $\varepsilon > 0$. We fix $\beta > 0$ small enough and define

$$f_t = f_0 * \gamma_{t^{-\beta}} \quad \text{for} \quad t \geq 1 \quad [\text{where } \gamma_{t^{-\beta}}(\lambda) = t^\beta\, \gamma_1(t^\beta \lambda)]. \tag{5.4}$$

It is easy to check (*cf.* Appendix) that f_t is an approximation of f_0 satisfying

(5.5)
$$|(f_t - f_0)(\lambda)| + \left|\lambda \frac{d}{d\lambda}(f_t - f_0)(\lambda)\right| \leq C_0\, t^{-\beta} \quad \text{for all } t \geq 1, \quad \lambda \in \mathbb{R}.$$

We denote

$$\tilde{f}_t(\lambda) = \lambda\, f_t(\lambda), \qquad \tilde{\tilde{f}}_t(\lambda) = \lambda \frac{d}{d\lambda} f_t(\lambda) \tag{5.6}$$

and remark (*cf.* Appendix) that (5.5-6) imply the existence of C_0, $c_0 > 0$ such that

$$(\tilde{f}_t + \tilde{\tilde{f}}_t)(\lambda) \geq c_0\, F_+(\lambda) - C_0\, t^{-\beta} \quad \text{for all } t \geq 1, \quad \lambda \in \mathbb{R}. \tag{5.7}$$

Moreover $f_0' \in L^1(\mathbb{R}) \Rightarrow \widehat{f_0'} \in L^\infty(\mathbb{R}) \Rightarrow \|\widehat{f_t^{(k+1)}}\|_{L^1(\mathbb{R})} \leq \|\widehat{f_0'}\|_{L^\infty(\mathbb{R})} \|\widehat{\gamma_{t^{-\beta}}^{(k)}}\|_{L^1(\mathbb{R})} \leq C_k\, t^{\beta(k+1)}$for every $k \in \mathbb{N}$ [where $f_t^{(n)}$ denotes the n-th derivatives of f_t with respect to λ and $\hat{\varphi}$ denotes the Fourier transform of φ]. Hence Lemma A.5 gives

$$\|[B, f_t^{(k)}(A)]\| \leq C t^{\beta(k+1)}\, \|[B, A]\|. \tag{5.8}$$

Proof of Proposition 5.1. – We shall prove $M_\pm^j \in \mathcal{G}(H)$ where

$$M_\pm^j(t) = \frac{1}{t} J_0\left(\frac{x}{t}\right) \eta(D)\, F_\pm\left(\frac{x_j}{t} - \nabla_j p(D)\right) \eta(D)\, J_0\left(\frac{x}{t}\right). \tag{5.9}$$

1a) We show $M_+^1 \in \mathcal{G}(H)$ in the case $-1 < \overline{\kappa}_- < \overline{\kappa}_+ < 1$.

Let $\overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+ < 1$, $J_\pm$ as in the point 1a) of the proof of Proposition 4.1 and set

(5.10)
$$M(t) = -J_- \, J_+ \left(\frac{x_1}{t}\right) \eta(D) \, f_t \left(\frac{x_1}{t} - \nabla_1 p(D)\right) \eta(D) \, J_- \, J_+ \left(\frac{x_1}{t}\right).$$

Then we have

$$(5.11) \quad \begin{cases} \mathbb{D}_{H(t)} M(t) = \mathbb{D}_{H_0} M(t) + [i V(t), M(t)], \\ \quad\text{with} \quad [iV(t), M(t)] = O(\Phi(t)) \\ \text{and} \quad \mathbb{D}_{H_0} M = M^0 + M^+ + M^- + O(t^{-1-\varepsilon}), \quad \varepsilon > 0, \end{cases}$$

and

$$(5.12) \quad M^0(t) = -J_- \, J_+ \left(\frac{x_1}{t}\right) \eta(D) \left(\mathbb{D}_{H_0} f_t \left(\frac{x_1}{t} - \nabla_1 p(D)\right)\right) \times \eta(D) \, J_- \, J_+ \left(\frac{x_1}{t}\right),$$

$$(5.13) \quad M^+(t) = \frac{2}{t} J_-^2 \, J_+ \, J'_+ \left(\frac{x_1}{t}\right) \eta^2(D) \, \tilde{f}_t \left(\frac{x_1}{t} - \nabla_1 p(D)\right) + hc,$$

$$(5.14) \quad M^-(t) = \frac{2}{t} J_+^2 \, J_- \, J'_- \left(\frac{x_1}{t}\right) \eta^2(D) \, \tilde{f}_t \left(\frac{x_1}{t} - \nabla_1 p(D)\right) + hc,$$

where we used $[f_t(x_1/t - \nabla_1 p(D)), J_- \, J_+(x_1/t) \, \eta(D)] = O(t^{-\varepsilon})$, being a consequence of (5.8).

Let $r > 0$ and let $J_1(x) = \tilde{J}_1(x_j)$ with $\tilde{J}_1 \in C_0^\infty(]\kappa_-; 1-r[)$ such that $-J_+^2 \, J_- \, J'_- \leq \tilde{J}_1^2$. Then using (A.2) of Appendix with commutators estimated by $O(t^{-\varepsilon})$ due to (5.8), we obtain

(5.15)
$$-M^-(t) \leq -\frac{C}{t} J_+^2 \, J_- \, J'_- \left(\frac{x_1}{t}\right) \eta^2(D) + Ct^{-1-\varepsilon} \leq C M_{g, J_1}^0(t) + C' \, t^{-1-\varepsilon}.$$

Since $g \in C_0^\infty(\mathcal{U}_1(\overline{\kappa}_-; \overline{\kappa}_+))$, the right hand side of (5.15) belongs to $\mathcal{G}(H)$ due to Proposition 4.1, hence $-M^- \in \mathcal{G}(H)$. Afterward

$J_0^2(x/t) \leq C J_-^2 \, J_+ \, J_+'(x_1/t)$ for a certain $C > 0$ implies

$$(5.16)\quad \tilde{M}^+(t) = \frac{1}{t} J_0\left(\frac{x}{t}\right) \eta(D)\, \tilde{f}_t\left(\frac{x_1}{t} - \nabla_1 p(D)\right) \eta(D)\, J_0\left(\frac{x}{t}\right) \leq C M^+(t) + C t^{-1-\varepsilon}.$$

It is easy to check (*cf.* Appendix) that

$$(5.17)\quad \left| \frac{d}{dt} f_t(\lambda) \right| \leq C_0\, t^{-1-\beta} \quad \text{for every } t \geq 1, \quad \lambda \in \mathbb{R},$$

hence using (5.17) and (5.2) we obtain

$$(5.18)\quad \mathbb{D}_{H_0}\, f_t\left(\frac{x_1}{t} - \nabla_1 p(D)\right) = -\frac{1}{t} \tilde{\tilde{f}}_t\left(\frac{x_1}{t} - \nabla_1 p(D)\right) + O(t^{-1-\beta}).$$

Since $J_0^2(x/t) \leq C J_- \, J_+(x_1/t)$, applying (A.2) of Appendix similarly as before we get

$$(5.19)\quad \tilde{M}^0(t) = \frac{1}{t} J_0\left(\frac{x}{t}\right) \eta(D) \tilde{\tilde{f}}_t\left(\frac{x_1}{t} - \nabla_1 p(D)\right) \eta(D)\, J_0\left(\frac{x}{t}\right) \leq C M^0(t) + C t^{-1-\varepsilon}$$

with $\varepsilon > 0$. We complete the proof of $M_+^1 \in \mathcal{G}(H)$ noting that (5.7), (5.16), (5.19) imply

$$\begin{aligned}(5.20)\quad M_+^1(t) &\leq \tilde{C}\,(\tilde{M}^0(t) + \tilde{M}^+(t) + t^{-1-\varepsilon}) \\ &\leq C\,(M^0(t) + M^+(t) + t^{-1-\varepsilon}) \\ &\leq C\,(\mathbb{D}_{H(t)}\, M(t) - M^-(t)) + C'\,\Phi(t) + C'\, t^{-1-\varepsilon}\end{aligned}$$

and the right hand side of (5.20) belongs to $\mathcal{G}(H)$.

1b) We show $M_+^1 \in \mathcal{G}(H)$ in the case $1 < \overline{\kappa}_- < \overline{\kappa}_+$ following the analogical reasoning where $J_\pm$ are associated with $1 < \overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+$ as in the point 2a) of the proof of Proposition 4.1. Similarly in the case $\overline{\kappa}_- < \overline{\kappa}_+ < -1$ we use $J_\pm$ associated with $\overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+ < -1$ as in the point 2d) of the proof of Proposition 4.1.

1c) In order to show $M_-^1 \in \mathcal{G}(H)$ in the case $-1 < \overline{\kappa}_- < \overline{\kappa}_+ < 1$ we use $J_\pm$ associated with $-1 < \kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+ < 1$ as in the point 1b) of the proof of Proposition 4.1 and we replace $x_1/t - \nabla_1 p(D)$ by $\nabla_1 p(D) - x_1/t$ in the definition of M given in (5.9). Then (5.15-20) hold with M^+, M^-, M^1_- instead of M^-, M^+, M^1_+ and $-M^+ \in \mathcal{G}(H)$ due to Proposition 4.1.

1d) We show $M^1_- \in \mathcal{G}(H)$ in the case $1 < \overline{\kappa}_- < \overline{\kappa}_+$, following the analogical reasoning as in the point 1c) above where $J_\pm$ are associated with $1 < \kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+$ as in the point 2c) of the proof of Proposition 4.1. Similarly in the case $\overline{\kappa}_- < \overline{\kappa}_+ < -1$ we use $J_\pm$ associated with $\kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+ < -1$ as in the point 2b) of the proof of Proposition 4.1.

2a) We show $M^j_+ \in \mathcal{G}(H)$ for $j \neq 1$ in the case $0 < \overline{\kappa}_- < \overline{\kappa}_+$ following the analogical reasoning as in the point 1a) above where x_1/t, $\nabla_1 p(D)$ are replaced by x_j/t, $\nabla_j p(D)$ and we use $J_\pm$ associated with $0 < \overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+$ as in the point 3a) of the proof of Proposition 4.1. Clearly to obtain $-M^- \in \mathcal{G}(H)$ instead of using Proposition 4.1 we note simply that $J_- = 1$ identically implies $M^-(t) = 0$ identically.

Similarly in the case $\overline{\kappa}_- < \overline{\kappa}_+ < 0$ we use $J_\pm$ associated with $\overline{\kappa}_- < \overline{\kappa}_+ < \kappa_- < \kappa_+ < 0$ as in the point 3d) of the proof of Proposition 4.1.

2b) We show $M^j_- \in \mathcal{G}(H)$ for $j \neq 1$ in the case $0 < \overline{\kappa}_- < \overline{\kappa}_+$ following the analogical reasoning as in the points 1c-d) above where x_1/t, $\nabla_1 p(D)$ are replaced by x_j/t, $\nabla_j p(D)$ and we use $J_\pm$ associated with $0 < \kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+$ as in the point 3c) of the proof of Proposition 4.1. Now we have $-M^+ \in \mathcal{G}(H)$, because $J_+ = 1$ identically implies $M^+(t) = 0$.

Similarly in the case $\overline{\kappa}_- < \overline{\kappa}_+ < 0$ we use $J_\pm$ associated with $\kappa_- < \kappa_+ < \overline{\kappa}_- < \overline{\kappa}_+ < 0$ as in the point 3b) of the proof of Proposition 4.1. ■

PROPOSITION 5.2. – *Theorem* 3.4 *holds if* $J_0 \in C_0^\infty(\mathbb{R}^d \setminus [-e_1; e_1])$.

Proof. – As we noted at the end of the section 3, due to Corollary 4.4 it suffices to fix $\overline{v}_0 \notin [-e_1; e_1]$ and to show that for $r_0 > 0$ small enough $M^\eta_{j, J_0} \in \mathcal{G}(H)$ holds for every $J_0 \in C_0^\infty(B(\overline{v}_0; r_0))$ and $\eta(\xi) = g(\nabla p(\xi))$ with $g \in C_0^\infty(B(\overline{v}_0; r_0))$ such that $g = 1$ on supp J_0.

1) Consider $\overline{v}_0 \notin \ell(-e_1; e_1)$. The condition $v_0 \notin \ell(-e_1; e_1)$ implies the existence of $k \in \{2, ..., d\}$ such that $v_0 \in \mathcal{U}_k(-\infty; 0) \cup \mathcal{U}_k(0; \infty)$. If $k = j$ then $M^\eta_{j, J_0} \in \mathcal{G}(H)$ follows from Proposition 5.1. In order to show $M^j_+ \in \mathcal{G}(H)$ in the case $k \neq j$ we follow the analogical reasoning as in the points 1a-b) (if $j = 1$) or 2a) (if $j \neq 1$) of the proof of Proposition 5.1, where instead of using $J_- = 1$ we take $J_- \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that $J_-(x_k/t) J_0(x/t) = J_0(x/t)$ and use $J_-(x_k/t)$ instead of $J_-(x_j/t)$ in

(5.10-13). Then instead of (5.14-15) we have

$$(5.14') \qquad M^{-}(t) = \frac{2}{t} J_{-} J'_{-}\left(\frac{x_k}{t}\right)\left(\frac{x_k}{t} - \nabla_k p(D)\right)\eta^2(D) \times J_{+}^2\left(\frac{x_j}{t}\right) f_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) + hc$$

and estimating similarly as in the proof of Theorem 3.3,

$$(5.15') \qquad -M^{-}(t) \leq \frac{C}{t} J_{-} J'_{-}\left(\frac{x_k}{t}\right)\left(\frac{x_k}{t} - \nabla_k p(D)\right)\eta^2(D) + Ct^{-1-\varepsilon} \leq C M^{\eta}_{k,\tilde{J}_0}(t) + C' t^{-1-\varepsilon}$$

with $\tilde{J}_0 \in C_0^{\infty}(\mathcal{U}_k(-\infty; 0) \cup \mathcal{U}_k(0; \infty))$ such that $\tilde{J}_0(x/t) J_{-}(x_k/t) = J_{-}(x_k/t)$. Since we already know $M^{\eta}_{k,\tilde{J}_0} \in \mathcal{G}(H)$, hence $-M^{-} \in \mathcal{G}(H)$ and we complete the proof as before.

Similarly in order to show $M^{j}_{-} \in \mathcal{G}(H)$ in the case $k \neq j$ we follow the analogical reasoning as in the points 1c-d) (if $j = 1$) or 2b) (if $j \neq 1$) of the proof of Proposition 5.1, where instead of using $J_{+} = 1$ we take $J_{+} \in C_0^{\infty}(\mathbb{R}\setminus\{0\})$ such that $J_{+}(x_k/t) J_0(x/t) = J_0(x/t)$ and use $J_{+}(x_k/t)$ instead of $J_{+}(x_j/t)$ in (5.11-13). Then (5.15′) holds with J_{+}, M^{+} instead of J_{-}, M^{-}, hence $-M^{+} \in \mathcal{G}(H)$ and we complete the proof as before.

2) Consider $\overline{v}_0 \in \ell(-e_1; e_1)\setminus[-e_1; e_1]$. If $j = 1$ then the statement follows from Proposition 5.1. If $j \neq 1$ then due to $\overline{v}_0 \in \mathcal{U}_1(-\infty; -1) \cup \mathcal{U}_1(1; \infty)$, in order to show $M^{j}_{+} \in \mathcal{G}(H)$ it suffices to follow the analogical reasoning as in 1) where we take $J_{-} \in C_0^{\infty}(\mathbb{R}\setminus[-1; 1])$ such that $J_{-}(x_1/t) J_0(x/t) = J_0(x/t)$ and use $J_{-}(x_1/t)$ instead of $J_{-}(x_k/t)$. Similarly in order to show $M^{j}_{-} \in \mathcal{G}(H)$ it suffices to follow the analogical reasoning as in 1) where we take $J_{+} \in C_0^{\infty}(\mathbb{R}\setminus[-1; 1])$ such that $J_{+}(x_1/t) J_0(x/t) = J_0(x/t)$ and use $J_{+}(x_1/t)$ instead of $J_{-}(x_k/t)$. ■

Proof of Theorem 3.3. – Using $\eta(\xi) = \langle\xi\rangle^{s/2}$ with s sufficiently negative and replacing H, p by $H_{\pm1}, p_{\pm1}$ in the proof of the point 2 of Proposition 5.1 we obtain $M^{s,\pm1}_{j,J_0} \in \mathcal{G}(H_{\pm1})$ for $J_0 \in C_0^{\infty}(\mathcal{U}_j(-\infty; 0) \cup \mathcal{U}_j(0; \infty))$. It remains to prove the analogical statement for $J_0 \in C_0^{\infty}(\mathcal{U}_k(-\infty; 0) \cup \mathcal{U}_k(0; \infty))$ with $k \neq j$, following the reasoning analogical as in the point 1 of the proof of Proposition 5.2. In fact, it suffices to replace p and $M^{\eta}_{k,\tilde{J}_0}$ in (5.14′-15′) by $p_{\pm1}$ and $M^{s,\pm1}_{j,\tilde{J}_0}$. ■

6. END OF THE PROOF

We begin by proving the following energy estimate

THEOREM 6.1. – *If* $\varphi \in L^2(\mathbb{R}^d)$ *then* $\lim_{\overline{r}\to\infty} \sup_{t\geq 1} \|E_{[\overline{r};\infty[}(H_0)\,U(t)\,\varphi\| = 0.$

We fix $r_0 > 0$ and introduce $\tilde{J}_1^+, \tilde{J}_1^- \in C^\infty(\mathbb{R})$ such that $\tilde{J}_1^+, \tilde{J}_1^- \geq 0$, $\tilde{J}_1^+(\lambda) = 0$ and $\tilde{J}_1^-(\lambda) = 1$ for $\lambda < -1 + 3r_0$, $\tilde{J}_1^+(\lambda) = 1$ and $\tilde{J}_1^-(\lambda) = 0$ for $\lambda > 1 - 3r_0$, the derivative $\tilde{J}_1^{+\prime} \geq 0$ and $\tilde{J}_1^{-\prime} \leq 0$. For $j = 2, \ldots, d$, we fix $\tilde{J}_j \in C_0^\infty(]-2r_0; 2r_0[)$ such that $\tilde{J}_j \geq 0$, $\tilde{J}_j = 1$ on $[-r_0; r_0]$ and set

$$(6.1)\qquad \tilde{J}_\pm(v^1, v^2, \ldots, v^d) = \tilde{J}_1^\pm(v^1)\,\tilde{J}_2(v^2)\ldots\tilde{J}_d(v^d).$$

Let $\xi^{cr} = (\xi_1^{cr}, \ldots, \xi_d^{cr})$ be as in (1.6′e) and set

$$(6.2)\qquad \begin{cases} G(t) = G_-(t) + G_+(t) \\ \text{with } G_\pm(t) = \tilde{J}_\pm\left(\dfrac{x}{t}\right)\dot{\chi}_{\pm 1}(t)\cdot(D - \xi^{cr}) + hc, \end{cases}$$

$$(6.3)\qquad \tilde{H}(t) = H(t) - G(t) + C_0 I, \qquad \tilde{H}_0(t) = H_0 - G(t) + C_0 I$$

where the constant $C_0 > 0$ is large enough to guarantee $\tilde{H}(t) \geq I + c_0 H_0$ and $\tilde{H}_0(t) \geq I + c_0 H_0$ with $c_0 > 0$ for all $t \geq C_0$. Such a choice of C_0 is clearly possible when $m > 1$. In the case $m = 1$ the ellipticity of $H_{\pm 1}$ implies $p(\xi) \geq |\xi_1| + c|\xi| - C$ with $c > 0$ and we note that $\pm G(t) \leq |\xi_1| + \Phi(t)|\xi| + C$.

Theorem 6.1 is a consequence of the following

PROPOSITION 6.2. – *If* $h(\lambda) = (1+\lambda)^{-1}$, *then there exists* $\tilde{M} \in \mathcal{G}(H)$ *such that for all* $t \geq 1$ *and* $0 < r \leq 1$ *we have* $\mathbb{D}_{H(t)}\,h(r\tilde{H}(t)) \geq -\tilde{M}(t)$.

Indeed, denoting $\varphi_t = U(t)\,\varphi$, $\overline{h} = 1 - h$, we have $\mathbb{D}_{H(t)}\,\overline{h}(r\tilde{H}(t)) = -\mathbb{D}_{H(t)}\,h(r\tilde{H}(t)) \leq \tilde{M}(t) \in \mathcal{G}(H(t))$, which implies that for every $\varepsilon > 0$ there is T_0 such that for $T \geq T_0$ and $0 < r \leq 1$,

$$[(\overline{h}(r\tilde{H}(t))\,\varphi_t, \varphi_t)]_{T_0}^T = \int_{T_0}^T dt\,(\mathbb{D}_{H(t)}\,\overline{h}(r\tilde{H}(t))\,\varphi_t, \varphi_t)$$
$$\leq \int_{T_0}^T dt\,(\tilde{M}(t)\,\varphi_t, \varphi_t) \leq \varepsilon.$$

Moreover $r \to 0 \Rightarrow \overline{h}(r\tilde{H}(t)) \to 0$ strongly, *i.e.* there is $r > 0$ such that $\sup_{1\leq t\leq T_0} \|\overline{h}(r\tilde{H}(t))^{1/2}\,\varphi_t\| < \varepsilon$ and the assertion of Theorem 6.1 follows from

(6.4)
$$\|E_{[\bar{r};\infty[}(H_0)\,\varphi_t\| \le \|E_{[\bar{r};\infty[}(H_0)\,h(r\,\tilde{H}(t))^{1/2}\|\,\|\varphi\| + \|\bar{h}(r\,\tilde{H}(t))^{1/2}\,\varphi_t\|,$$

because the first term of the right hand side of (6.4) goes to zero when $\bar{r} \to \infty$.

LEMMA 6.3. – *We have* $\mathbb{D}_{G(t)}\,V(t) = [i\,G(t),\,V(t)] + V'(t) = O(\Phi(t))$.

Proof. – Since

$$\mathbb{D}_{G(t)}\,V_{-1}(x - \chi_{-1}(t)) = \left(\left(\tilde{J}_-\left(\frac{x}{t}\right) - 1\right)\dot{\chi}_{-1}(t) + \tilde{J}_+\left(\frac{x}{t}\right)\dot{\chi}_1(t)\right)\cdot\nabla V_{-1}(x - \chi_{-1}(t)), \tag{6.5}$$

it is clear that for $x/t \in B(-e_1,\,r_0)$ we have $\tilde{J}_-(x/t) = 1$ and $\tilde{J}_+(x/t) = 0$, *i.e.* the right hand side of (6.5) is zero, and for $x/t \notin B(-e_1,\,r_0)$ it is $O(\Phi(t))$. Similarly

$$\mathbb{D}_{G(t)}\,V_1(x - \chi_1(t)) = \left(\tilde{J}_-\left(\frac{x}{t}\right)\dot{\chi}_{-1}(t) + \left(\tilde{J}_+\left(\frac{x}{t}\right) - 1\right)\dot{\chi}_1(t)\right)\cdot\nabla V_1(x - \chi_1(t)), \tag{6.5$'$}$$

i.e. for $x/t \in B(e_1,\,r_0)$ the right hand side of (6.5′) is 0 and for $x/t \notin B(e_1,\,r_0)$ it is $O(\Phi(t))$. ■

LEMMA 6.4. – *If* $h(\lambda) = (1+\lambda)^{-1}$ *and* $C > 0$ *is large enough then for* $0 < r \le 1$ *we have*

$$\|\mathbb{D}_{H(t)}\,h(r\,\tilde{H}(t)) - h(r\,\tilde{H}_0(t))\,(r\,\mathbb{D}_{H_0}\,G(t))\,h(r\,\tilde{H}_0(t))\| \le C\,(\Phi(t) + t^{-1-\varepsilon_0}). \tag{6.6}$$

Proof. – Indeed, we have $\mathbb{D}_{H(t)}\,h(r\,\tilde{H}(t)) = -h(r\,\tilde{H}(t))\,(r\,\mathbb{D}_{H(t)}\,\tilde{H}(t))\,h(r\,\tilde{H}(t))$ and $\mathbb{D}_{H(t)}\,\tilde{H}(t) = -\mathbb{D}_{H_0}\,G(t) + \mathbb{D}_{G(t)}\,V(t)$, hence Lemma 6.3 gives $\mathbb{D}_{H(t)}\,\tilde{H}(t) = -\mathbb{D}_{H_0}\,G(t) + O(\Phi(t))$ and it remains to justify the substitution of $h(r\,\tilde{H}_0(t))$ and $h(r\,\tilde{H}(t))$. We note that

(6.7)
$$\begin{cases} \mathbb{D}_{H_0}\,G_\pm(t) = \sum\limits_{1\le j\le d} M_j^\pm(t) + O(t^{-2}\langle D\rangle^{m-1} + t^{-1-\varepsilon_0}\langle D\rangle), \text{ where} \\ M_j^\pm(t) = \dfrac{1}{t}\,\nabla_j\,\tilde{J}_\pm\left(\dfrac{x}{t}\right)\left(\nabla_j\,p(D) - \dfrac{x_j}{t}\right)\dot{\chi}_{\pm 1}(t)\cdot(D - \xi^{cr}) + hc. \end{cases}$$

Let $\tilde{J}_0 \in C_0^\infty(]-1+3r_0;\ 1-3r_0[\times]-3r_0;\ 3r_0[^{d-1})$ be such that $\tilde{J}_0 = 1$ on supp $\nabla_j \tilde{J}_\pm$. Then

$$(6.8)\qquad (1-\tilde{J}_0)\left(\frac{x}{t}\right) M_j^\pm(t) = O(t^{-n}) \quad \text{for every } n \in \mathbb{N},$$

$$(6.9)\qquad \left[\tilde{J}_0\left(\frac{x}{t}\right) M_j^\pm(t),\ \tilde{H}_0(t)\right] = O(t^{-2}\langle D\rangle^{2m-1}),$$

and since there is $C_0 > 0$ such that $rh(r\tilde{H}(t)) \le C_0\langle D\rangle^{-m}$ holds for $0 < r \le 1$, we have

$$\begin{aligned} & h(r\tilde{H}(t))\,\tilde{J}_0\left(\frac{x}{t}\right) M_j^\pm(t) - \tilde{J}_0\left(\frac{x}{t}\right) M_j^\pm(t)\, h(r\tilde{H}_0(t)) \\ &= rh(r\tilde{H}(t))\left(\tilde{J}_0\left(\frac{x}{t}\right) M_j^\pm(t)\tilde{H}_0(t) - \tilde{H}(t)\tilde{J}_0\left(\frac{x}{t}\right) M_j^\pm(t)\right) h(r\tilde{H}_0(t)) \\ &= O(t^{-1}\Phi(t) + t^{-2}\langle D\rangle^{m-1}) \end{aligned}$$

and similarly $[\tilde{J}_0(x/t)\, M_j^\pm(t),\ h(r\tilde{H}_0(t))] = O(t^{-2}\langle D\rangle^{m-1})$. ■

Proof of Proposition 6.2. – Due to Lemma 6.4 and (6.7) we have

$$(6.10)\qquad \begin{aligned} & \|\mathbb{D}_{H(t)}\, h(r\tilde{H}_0(t)) - \sum_{1\le j\le d} rh(r\tilde{H}_0(t))\,(M_j^+ + M_j^-)(t)\, h(r\tilde{H}_0(t))\| \\ & \qquad \le C(\Phi(t) + t^{-1-\varepsilon_0}). \end{aligned}$$

Let J_0, $g_0 \in C_0^\infty(\mathbb{R}^d \setminus \ell(-e_1, e_1))$ be such that $J_0 = 1$ on supp $\nabla_j \tilde{J}_\pm$ for $2 \le j \le d$, $g_0 = 1$ on supp J_0 and $\overline{g} \in S^0(\mathbb{R}^d \setminus \operatorname{supp} J_0)$ such that $g_0^2 + \overline{g}^2 = 1$. Since $rh(r\tilde{H}(t)) \le C_0\langle D\rangle^{-m}$ we estimate

$$(6.11)\qquad \begin{aligned} & -rh(r\tilde{H}_0(t))\, M_j^\pm(t)\,\overline{g}(\nabla p(D))^2\, h(r\tilde{H}_0(t)) + hc \\ & \qquad \le CM_{\overline{g},\,J_0}^0(t) + Ct^{-2} \\ & \qquad \text{for} \quad j = 2, \dots, d. \end{aligned}$$

Introducing $\eta_0 \in C_0^\infty(\mathbb{R}^d)$ such that $\eta_0(\xi) = g_0(\nabla p(\xi))$, we get

$$(6.12)\qquad \begin{aligned} & -rh(r\tilde{H}_0(t))\, M_j^\pm(t)\, g_0(\nabla p(D))^2\, h(r\tilde{H}_0(t)) + hc \\ & \qquad \le CM_{j,\,J_0}^{\eta_0}(t) + Ct^{-3/2} \\ & \qquad \text{for} \quad j = 2, \dots, d. \end{aligned}$$

Indeed, setting $\eta(\xi) = \tilde{g}_0(\nabla p(\xi))$ with $\tilde{g}_0 \in C_0^\infty(\mathbb{R}^d)$ such that $\tilde{g}_0 = 1$ on supp g_0 we may write the left hand side of (6.12) as

$$\frac{1}{t} J_0\left(\frac{x}{t}\right) \eta_0(D)(B(t)A(t) + hc)\eta_0(D) J_0\left(\frac{x}{t}\right) + O(t^{-2})$$
$$\text{with } A(t) = x_j/t - \nabla_j p(D)$$
$$\text{and } B(t) = \Big(\nabla_j \tilde{J}_+\left(\frac{x}{t}\right)\dot{\chi}_1(t)\cdot(D-\xi^{cr}) + \nabla_j \tilde{J}_-\left(\frac{x}{t}\right)\dot{\chi}_{-1}(t) \cdot(D-\xi^{cr})\Big)\eta(D)\, rh(r\tilde{H}_0(t))^2 + hc,$$

and (6.12) follows from (A.3) of Lemma A.2.

Consider now $j = 1$, $\tilde{J}_0$ as in the proof of Lemma 6.4, $g \in S^0(\mathcal{U}(r_0)\backslash\text{supp}\,\tilde{J}_0)$, $g_1 \in C_0^\infty(\mathcal{U}_1(-1;1))$ and $\tilde{g} \in S^0(\tilde{\mathcal{U}}(r_0))$ such that $g_1^2 + g^2 + \tilde{g}^2 = 1$. Then we have

(6.14)
$$-rh(r\tilde{H}_0(t))\, M_1^{\pm}(t)\, g(\nabla p(D))^2\, h(r\tilde{H}_0(t)) + hc \le C M^0_{g,\tilde{J}_0}(t) + C t^{-3/2},$$

(6.15)
$$-rh(r\tilde{H}_0(t))\, M_1^{\pm}(t)\, g_1(\nabla p(D))^2\, h(r\tilde{H}_0(t)) + hc \le C M^{\eta_1}_{1,\tilde{J}_0}(t) + C t^{-3/2}$$

with $\eta_1 \in C_0^\infty(\mathbb{R}^d)$ such that $\eta_1(\xi) = g_1(\nabla p(\xi))$ similarly as (6.11-12). Since

(6.16)
$$\tilde{M}(t) = C\Big(M^0_{g,\tilde{J}_0} + M^{\eta_1}_{1,\tilde{J}_0} + M^0_{\tilde{g},J_0} + \sum_{2\le j\le d} M^{\eta_0}_{j,J_0}\Big)(t) + C(\Phi(t) + t^{-1-\varepsilon_0}) \in \mathcal{G}(H(t))$$

due to Corollary 4.4 and Proposition 5.1, we complete the proof of Proposition 6.2 noting that for C great enough in (6.16) we have $\mathbb{D}_{H(t)}\, h(r\tilde{H}(t)) \ge -\tilde{M}(t)$ due to

LEMMA 6.5. – *If $\tilde{g} \in S^0(\tilde{\mathcal{U}}(r_0))$ then*

(6.17)
$$rh(r\tilde{H}_0(t))\, M_1^{\pm}(t)\, \tilde{g}(\nabla p(D))^2\, h(r\tilde{H}_0(t)) + hc \ge -Ct^{-2}.$$

Proof. – Similarly as before it suffices to prove

$$M_1^{\pm}(t)\,\tilde{g}\,(\nabla\,p\,(D))^2 + hc \geq -Ct^{-2}\,\langle D\rangle^m. \tag{6.18}$$

Due to (1.6′e) there is $c_1 > 0$ such that

$$(\xi_1 - \xi_1^{cr})\,\nabla_1\,p\,(\xi)\,\tilde{g}\,(\nabla\,p\,(\xi))^2 \geq c_1\,\langle\xi\rangle^m\,\tilde{g}\,(\nabla\,p\,(\xi))^2. \tag{6.19}$$

Using (6.19) and the fact that $(v^1, ..., v^d) \in \operatorname{supp}\tilde{g} \subset \tilde{\mathcal{U}}(r_0) \Rightarrow |v^1| \geq 1 - 2\,r_0$, we get

(6.20)
$$\tilde{g}(\nabla p(\xi)) \neq 0 \Rightarrow \operatorname{sign}(\xi_1 - \xi_1^{cr}) = \operatorname{sign}\nabla_1 p(\xi) = \operatorname{sign}(\nabla_1 p(\xi) \pm (1 - 2r_0)),$$

(6.21)
$$\nabla_1\,p\,(\xi)\,(\xi_1 - \xi_1^{cr})\,\tilde{g}\,(\nabla\,p\,(\xi))^2 \geq ((1 - 3\,r_0)\,|\xi_1 - \xi_1^{cr}| + c\,\langle\xi\rangle^m)\,\tilde{g}\,(\nabla\,p\,(\xi))^2$$

with $c > 0$. Taking $J_1 \in C_0^\infty\,(]-1 + 3\,r_0;\,1 - 3\,r_0[)$ such that $0 \leq J_1 \leq 1$, we get

$$\begin{aligned} \pm J_1\left(\frac{x_1}{t}\right)\left(-\frac{x_1}{t}\right)&(D_1 - \xi_1^{cr})\,\tilde{g}\,(\nabla\,p\,(D))^2 \\ &+ hc \leq (1 - 3\,r_0)\,|D_1 - \xi_1^{cr}|\,\tilde{g}\,(\nabla\,p\,(D))^2 + C\,t^{-1} \end{aligned} \tag{6.22}$$

and due to (6.21),

(6.23)
$$\begin{aligned} M_\pm^0\,(t) = &\left(\nabla_1\,p\,(D) - J_1\left(\frac{x_1}{t}\right)\frac{x_1}{t}\right)(D_1 - \xi_1^{cr})\,\tilde{g}\,(\nabla\,p\,(D))^2 + hc \\ &\geq c\,\langle D\rangle^m\,\tilde{g}\,(\nabla\,p\,(D))^2 - C\,t^{-1} \geq -Ct^{-1}. \end{aligned}$$

Since $\dot{\chi}_{\pm 1}\,(t)\cdot(\xi - \xi^{cr}) = \pm(\xi_1 - \xi_1^{cr}) + O\,(\Phi\,(t)\,\langle\xi\rangle)$, using J_1 such that $\nabla_1\,\tilde{J}_\pm = J_1\,\nabla_1\,\tilde{J}_\pm$ we get

$$\begin{aligned} M_1^{\pm}\,(t)\,\tilde{g}(\nabla\,p\,(D))^2 + hc = \pm\,&\frac{1}{t}\,\nabla_1\,\tilde{J}_\pm\left(\frac{x}{t}\right)M_\pm^0\,(t) \\ &+ hc + O\,((\Phi\,(t) + t^{-2})\,\langle D\rangle^m), \end{aligned} \tag{6.24}$$

which implies (6.18) due to (6.23) and $\pm\nabla_1\,\tilde{J}_\pm \geq 0$. ■

End of the proof of Theorem 3.5. – Due to Corollary 4.4 it remains to prove that $M^{s}_{\tilde{g},\,J_0} \in \mathcal{G}(H)$ with $s \leq -m$, $\tilde{g} \in S^0(\tilde{\mathcal{U}}(r_0))$, $J_0 \in C_0^\infty(B(\overline{v}_0;\, r_0))$ and $\overline{v}_0 \in]-(1-5\,r_0)\,e_1;\,(1-5\,r_0)\,e_1[$. Taking $\tilde{J}_\pm$ such that $\pm\nabla_1 \tilde{J}_\pm > 0$ on $B(\overline{v}_0;\, r_0)$ in (6.23-24), we note that instead of (6.18) we have the following stronger estimate

$$(6.18') \qquad M_1^\pm(t)\,\tilde{g}\,(\nabla p(D))^2 + hc \geq c\,M^m_{\tilde{g},\,J_0}(t) - Ct^{-2}\langle D\rangle^m,$$

where $c > 0$ and similarly as before (6.18′) implies

(6.17′)
$$h(\tilde{H}_0(t))\,M_1^\pm(t)\,\tilde{g}\,(\nabla p(D))^2\,h(\tilde{H}_0(t)) + hc \geq c\,M^{-m}_{\tilde{g},\,J_0}(t) - Ct^{-2}.$$

We complete the proof using (6.16) in the estimate

$$(6.25) \qquad c\,M^{-m}_{\tilde{g},\,J_0}(t) \leq \mathbb{D}_{H(t)}\,h(\tilde{H}(t)) + \tilde{M}(t) \in \mathcal{G}(H(t)). \quad \blacksquare$$

End of the proof of Theorem 3.4. – It suffices to follow the proof of Proposition 5.2, where instead of assuming $\overline{v}_0 \in \ell(-e_1,\, e_1)\backslash[-e_1;\, e_1]$ in the point 2) we may assume $\overline{v}_0 \in \ell(-e_1,\, e_1)\backslash\{-e_1,\, e_1\}$, because Theorem 3.5 has been already proved without restrictions on supports considered in Corollary 4.4. $\blacksquare$

Proof of Theorem 2.2. – a) Similarly as in the reasoning from the section 3, it suffices to prove the existence of $\Omega(\hat{H}_k,\, H_k,\, J)$ on vectors $\varphi \in L^2(\mathbb{R}^d)$ such that $\varphi = h_r^2(H_k)\,\varphi$ for a certain $r > 0$, *i.e.* it suffices to prove the existence of the limit $\Omega(\hat{H}_k,\, H_k,\, J)\,h_r^2(H_k)$ expressed by the right hand side of (3.14) with $H_1(t) = H_k$, $H_2(t) = \hat{H}_k(t)$ and M given by (3.19) with h_r as in (3.18). Similarly as in the section 3 it suffices to consider $J \in C_0^\infty(B(0,\,1))$ with $0 \notin \operatorname{supp} \nabla_j J$ for $j = 1, \ldots, d$, and we have

$$(6.26) \qquad V(t,\, x + \chi_k(t))\,M(t) - M(t)\,V_k(x) = O(\Phi(t)),$$

$$(6.27) \qquad \mathbb{D}_{\hat{H}_k(t)}\,M(t) = \mathbb{D}_{H_k}\,M(t) + O(t^{-2}) = \tilde{M}(t) + O(t^{-2}),$$

where $\tilde{M}(t) \leq \tilde{M}_0(t)$ with $\tilde{M}_0$ expressed by (3.26) with $J_0 \in C_0^\infty(\mathbb{R}^d\backslash\{0\})$. Since $\tilde{M}_0 \in \mathcal{G}(H_k)$ due to Theorem 3.3, in order to complete the proof of the existence of $\Omega(\hat{H}_k,\, H_k,\, J)\,h_r^2(H_k)$ via Proposition 3.1 it remains to show $\tilde{M}_0 \in \mathcal{G}(\hat{H}_k)$, which is equivalent to

$$(6.28) \qquad e^{-i\,\chi_k(t)\cdot D}\,\tilde{M}_0(t)\,e^{i\,\chi_k(t)\cdot D} \in \mathcal{G}(H(t)).$$

However (2.8) implies

(6.29) $$e^{-i\chi_k(t)\cdot D}\, M^{s,k}_{j,J_0}(t)\, e^{i\chi_k(t)\cdot D} = \frac{1}{t}\, J_0\left(\frac{x-\chi_k(t)}{t}\right)\langle D\rangle^{s/2}\left|\frac{x_j-\chi_{k,j}(t)}{t} - \nabla_j p_k(D)\right| \times \langle D\rangle^{s/2} J_0\left(\frac{x-\chi_k(t)}{t}\right) = M^s_{j,J_k}(t) + O(t^{-1-\varepsilon_0}),$$

where $J_k \in C_0^\infty(\mathbb{R}^d\backslash\{e_1, -e_1\})$ is expressed by $J_{\pm 1}(v \pm e_1) = J_0(v)$. Then $M^s_{j,J_k} \in \mathcal{G}(H)$ due to Theorem 3.4 and (6.28) follows.

To prove the existence of $\Omega(H_k, \hat{H}_k, J)$ we take h_r defined as in (3.18) and note that

(6.30)
$$\lim_{r\to 0}\sup_{t\geq 1} \|(1-h_r^2(H_k))\,\hat{U}_k(t)\,\varphi\| = \lim_{r\to\infty}\sup_{t\geq 1} \|E_{[r;\,\infty[}(H_0)\,\hat{U}_k(t)\,\varphi\| = 0$$

due to Theorem 6.1. Therefore it suffices to prove the existence of the limit expressed by the right hand side of (3.14) with $H_1(t) = \hat{H}_k(t)$, $H_2(t) = H_k$, M given by (3.19) with $k = 0$. Now it is clear that the proof is analogical as above.

b) It suffices to prove the existence of $\Omega(H, H_0, \overline{J})$ on vectors $\varphi \in L^2(\mathbb{R}^d)$ such that $\varphi = h_r^2(H_0)\,\varphi$ for a certain $r > 0$, *i.e.* it suffices to prove the existence of the limit $\Omega(H, H_0, \overline{J})\, h_r^2(H_0)$ expressed by the right hand side of (3.14) with $H_1(t) = H_0$, $H_2(t) = H(t)$,

(6.31) $$M(t) = \overline{J}\left(\frac{x}{t}\right) h_r^2(H_0),$$

where h_r is as in (3.18). Moreover it suffices to consider $\overline{J} \in C^\infty(\mathbb{R}^d)$ such that supp $\nabla_j \overline{J}$ is compact for $j = 1, \ldots, d$ and $\pm e_1 \notin$ supp $\overline{J}$. Then clearly $V(t, x)\, M(t) = O(\Phi(t))$ and $\mathbb{D}_{H_0} M(t) = \tilde{M}(t) + O(t^{-2})$ with $\tilde{M}(t) \leq \tilde{M}_0(t)$ where

(6.32) $$\tilde{M}_0(t) = C \sum_{1\leq j\leq d} M^s_{j,J_0}(t) + Ct^{-3/2}$$

with certain $J_0 \in C_0^\infty(\mathbb{R}^d\backslash\{e_1, -e_1\})$, hence $\tilde{M}_0 \in \mathcal{G}(H)$ due to Theorem 3.4 (and clearly $\tilde{M}_0 \in \mathcal{G}(H_0)$ holds as well). Finally in order to prove the existence of $\Omega(H_0, H, \overline{J})$ we use (6.30) with $U(t)$ instead of $\hat{U}_k(t)$ and note as before that it suffices to prove the existence of the limit given by the right hand side of (3.14) with $H_1(t) = H(t)$, $H_2(t) = H_0$ and M given by (6.31). Clearly this prove is analogical as above. ■

Proof of Theorem 2.1. – In order to show the existence of $\Omega(H, J)$ it suffices to consider $J = J_1 + J_{-1}$ with $J_1 \in C_0^\infty(\mathbb{R}^d \backslash \{-e_1\})$, $J_{-1} \in C_0^\infty(\mathbb{R}^d \backslash \{e_1\})$ such that $J_1 = \text{const}$ in a neighbourhood of e_1 and $J_{-1} = \text{const}$ in a neighbourhood of $-e_1$. Using (6.30) with $U(t)$ instead of $\hat{U}_k(t)$, it suffices to prove the existence of the limit expressed by the right hand side of (3.14) with $H_1(t) = H_2(t) = H(t)$ and

$$M(t) = J_k\left(\frac{x}{t}\right) h_r^2(H_k), \tag{6.33}$$

for $k = \pm 1$ and h_r as in (3.18). Then $\mathbb{D}_{H(t)} M(t) = \tilde{M}(t) + O(t^{-2})$ with $\tilde{M}(t) \le \tilde{M}_0(t)$ and $\tilde{M}_0$ given by (6.32) with certain $J_0 \in C_0^\infty(\mathbb{R}^d \backslash \{e_1, -e_1\})$. As before we complete the proof noting that $\tilde{M}_0 \in \mathcal{G}(H)$ due to Theorem 3.4. ■

Proof of Theorem 2.4. – Let $J_1 \in C_0^\infty(B(0, 1))$ be such that $J_1 = 1$ on $B(0, 1/2)$ and for $0 < r \le 1$ we set $J_r(x) = J_1(x/r)$. If $\varphi \in L^2(\mathbb{R}^d)$ is an eigenvector of $H_{\pm 1}$ associated with the eigenvalue λ, *i.e.* $H_{\pm 1}\varphi = \lambda\varphi$, then due to the Lebesgue dominated convergence theorem,

$$\begin{aligned}\left\| e^{itH_{\pm 1}}(1 - J_r)\left(\frac{x}{t}\right) e^{-itH_{\pm 1}}\varphi \right\| &= \left\| e^{itH_{\pm 1}}(1 - J_r)\left(\frac{x}{t}\right) e^{-it\lambda}\varphi \right\| \\ &= \left\| (1 - J_r)\left(\frac{x}{t}\right)\varphi \right\| \to 0\end{aligned}$$

when $t \to \infty$, *i.e.* $\Omega(H_{\pm 1}, 1 - J_r)\varphi = (I - \Omega(H_{\pm 1}, J_r))\varphi = 0$ if φ is an eigenvector of $H_{\pm 1}$ or more generally if $\varphi \in \mathcal{H}_p(H)$, hence $\mathcal{H}_p(H) \subseteq \text{Ran}\, E_{\{0\}}(\vec{V}(H_{\pm 1}))$.

To prove the inverse inclusion it suffices to fix $\lambda_0 \in \mathbb{R}\backslash(\mathcal{C}_p \cup \sigma_p(H_{\pm 1}))$ [where $\mathcal{C}_p$ is the set of critical values of p defined by (1.6'd) and $\sigma_p(H_{\pm 1})$ is the set of eigenvalues of $H_{\pm 1}$] and to show that there is $r_0 > 0$ such that $\Omega(H_{\pm 1}, J_r)\, h(H_{\pm 1}) = 0$ holds for $h \in C_0^\infty(]\lambda_0 - r_0; \lambda_0 + r_0[)$ and $0 < r < r_0$. Following Graf [11] we note that $\Omega(H_{\pm 1}, J_r)\, h(H_{\pm 1}) = 0$ follows from

$$M_h^r(t) = \frac{1}{t}\, h(H_{\pm 1})\, J_r^2\left(\frac{x}{t}\right) h(H_{\pm 1}) \in \mathcal{G}(H_{\pm 1}). \tag{6.34}$$

Indeed, assuming that $\Omega(H_{\pm 1}, J_r)\, h(H_{\pm 1})\varphi \neq 0$ and denoting $\varphi_t = e^{-itH_{\pm 1}}\varphi$ we have

$$\lim_{t\to\infty} t\,(M_h^r(t)\varphi_t, \varphi_t) = \|\Omega(H_{\pm 1}, J_r^2)\, h(H_{\pm 1})\varphi\|^2 > 0,$$

hence $t \longrightarrow (M_h^r(t)\,\varphi_t,\, \varphi_t)$ is not integrable on $[1; \infty[$, *i.e.* $M_h^r \notin \mathcal{G}(H_{\pm 1})$. Denote

(6.35)
$$M^r(t) = J_r\left(\frac{x}{t}\right) h(H_{\pm 1}) \left(\frac{x}{t} \cdot \nabla p_{\pm 1}(D) + hc\right) h(H_{\pm 1})\, J_r\left(\frac{x}{t}\right)$$

If $\tilde{h} \in C_0^\infty(\mathbb{R})$ is such that $h(\lambda) = \tilde{h}((\lambda + C_0))^{-1})$, then Lemma A.5 of Appendix implies

$$\left\| \left[h(H_{\pm 1}),\, J\left(\frac{x}{t}\right)\right] \right\| = \left\| \left[\tilde{h}((H_{\pm 1} + C_0)^{-1}),\, J\left(\frac{x}{t}\right)\right] \right\| \le C \left\| \left[(H_{\pm 1} + C_0)^{-1},\, J\left(\frac{x}{t}\right)\right] \right\| = O(t^{-1}) \tag{6.36}$$

for every $J \in C_0^\infty(\mathbb{R}^d)$. Therefore writing $J_r = J_r\, J_{2r}$ we have

$$\begin{aligned} \frac{1}{t} M^r(t) = & \frac{1}{t} J_r\left(\frac{x}{t}\right) h(H_{\pm 1})\, J_{2r}\left(\frac{x}{t}\right) \\ & \times \left(\frac{x}{t} \cdot \nabla p_{\pm 1}(D) + hc\right) \\ & \times J_{2r}\left(\frac{x}{t}\right) h(H_{\pm 1})\, J_r\left(\frac{x}{t}\right) + O(t^{-2}) \\ & \le r\, \frac{\tilde{C}}{t} J_r\left(\frac{x}{t}\right) h^2(H_{\pm 1})\, J_r\left(\frac{x}{t}\right) \\ & + \tilde{C}_r\, t^{-2} = \tilde{C} r\, M_h^r(t) + O(t^{-2}). \end{aligned} \tag{6.37}$$

We have $\mathbb{D}_{H_{\pm 1}} M^r = M_1^r + M_2^r$ with

$$\begin{aligned} M_1^r(t) = & \frac{1}{t} J_r\left(\frac{x}{t}\right) h(H_{\pm 1}) [i H_{\pm 1},\, x \cdot \nabla p_{\pm 1}(D) + hc] \\ & \times h(H_{\pm 1})\, J_r\left(\frac{x}{t}\right) - \frac{1}{t} M^r(t), \end{aligned} \tag{6.38}$$

$$\begin{aligned} M_2^r(t) = & 2\, J_r\left(\frac{x}{t}\right) h(H_{\pm 1}) \left(\frac{x}{t} \cdot \nabla p_{\pm 1}(D) + hc\right) \\ & \times h(H_{\pm 1})\, \mathbb{D}_{H_{\pm 1}} J_r\left(\frac{x}{t}\right) + hc. \end{aligned} \tag{6.39}$$

However for $r_0 > 0$ small enough there is $c > 0$ such that the Mourre estimate

$$h(H_{\pm 1})[i H_{\pm 1}, x \cdot \nabla p_{\pm 1}(D) + hc] h(H_{\pm 1}) \geq ch^2(H_{\pm 1}) \tag{6.40}$$

holds (*cf.* Appendix for the proof), hence $M_1^r(t) \geq (c - C' r) M_h^r(t) - C_r t^{-2}$ due to (6.38) and (6.37). Therefore for $0 < r < r_0 \leq C'/(2c)$ we have

$$M_h^r(t) \leq C(M_1^r(t) + t^{-2}) = C(\mathbb{D}_{H_{\pm 1}} M_0^r(t) - M_2^r(t) + t^{-2}) \tag{6.41}$$

and to complete the proof of $M_h^r \in \mathcal{G}(H_{\pm 1})$, it remains to check that $-M_2^r \in \mathcal{G}(H_{\pm 1})$.

However due to Lemma A.5 of Appendix, (3.22) still holds with h_r replaced by $h \in C_0^\infty(\mathbb{R})$ and $\eta(\xi) = h(p_{\pm 1}(\xi))$. Now it is clear that taking $\tilde{J}_r \in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that $\tilde{J}_r = 1$ on supp $\nabla_j J_r$ for $j = 1, ..., d$, we have $-M_2^r(t) \leq C(\sum_{1\leq j\leq d} M_{j, \tilde{J}_r}^{s, \pm 1}(t) + t^{-2}) \in \mathcal{G}(H_{\pm 1})$ due to Theorem 3.3 if s is fixed sufficiently negative. ■

APPENDIX

Remark A.1. – a) Let $A_1(t), A_2(t) \in B(L^2(\mathbb{R}^d))$ and $\varphi_1, \varphi_2, \varphi_3 \in L^2(\mathbb{R}^d)$. Then

(A.1.a)
$$\lim_{t\to\infty} A_k(t)\varphi_k = \varphi_{k+1} \quad \text{for } k = 1, 2 \Rightarrow \lim_{t\to\infty} A_2(t) A_1(t)\varphi_1 = \varphi_3$$

where the existence of left hand side limits implies the existence of the right hand side limit. b) Let $\varphi \in L^2(\mathbb{R}^d)$, $A(t)$, $A_\varepsilon(t) \in B(L^2(\mathbb{R}^d))$ for $t \geq 1$, $\varepsilon > 0$ and for $\varepsilon \to 0$ we have $A_\varepsilon(t) \to A(t)$ in $B(L^2(\mathbb{R}^d))$ uniformly with respect to $t \geq 1$. Then

$$\lim_{t\to\infty} A_\varepsilon(t)\varphi \text{ exists for every } \varepsilon > 0 \Rightarrow \lim_{t\to\infty} A(t)\varphi \text{ exists.} \tag{A.1.b}$$

LEMMA A.2. – *Assume that A, B are self-adjoint in $L^2(\mathbb{R}^d)$, $B \in B(L^2(\mathbb{R}^d))$ and the sesquilinear forms $[A, B]$ and $[A, [A, B]]$ are bounded in $L^2(\mathbb{R}^d)$. Then there exists a constant C (independent on operators A and*

B) such that

(A.2)
$$A \geq 0,\ B \geq 0 \Rightarrow AB + hc \geq -C\,\|B\|^{1/4}\,\|[A,\,B]\|^{1/2}\,\|[A,\,[A,\,B]]\|^{1/4}.$$

$$\text{(A.3)}\qquad AB + hc \leq \|B\|\,|A| + 2\,C\,\|B\|^{1/4}\,\|[A,\,B]\|^{1/2}\,\|[A,\,B]]\|^{1/4}.$$

Proof. – For $Z \subset \mathbb{R}$ let $\mathbf{1}_Z(\cdot)$ denote the characteristic function of Z on $\mathbb{R}$ and set

$$\text{(A.4)}\qquad F_+(\lambda) = \lambda\,\mathbf{1}_{[0;\,\infty[}(\lambda), \qquad F_-(\lambda) = -\lambda\,\mathbf{1}_{]-\infty;\,0[}(\lambda).$$

Since

$$F_+(A)\,B + hc = F_+(A)^{1/2}\,BF_+(A)^{1/2} - \frac{1}{2}\,[F_+(A)^{1/2},\,[F_+(A)^{1/2},\,B],$$

it is clear that (A.2) follows from

LEMMA A.3 (Nirenberg-Trèves). – *Let A, B be as in Lemma A.2. Then there exists a constant C (independent on operators A and B) such that*

(A.5)
$$\|[F_+(A)^{1/2},\,[F_+(A)^{1/2},\,B]]\| \leq C\,\|B\|^{1/4}\,\|[A,\,B]\|^{1/2}\,\|[A,\,[A,\,B]]\|^{1/4}.$$

The proof of Lemma A.3 may be found in [20] or [13], § 26.8.

To show (A.3) we replace B by $\|B\| - B \geq 0$, getting [due to (A.5)]

$$\text{(A.6)}\quad F_+(A)\,(\|B\| - B) + hc \geq -C\,\|B\|^{1/4}\,\|[A,\,B]\|^{1/2}\,\|[A,\,[A,\,B]]\|^{1/4}.$$

Then it remains to add (A.6) and the analogical estimate with F_- and $\|B\| + B \geq 0$, *i.e.*

(A.7)
$$F_-(A)\,(\|B\| + B) + hc \geq -C\,\|B\|^{1/4}\,\|[A,\,B]\|^{1/2}\,\|[A,\,[A,\,B]]\|^{1/4}. \quad \blacksquare$$

COROLLARY A.4. – *Let $\nu \in \mathbb{R}$, $\eta \in S^\nu(\mathbb{R}^d)$ and $J \in S^0(\mathbb{R}^d)$. Then we have the following Garding type inequality*:

$$\text{(A.8)}\qquad \eta \geq 0, \quad J \geq 0 \Rightarrow \eta(D)\,J(x/t) + hc \geq -Ct^{-1}\,\langle D\rangle^{\nu-1}.$$

Proof. – Introducing $\eta_1 \in S^1(\mathbb{R}^d)$ given by $\eta_1(\xi) = \eta(\xi)\langle\xi\rangle^{1-\nu}$, we have

$$\langle D\rangle^{(1-\nu)/2}(\eta(D)\,J(x/t)+hc)\,\langle D\rangle^{(1-\nu)/2} = \eta_1(D)\,J(x/t)+hc+O(t^{-1})$$

and it remains to use (A.2) with $A(t) = \eta_1(D)$ and $B(t) = J(x/t)$, noting that

$$B(t) = O(1), \quad [A(t), B(t)] = O(t^{-1}),$$
$$[A(t), [A(t), B(t)]] = O(t^{-2}). \quad \blacksquare$$

Proof of Proposition 3.1. – We denote $\Omega_t = U_2(t)^* M(t)\,U_1(t)$ and check that

$$\text{(A.9)} \quad \|\Omega_{t''}\varphi - \Omega_{t'}\varphi\| \leq \sup_{\psi\in D(H_0),\,\|\psi\|\leq 1} \int_{t'}^{t''} dt \left|\frac{d}{dt}(\Omega_t\varphi, \psi)\right| \to 0 \text{ for } t', t'' \to \infty,$$

if $\varphi \in D(H_0)$. However denoting $\varphi_t^1 = U_1(t)\varphi$, $\psi_t^2 = U_2(t)\psi$, (3.11-12) give

$$\text{(A.10)} \quad \left|\frac{d}{dt}(\Omega_t\varphi, \psi)\right| = |((i(H_2(t)\,M(t) - M(t)\,H_1(t)) + M'(t))\,\varphi_t^1, \psi_t^2)|$$
$$\leq 4\,\tilde{M}_0(t)\,[\varphi_t^1, \varphi_t^1]^{1/2}\,\tilde{M}_0(t)\,[\psi_t^2, \psi_t^2]^{1/2} + C\,\Phi(t)\,\|\varphi\|\,\|\psi\|.$$

Since $\int_{t'}^{t''}\Phi(t)\,dt \to 0$ for $t', t'' \to \infty$ we get (A.9) using (3.13) to estimate

$$\int_{t'}^{t''} \tilde{M}_0(t)\,[\varphi_t^1, \varphi_t^1]^{1/2}\,\tilde{M}_0(t)\,[\psi_t^2, \psi_t^2]^{1/2}\,dt \leq$$
$$\leq \left[\int_{t'}^{t''} \tilde{M}_0(t)\,[\varphi_t^1, \varphi_t^1]\,dt\right]^{1/2} \left[\int_{t'}^{t''} \tilde{M}_0(t)\,[\psi_t^2, \psi_t^2]\,dt\right]^{1/2}. \quad \blacksquare$$

Proof of (5.5), (5.7), (5.17). – By the definition of f_t we have

$$(f_t - f_0)(\lambda) = \int (f_0(\lambda+\lambda') - f_0(\lambda))\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda',$$

$$\lambda\frac{d}{d\lambda}(f_t - f_0)(\lambda) = \int (f_0'(\lambda+\lambda')\,\lambda - f_0'(\lambda)\,\lambda)\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda',$$

$$\frac{d}{d\lambda}f_t(\lambda) = \frac{d}{dt}(f_t - f_0)(\lambda) = \int (f_0(\lambda+\lambda') - f_0(\lambda))\,\frac{d}{dt}\gamma_{t^{-\beta}}(\lambda')\,d\lambda'.$$

Since f_0 and $\lambda \to f_0'(\lambda)\,\lambda$ are Lipschitz continuous,

$$|(f_t - f_0)(\lambda)| + \left| \int (f_0'(\lambda+\lambda')(\lambda+\lambda') - f_0'(\lambda)\,\lambda)\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' \right| \leq C \int |\lambda'|\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' = C_1\,t^{-\beta}$$

and we complete the proof of (5.5) noting that

$$\left| \int f_0'(\lambda+\lambda')\,\lambda'\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' \right| \leq C \int |\lambda'|\gamma_{t^{-\beta}}(\lambda')\,d\lambda' = C_1\,t^{-\beta}.$$

The convolution of f_0 with first term of $\frac{d}{dt}\gamma_{t^{-\beta}}(\lambda') = \beta\,t^{\beta-1}\,\gamma_1(t^\beta\,\lambda') + \beta\,t^{\beta-1}\,\lambda'\,\gamma_1'(t^\beta\,\lambda')\,t^\beta$ may be estimated by the same integral as before with additional factor t^{-1} and with the second term estimated by $C \int |\lambda'|^2\,t^{\beta-1}\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' = C_2\,t^{-1-\beta}$, completing the proof of (5.17).

Since $\operatorname{supp} f_t(\lambda) \subset [-t^{-\beta}; \infty) \Rightarrow \lambda\,f_t(\lambda) \geq -Ct^{-\beta}$, using (5.5) we may write $\lambda\,\frac{d}{d\lambda}\,f_t(\lambda) = \lambda\,f_0'(\lambda) + O(t^{-\beta}) = F_+(\lambda) + O(t^{-\beta})$ for $\lambda \leq \frac{1}{4}$. Hence (5.7) holds for $\lambda \leq \frac{1}{4}$ and to complete the proof of (5.7) we note that $\lambda \geq \frac{1}{4} \Rightarrow \lambda\,f_t(\lambda) \geq \lambda\,f_t\left(\frac{1}{4}\right) \geq \lambda\,c_1$ with $c_1 = \frac{1}{8}\int_0^{1/8}\gamma_1 > 0$. ■

LEMMA A.5. – *Let $f \in S^0(\mathbb{R})$ be such that $f' \in C_0^\infty(\mathbb{R})$. Then*

$$\|Bf(A) - f(\tilde{A})\,B\| \leq \frac{1}{2\pi}\,\|\hat{f}'(\lambda)\|_{L^1(\mathbb{R},\,d\lambda)}\,\|BA - \tilde{A}B\|, \tag{A.11}$$

where $\hat{f}'$ denotes the Fourier transform of f'.

Proof. – If $f \in C_0^\infty(\mathbb{R})$, then $Bf(A) - f(\tilde{A})\,B$ may be expressed in the form

$$\int \frac{d\lambda}{2\pi}\,\hat{f}(\lambda)\,(Be^{i\lambda A} - e^{-i\lambda\tilde{A}}\,B) = \int \frac{d\lambda}{2\pi}\,\hat{f}(\lambda)\,i\,\lambda \int_0^1 d\sigma\,e^{i\,(1-\sigma)\,\lambda\tilde{A}}\left(BA - \tilde{A}B\right)e^{i\sigma\lambda A}$$

and (A.11) follows due to $i\,\lambda\,\hat{f}(\lambda) = \hat{f}'(\lambda)$. To complete the proof for a general f, it suffices to pass to the limit $\varepsilon \to 0$ in estimates for the sequence of functions $f(x)\,\gamma(\varepsilon\,x)$ with $\gamma \in C_0^\infty(\mathbb{R})$, $\gamma = 1$ in a neighbourhood of 0. ■

Proof of (6.40). – Since $\mathcal{C}_p$ is closed, we have $[\lambda_0 - r_0;\ \lambda_0 + r_0] \cap \mathcal{C}_p = \emptyset$ for $r_0 > 0$ small enough, implying

(A.12)
$$\begin{cases} h(H_0)\,[ix \cdot \nabla p_{\pm 1}(D) + hc,\ H_0]\,h(H_0) \\ \quad = h(H_0)\,|\nabla p_{\pm 1}(D)|^2\,h(H_0) \geq 2\,c_0\,h^2(H_0) \\ \text{with } 2\,c_0 = \inf\{|\nabla p_{\pm 1}(\xi)|^2 : p_{\pm 1}(\xi) \in]\,\lambda_0 - r_0;\ \lambda_0 + r_0[\} > 0 \end{cases}$$

and the hypothesis of H_0-compactness of $\langle x \rangle V_{\pm 1}(x)$ implies the compactness of $h(H_{\pm 1}) - h(H_0)$ for $h(\lambda) = (\pm i + \lambda)^{-1}$ (via the first resolvent identity) and consequently for every $h \in C_0^\infty(\mathbb{R})$ (via Stone-Weierstrass theorem), hence the difference of left hand sides of (6.40) and (A.12) is compact and to get rid of the compact term it remains to note that shrinking the size of $]\lambda_0 - r_0;\ \lambda_0 + r_0[$ implies the strong convergence of spectral projectors $E_{]\lambda_0 - r_0;\,\lambda_0 + r_0[}(H_{\pm 1})$ to $E_{\{\lambda_0\}}(H_{\pm 1}) = 0$, *i.e.* we have $\|E_{]\lambda_0 - r_0;\,\lambda_0 + r_0[}(H_{\pm 1})\,K\| \to 0$ when $r_0 \to 0$ for any compact operator K. ■

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